GENERIC DIFFERENTIAL EQUATIONS ARE STRONGLY MINIMAL

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ABSTRACT. In this manuscript we develop a new technique for showing that a nonlinear algebraic differential equation is strongly minimal based on the recently developed notion of the degree of nonminimality of Freitag and Moosa. Our techniques are sufficient to show that generic order h differential equations with nonconstant coefficients are strongly minimal, answering a question of Poizat (1980).

1. INTRODUCTION

Let $f(x)=0$ be an algebraic differential equation of in a single indeterminant x with coefficients in a differential field (K,δ) of characteristic zero. In this manuscript, we are particularly interested in the case that $f(x)$ is nonlinear and of order ≥ 2. The central property we study is the *strong minimality* of the solution set of $f(x)=0$. The notion of strong minimality comes from model theory; in general, a *definable set* X is *strongly minimal* if every definable subset is finite or cofinite, uniformly in parameters. In our setting, we are interested in the situation $X=\{x\in\mathcal{U}\mid f(x)=0\}$ is the set of solutions to an algebraic differential equation where $\mathcal{U}$ is a differentially closed field. Let h be the order of f – that is, the highest derivative of x appearing in f. The strong minimality of X is equivalent to:

- f is irreducible as a (multivariate) polynomial over K^{alg} and given any $a\in\mathcal{U}$ with $f(a)=0$, and any differential field $K_1\leq\mathcal{U}$ with $K\leq K_1$, the transcendence degree of $K_1\langle a\rangle$ over K_1 is either 0 or h.

Strong minimality is an intensively studied property of definable sets, and has been at the center of many important number theoretic applications of model theory and differential algebra [**?**, **?**, **?**, **?**]. Despite this, there are relatively few (classes of) equations which have been shown to satisfy the property - so few, that we are in fact able to give below what we believe to be a (at the moment) comprehensive list of those equations which have been shown to be strongly minimal. Showing the strong minimality of a given equation is itself sometimes a motivational goal, but often it is an important piece of a more elaborate application, since it allows one to use powerful tools from geometric stability theory. The existing strategies to prove strong minimality are widely disparate but apply only to very special cases. In roughly chronological order:

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(1) Poizat established that the set of non-constant solutions of $x \cdot x'' = x'$ is strongly minimal (see [**?**] for an explanation). Poizat's arguments were generalized by Brestovski [**?**] to a class of very specifically chosen order two differential equations with constant coefficients.

(2) Hrushovski's work [**?**] around the Mordell-Lang conjecture proved the strong minimality of Manin kernels of nonisotrivial simple abelian varieties. It uses specific properties of abelian varieties as well as model-theoretic techniques around *modularity* of strongly minimal sets.

(3) Nagloo and Pillay [**?**] show that results of the Japanese school of differential algebra [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] imply that Painlevé equations with generic coefficients are strongly minimal. The techniques employed are differential algebraic and valuation theoretic, relying on very specific properties of the equations.

(4) Work of Freitag and Scanlon [**?**] for the differential equation satisfied by the j-function ultimately relies on point-counting and o-minimality via the Pila-Wilkie theorem as applied in [**?**, **?**]; the argument there is very specific to the third order nonlinear differential equation satisfied by the j-function. Later, Aslanyan [**?**] produced another proof, ultimately relying on similar (stronger) inputs of [**?**].

(5) Casale, Freitag and Nagloo [**?**] show that equations satisfied by Γ-automorphic functions on the upper half-plane for Γ a Fuchsian group of the first kind are strongly minimal. The arguments use differential galois theory with some additional analytic geometry, and the techniques again are very specific to the third order equations of this specific form.

(6) Jaoui shows that generic planar vector fields over the constants give rise to strongly minimal order two differential varieties [**?**]. The techniques rely on various sophisticated techniques including o-minimality and results from foliation theory, some of which are particular to the specific class of equations considered.

(7) Blázquez-Sanz, Casale, Freitag, and Nagloo [**?**] prove the strong minimality of certain general Schwarzian differential equations.

We should also mention that strong minimality in this context was perhaps first studied by Painlevé using different language in [**?**]. Painlevé conjectured the strong minimality of various classes of differential equations, where the notion is equivalent to *Umemura's Condition (J).* See [**?**] for a discussion of these connections. We believe that the above list, together with a specific example of [**?**] constitutes the entire list of differential equations (of order at least two) which have been proven to be strongly minimal. Most of the techniques in the above listed results apply only to specific equations or narrow classes of equations and rely on specific properties of those classes in proving strong minimality. Our goal in this article will be to develop a rather more general approach which applies widely to equations with nonconstant coefficients.

1.1. **Our approach and results.**

Let $f \in k\{x\}$. Generally speaking, when attempting to prove strong minimality[1] of some differential variety

$$V = Z(f) = \{a \in \mathcal{U} \mid f(a) = 0\},$$

there are two phenomena which make the task difficult:

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[1] Equivalently, there are no infinite differential subvarieties.

(1) There is no a priori upper bound on the degree of the differential polynomials which define a differential subvariety of V.
(2) The differential polynomials used to define a differential subvariety might (necessarily) have coefficients from a differential field extension of the field of k.

There are structure theorems related to (1) but only in special cases. See for instance [**?**] when the subvarieties are co-order one in V. Controlling the field extension in (2) is a key step in various recent works [**?**, **?**, **?**]. This is most often accomplished by noting that *stable embeddedness* of the generic type of V implies that the generators of the field of definition of a forking extension can be assumed to themselves realize the generic type of V – see explanations in [**?**, **?**]. In recent work, Freitag and Moosa [**?**] introduce a new invariant of a type, which more closely controls the structure over which the forking extension of a type is defined:

Definition 1.1. Suppose $p \in S(A)$ is a stationary type of U-rank > 1. By the *degree of nonminimality* of p, denoted by $\operatorname{nmdeg}(p)$, we mean the least positive integer k such that for some sequence of realizations of p of length k, say $(a_1, \dots, a_k)$, p has a nonalgebraic forking extension over $A, a_1, \dots, a_k$. If $U(p) \leq 1$ then we set $\operatorname{nmdeg}(p) = 0$.

In the theory of differentially closed fields of characteristic zero, Freitag and Moosa [**?**] give an upper bound for the degree of nonminimality in terms of Morley rank:

Theorem 1.2. *Let $p \in S(k)$ have finite rank. Then $\operatorname{nmdeg}(p) \leq RU(p) + 1$.*

Let $a \models p$, we will call the transcendence degree of the differential field $k\langle a\rangle/k$ the order of p. When p is the generic type of a differential variety V, we also call this the order of V. The order of p is an upper bound for the Morley rank of p. The Morley rank of p is a bound for the Lascar rank of p. For proofs of these facts, see [**?**]. It follows that *if* the type p of a generic solution of an order n differential equation over k has a nonalgebraic forking extension over some differential field extension, then already p has such a forking extension over $k\langle a_1, \dots, a_{n+1}\rangle$ where the a_i are from a Morley sequence in the type of p over k. This consequence of Theorem 1.2 will be essential to our approach to handling issue (2) above.

Our approach to issue (1) follows a familiar general strategy of reducing certain problems for nonlinear differential equations to related problems for associated linear differential equations. For instance, [**?**] applies a strategy of this nature to establish results around the Zilber trichotomy, while [**?**, **?**] use this strategy to establish irreducibility of solutions to automorphic and Painlevé equations using certain associated Riccati equations. Our technique fits into this general framework and relies on Kolchin's differential tangent space, which will provide the linear equations associated with the original nonlinear differential variety V. Our approach to the associated linear equations has been under development in the thesis of Wolf [**?**] and the forthcoming thesis of DeVilbiss which gives an approach to calculating the Lascar rank of underdetermined systems of linear differential equations.

Our main theorem is:

Theorem 1.3. *Let $f(x)$ be a generic differential polynomial of order $h > 1$ and degree d. Let p be the type of a generic solution to $Z(f)$. If $d \geq 2 \cdot (\operatorname{nmdeg}(p) + 1)$, then $Z(f)$ is strongly minimal. In particular, if $d \geq 2 \cdot (h + 2)$, then $Z(f)$ is strongly minimal.*

This answers Question 7 of [**?**] for sufficiently large degree, any order, and nonconstant coefficients. As described above, Jaoui [**?**] has recently answered the order two case of Question 7 of [**?**]

for constant coefficients. In this paper, our techniques are applied to equations with differentially transcendental coefficients, but this is not an inherent restriction of the methods. For instance, in forthcoming work using these techniques joint with Casale and Nagloo, we give a fundamentally new proof of the main theorem of [**?**], proving that the equation satisfied by the j-function is strongly minimal. There we also establish new results for several other equations of Schwarzian type.

1.2. **Organization.** In section 2, we set up the notation and background results we require. Section 3 gives a new sufficient condition for the strong minimality of a differential variety. Section 4 applies this condition to show that generic differential equations are strongly minimal. Section 5 shows how one can establish a weaker condition than strong minimality in a more computationally straightforward manner and gives some open problems.

2. NOTATION

Let $\mathcal{U}$ be a countably saturated differentially closed field of characteristic zero. All of the fields we consider will be subfields of $\mathcal{U}$. An affine *differential variety* is the zero set of a (finite) system of differential polynomial equations over (a finitely generated subfield of) $\mathcal{U}$. If X is a differential variety, we denote the *differential tangent space* of X at point $a \in X$ by $T_a^{\Delta}(X)$ as defined in [**?**, pg 198].

Let $\bar{a} \in \mathcal{U}$, and F a differential subfield of $\mathcal{U}$. Then $\omega(\bar{a}/F)$ denotes the *Kolchin polynomial* of $\bar{a}$ over F (see [**?**, Theorem 6, pg 115]). When X is a differential variety, that is, a closed irreducible set in the Kolchin topology, $\omega(X/F) := \omega(\bar{a}/F)$ where $\bar{a}$ is a generic point on X over F.

Let $(y_1, \dots y_n)$ be a finite set of differential indeterminants over $\mathcal{U}$ and let Θ denote the set of derivative operators on $\mathcal{U}$. Since we are interested in differential fields with a single derivation, $\Theta = \{\delta^k : k \geq 0\}$. A *ranking* on $(y_1, \dots y_n)$ is a total ordering on the derivatives $\{\theta y_j : \theta \in \Theta, 1 \leq j \leq n\}$ such that for all such derivatives u, v, and all $\theta \in \Theta$, we have

$$u \leq \theta u, \quad u \leq v \Rightarrow \theta u \leq \theta v.$$

A ranking is *orderly* if whenever the order of θ_1 is lower than the order of θ_2, we have $\theta_1 y_i < \theta_2 y_j$ for any i, j. An *elimination ranking* is a ranking in which $y_i < y_j$ implies $\theta_1 y_i < \theta_2 y_j$ for any $\theta_1, \theta_2 \in \Theta$. For a δ-polynomial $f(y_1, \dots, y_n)$, the highest ranking θy_j appearing in f is the *leader* of f, denoted u_f. If u_f has degree d in f, we can rewrite f as a polynomial in u_f, $f = \sum_{i=0}^{d} I_i u_f^i$, where the *initial* of f, I_d, is not zero. The *separant* of f is the formal derivative $\frac{\partial f}{\partial u_f}$. A detailed treatment of these definitions can be found in [**?**, pg 75].

3. A GENERAL SUFFICIENT CRITERION FOR STRONG MINIMALITY

Let $f(x)$ be an order $n \geq 1$ non-linear differential polynomial in one variable without a constant term. Let $\bar{\alpha}$ denote the coefficients of f and let α_0 be differentially transcendental over $\bar{\alpha}$. Let V be the differential variety corresponding to $f(x) = \alpha_0$. Our goal in this section is to find sufficient conditions under which such a variety V is strongly minimal. The following lemma is a corollary of [**?**, Theorem 1, pg 199].

Lemma 3.1. *Let F be a differential field, X a differential variety defined over F, and a a generic point of X over F. Then $\omega\left(X/F\right) = \omega\left(T_a^{\Delta}(X)/F\langle a\rangle\right)$.*

Our next proposition shows that when $\bar{\alpha}, \alpha_0$ are independent and differentially transcendental, there are no proper subvarieties of V which are defined over the field $\mathbb{Q}\langle\bar{\alpha}, \alpha_0\rangle$. Though the argument is simple, an elaboration of the technique in the proof will be used in the more difficult general case where one extends the field of coefficients.

Proposition 3.2. *Let f and V be as above. Then V has no infinite subvarieties that are defined over $\mathbb{Q}\langle\bar{\alpha}, \alpha_0\rangle$.*

Proof. Suppose towards a contradiction that W is an infinite proper subvariety of V defined over $\mathbb{Q}\langle\bar{\alpha}, \alpha_0\rangle$. Then W is given by some positive order δ-polynomial $g(x) \in \mathbb{Q}\langle\bar{\alpha}, \alpha_0\rangle\{x\}$. By clearing the denominators of α_0, we can write $g(x, \alpha_0) \in \mathbb{Q}\langle\bar{\alpha}\rangle\{x, \alpha_0\}$. For ease of notation, let $k = \mathbb{Q}\langle\bar{\alpha}\rangle$.

Let V_y be the δ-variety given by $f(x) = y$ and let W_y be given by $g(x, y) = 0$ so that each instance of α_0 is replaced with the variable y. These varieties are now defined by δ-polynomials in two variables with coefficients in k and $W_y \subsetneq V_y$. Let $a = (a_1, a_2)$ be a generic point of W_y over k. Since α_0 is differentially transcendental over $\bar{\alpha}$ the locus of y over k is $\mathbb{A}^1$, so it follows that W_y is an infinite rank (proper) subvariety of V_y. Consider the orderly ranking with x ranked higher than y.

We claim that the generic point a of W_y lies outside the locus on V_y where the separant of $f(x) - y$ vanishes (we will call this the singular locus of V_y). This follows because the locus of the separant of f inside of V_y is finite rank (to see this, note that the separant is a differential polynomial in $k\{x\}$ so its generic solution has x-coordinate differentially algebraic over k). From the fact that a lies outside the singular locus of V_y and the singular locus of W_y (since a is generic on W_y), it follows that the Kolchin polynomials of $T_a^{\Delta}(W_y)$ and $T_a^{\Delta}(V_y)$ are equal to W_y and V_y, respectively, and so $T_a^{\Delta}(W_y) \subsetneq T_a^{\Delta}(V_y)$.

For $0 \leq i \leq n$, let

$$\beta_i(x) = \frac{\partial f}{\partial x^{(i)}}(x)$$

denote the formal derivative of f with respect to the ith derivative of x. Using this notation, the differential tangent space $T_a^{\Delta}(V_y)$ is the set of (w, z) satisfying the linear differential equation

$$z = \sum_{i=0}^{n} \beta_i(a_1) w^{(i)}.$$

From this equation, we can see that z is determined by our choice of w, but w may be chosen freely. This gives a definable bijection between $T_a^{\Delta}(V_y)$ and $\mathbb{A}^1$. Further, it follows that $T_a^{\Delta}(V_y)$ has no infinite rank subspaces over $k\langle a\rangle$, since if it did, we could consider the image of this subvariety under the definable bijection to $\mathbb{A}^1$. However, $\mathbb{A}^1$ has no infinite rank subsets, so the image must have finite rank. Therefore, $\omega(T_a^{\Delta}(W_y)/k\langle a\rangle)$ is finite, a contradiction. □

Remark 3.3. The previous result shows that under very general circumstances, for instance when any single coefficient is differentially transcendental over the others, the equation has no subvarieties over the coefficients of the equation itself. We state the following result, but omit its proof, as it is analogous to the previous proof and will not be used later in this paper.

Proposition 3.4. *Let f be a differential polynomial in one variable and V the zero set of f. Let $\bar{a}$ denote the tuple of coefficients in f. If $\bar{a}$ has some element a_1 such that a_1 is differentially transcendental over $\mathbb{Q}\langle \bar{a}_{-1}\rangle$,*[2] *then V has no differential subvarieties over $\mathbb{Q}\langle \bar{a}\rangle$ except perhaps the zero set given by the monomial of which a_1 is a coefficient.*

The previous proposition works in such generality, in part because we have restricted the coefficient field. In various situations, identifying differential subvarieties defined over the field of definition of a variety V is a *much easier problem* than identifying differential subvarieties of V defined over differential field extensions. For instance, in [**?**], Nishioka shows that the equations corresponding to automorphic functions of dense subgroups of SL_2 have to differential subvarieties over $\mathbb{C}$. In the special case of genus zero Fuchsian functions, a much more difficult argument was required to extend the result to differential subvarieties over differential field extensions [**?**], answering a long-standing open problem of Painlevé.

There is one general purpose model theoretic tool which restricts the field extensions one needs to consider. We will use a principle in stability theory, generally related to *stable embeddedness* (see for instance see [**?**] where this general type of result is referred to as the *Shelah reflection principle*). For the following result see Lemma 2.28 [**?**]:

Lemma 3.5. *In a stable theory, let $A \subseteq B$ and $p \in S(B)$ which forks over A. Then there is an indiscernible sequence $(a_i : i \in \mathbb{N})$ in p, such that there is a finite initial segment $\{a_1, \dots, a_d\}$ such that the canonical base of p is contained in the definable closure of $A, a_1, \dots, a_d$.*

Let f and V be as before and let $d \in \mathbb{N}$. Consider V^d, the set of d-tuples so that each coordinate x_i satisfies $f(x_i) = \alpha_0$. As before, we can replace each instance of α_0 with a new variable y, resulting in a differential variety $(V^d)_y$ defined by the system of equations:

$$\begin{cases} f(x_1) &= y \\ f(x_2) &= y \\ & \vdots \\ f(x_d) &= y \end{cases}$$

Proposition 3.6. *Suppose that for all $d \in \mathbb{N}$ and indiscernible sequences $\bar{a}$ in the generic type of V, the differential tangent space $T^{\Delta}_{\bar{a}}\left((V^d)_y\right)$ has no definable proper infinite rank subspaces over $\mathbb{Q}\langle \bar{\alpha}, \bar{a}\rangle$. Then V is strongly minimal.*

Proof. Suppose V is not strongly minimal and let $p(x) \in S_1\left(\mathbb{Q}\langle \bar{\alpha}, \alpha_0\rangle\right)$ be the type of a generic solution of V. By Proposition 3.2, V does not have any infinite subvarieties defined over $\mathbb{Q}\langle \bar{\alpha}, \alpha_0\rangle$, so p has a forking extension q over a differential field extension $K > \mathbb{Q}\langle \bar{\alpha}, \alpha_0\rangle$. By Lemma 3.5, there is some finite d and a Morley sequence $(a_1, \dots, a_d)$ for q such that $(a_1, \dots, a_d)$ is not $\mathbb{Q}\langle \bar{\alpha}, \alpha_0\rangle$-independent. Consider the minimal such d. Then $\operatorname{tp}\left(a_1/\mathbb{Q}\langle \bar{\alpha}, \alpha_0, a_2, a_3, \dots, a_d\rangle\right)$ forks over $\mathbb{Q}\langle \bar{\alpha}, \alpha_0\rangle$. Since these are types over differential fields, this happens exactly when the Kolchin polynomial of $(a_1/\mathbb{Q}\langle \bar{\alpha}, \alpha_0, a_2, a_3, \dots, a_d\rangle)$ is strictly less than the Kolchin polynomial of $(a_1/\mathbb{Q}\langle \bar{\alpha}, \alpha_0\rangle)$.

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[2] By $\bar{a}_{-1}$, we mean the tuple $\bar{a}$ excluding a_1.

Thus, there is a differential polynomial $g(x) \in \mathbb{Q}\langle\bar{\alpha}, \alpha_0, a_2, \dots, a_d\rangle\{x\}$ so that $g(a_1) = 0$ and g has order strictly less than the order of f. By clearing denominators, we can write $g(x_1, \dots, x_d) \in \mathbb{Q}\langle\bar{\alpha}, \alpha_0\rangle\{x_1, \dots, x_d\}$ such that $g(a_1, \dots, a_d) = 0$. Let $U \subset V^d$ be the vanishing set of $g(x_1, \dots, x_d)$. Just as with V, we can replace α_0 with a new variable y after clearing denominators again, giving a $\mathbb{Q}\langle\bar{\alpha}\rangle$-polynomial $g(x_1, \dots, x_d, y)$ and the corresponding variety $U_y \subset (V^d)_y$. The Kolchin polynomial $\omega\left(U_y/\mathbb{Q}\langle\bar{\alpha}\rangle\right)$ is nonconstant. Let $\bar{a} = (a_1, \dots, a_d, \alpha_0)$ and notice that $\bar{a}$ is a generic point of U_y over $\mathbb{Q}\langle\alpha\rangle$. By Lemma 3.1, the Kolchin polynomial of the differential tangent space $\omega\left(T^{\Delta}_{\bar{a}}(U_y)/\mathbb{Q}\langle\bar{\alpha}, \bar{a}\rangle\right)$ is also nonconstant, so $T^{\Delta}_{\bar{a}}((V^d)_y)$ has an infinite rank subspace over $\mathbb{Q}\langle\bar{\alpha},\bar{a}\rangle$, a contradiction to our assumption. ☐

Remark 3.7. Using Lemma 3.5 together with Proposition 3.6 gives a strategy for establishing the strong minimality of nonlinear differential equations with generic coefficients, but *only if one can verify the hypothesis of Proposition 3.6*. A priori, this looks quite hard since it would require the analysis of systems of linear differential equations in n variables for all $n \in \mathbb{N}$. This may be possible via a clever inductive argument for specially selected classes of equations, but Theorem 1.2 gives a bound for the number of variables we need to consider.

Theorem 3.8. *Let p be the generic type of V. Suppose that for $d \leq \mathrm{nmdeg}\, p + 1 \leq ord(V) + 2$ and any indiscernible sequence $\bar{a} = (a_1, \dots, a_d)$ in the generic type of V, the differential tangent space $T^{\Delta}_{\bar{a}}\left((V^d)_y\right)$ has no definable proper infinite rank subspaces over $\mathbb{Q}\langle\bar{\alpha}, \bar{a}\rangle$. Then V is strongly minimal.*

Proof. By Proposition 3.2, there are no subvarieties of V defined over the differential field generated by the coefficients of f. So, we need only consider forking extensions of the generic type of V. By Theorem 1.2, if there is an infinite proper differential subvariety of V, then it is defined over (the algebraic closure of) a Morley sequence of length at most $\mathrm{nmdeg}(p)$ which is at most $h + 1$. Thus, there is a proper subvariety of $W \subset V^d$ which surjects onto the first $d - 1$ coordinates such that the fiber over a generic point in the first $d - 1$ coordinates is a forking extension of the generic type of V. But then by the argument of Proposition 3.6, there is a definable proper infinite rank subspace of $T^{\Delta}_{\bar{a}}\left((V^d)_y\right)$ over $\mathbb{Q}\langle\bar{\alpha}, \bar{a}\rangle$. ☐

4. STRONG MINIMALITY OF GENERIC EQUATIONS

4.1. A first example.

Theorem 4.1. *Let X be the differential variety given by*

$$x'' + \sum_{i=1}^{n} \alpha_i x^i = \alpha \tag{1}$$

for some $n \geq 8$, where $(\alpha, \alpha_0, \dots, \alpha_n)$ is a tuple of independent differential transcendentals over $\mathbb{Q}$. Then X is strongly minimal.

Proof. To show that equation 1 is strongly minimal, by the explanation following Theorem 1.2, we need only show that given any solutions $a_1, \dots, a_4$ to equation 1, we cannot have that the transcendence degree of $\mathbb{Q}\langle a_1, \alpha, \alpha_1, \dots, \alpha_n, a_2, \dots, a_4\rangle$ over $\mathbb{Q}\langle\alpha, \alpha_1, \dots, \alpha_n, a_2, \dots, a_4\rangle$ is one. Without loss of generality, assume that $a_1, \dots, a_4$ are *algebraically independent* over $\mathbb{Q}\langle\alpha, \alpha_1, \dots, \alpha_n\rangle$ (that is, they satisfy no polynomial relation over $\mathbb{Q}\langle\alpha, \alpha_1, \dots, \alpha_n\rangle$).

Observe that the differential tangent space $T_{\bar{a}}^{\Delta}\left((X^4)_y\right)$ after eliminating y is given by the system:

$$\left\{\begin{array}{lcl} u_0'' + \left(\sum_{i=1}^n i a_1^{i-1}\alpha_i\right) u_0 & = & v_0'' + \left(\sum_{i=1}^n i a_2^{i-1}\alpha_i\right) v_0 \\ u_0'' + \left(\sum_{i=1}^n i a_1^{i-1}\alpha_i\right) u_0 & = & w_0'' + \left(\sum_{i=1}^n i a_3^{i-1}\alpha_i\right) w_0 \\ u_0'' + \left(\sum_{i=1}^n i a_1^{i-1}\alpha_i\right) u_0 & = & z_0'' + \left(\sum_{i=1}^n i a_4^{i-1}\alpha_i\right) z_0 \end{array}\right.$$

For $j = 1, \ldots, 4$, we let $\beta_j = \sum_{i=0}^n i a_j^{i-1}\alpha_i$. We argue that $\beta_1, \ldots, \beta_4$ are independent differential transcendentals. Note that

$$\begin{pmatrix} n a_1^{n-1} & (n-1)a_1^{n-2} & \ldots & 2a_1 & 1 \\ n a_2^{n-1} & (n-1)a_2^{n-2} & \ldots & 2a_2 & 1 \\ n a_3^{n-1} & (n-1)a_3^{n-2} & \ldots & 2a_3 & 1 \\ n a_4^{n-1} & (n-1)a_4^{n-2} & \ldots & 2a_4 & 1 \end{pmatrix} \begin{pmatrix} \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

We claim that any four columns of the matrix of a_i's are linearly independent. To see this, note that if not then the vanishing of the corresponding determinant shows that there is a nontrivial polynomial relation which holds of $a_1, \ldots, a_4$.

This contradicts the fact that a_1 satisfies an order one equation over $a_2, \ldots, a_4$. By the independence of $\alpha_1, \ldots, \alpha_n$ there are at least four of the α_i which are independent differential transcendentals over the other α_i and $a_1, \ldots, a_4$. Without loss of generality, assume $\alpha_1, \ldots, \alpha_4$ are independent differential transcendentals over $\mathbb{Q}\langle \alpha_5, \ldots, \alpha_n, a_1, \ldots, a_4\rangle$. Then since the last four columns of the above matrix of a_i are linearly independent, it follows that $\alpha_1, \ldots, \alpha_4$ are interalgebraic with $\beta_1, \ldots, \beta_4$ over $\mathbb{Q}\langle a_1, \ldots, a_4, \alpha_5, \ldots, \alpha_n\rangle$. It follows that $\beta_1, \ldots, \beta_4$ are independent differential transcendentals over $\mathbb{Q}\langle a_1, \ldots, a_4, \alpha_5, \ldots, \alpha_n\rangle$.

Lemma 4.2. *A linear system of the form*

$$\left\{\begin{array}{lcl} u_0'' + \beta_1 u_0 & = & v_0'' + \beta_2 v_0 \\ u_0'' + \beta_1 u_0 & = & w_0'' + \beta_3 w_0 \\ u_0'' + \beta_1 u_0 & = & z_0'' + \beta_4 z_0 \end{array}\right. \tag{2}$$

with $\beta_1, \ldots, \beta_4$ independent differential transcendentals has no infinite rank subvarieties.

Proof. We will prove that this system has no infinite rank subvarieties by proving that the solution set is in definable bijection with $\mathbb{A}^1$. This is constructed by composing a series of linear substitutions.

First, we substitute u_1 for u_0 where $u_0 = u_1 + v_0$. This reduces the order of v_0 in the top equation, resulting in the system

$$\left\{\begin{array}{lcl} u_1'' + \beta_1 u_1 & = & (\beta_2 - \beta_1) v_0 \\ u_1'' + \beta_1 u_1 + v_0'' + \beta_1 v_0 & = & w_0'' + \beta_3 w_0 \\ u_1'' + \beta_1 u_1 + v_0'' + \beta_1 v_0 & = & z_0'' + \beta_4 z_0 \end{array}\right.$$

To reduce the order v_0 in the lower equations we substitute w_1, z_1 for w_0, z_0 where $w_0 = w_1 + v_0, z_0 = z_1 + v_0$. Then we have

$$\left\{\begin{array}{lcl} u_1'' + \beta_1 u_1 & = & (\beta_2 - \beta_1) v_0 \\ u_1'' + \beta_1 u_1 + (\beta_1 - \beta_3) v_0 & = & w_1'' + \beta_3 w_1 \\ u_1'' + \beta_1 u_1 + (\beta_1 - \beta_4) v_0 & = & z_1'' + \beta_4 z_1 \end{array}\right.$$

Solving the top equation for v_0 in terms of u_1 and plugging this in for v_0 allows us to eliminate v_0 from lower equations, resulting in the system

$$\left\{\begin{array}{lcl} A_{2,0}u_1'' + A_{0,0}u_1 & = & w_1'' + \beta_3 w_1 \\ C_{2,0}u_1'' + C_{0,0}u_1 & = & z_1'' + \beta_4 z_1 \end{array}\right.$$

where (after some simplification)

$$A_{2,0} := \frac{\beta_2 - \beta_3}{\beta_2 - \beta_1}, \quad A_{0,0} := \beta_1 A_{2,0}$$
$$C_{2,0} := \frac{\beta_2 - \beta_4}{\beta_2 - \beta_1}, \quad C_{0,0} := \beta_1 C_{2,0}.$$

We again reduce the order of the variable in the top equation by substituting u_2 for u_1 defined by $u_1 = u_2 + \frac{1}{A_{2,0}} w_1$ resulting in the system

$$\left\{\begin{array}{lcl} A_{2,0}u_2'' + A_{0,0}u_2 & = & B_{1,1}w_1' + B_{0,1}w_1 \\ C_{2,0}u_2'' + C_{0,0}u_2 + D_{2,1}v_1'' + D_{1,1}v_1' + D_{0,1}v_1 & = & z_1'' + \beta_4 z_1 \end{array}\right.$$

where

$$B_{1,1} := -2A_{2,0}\left(\tfrac{1}{A_{2,0}}\right)', \quad B_{0,1} := \beta_3 - A_{2,0}\left(\tfrac{1}{A_{2,0}}\right)'' - \tfrac{A_{0,0}}{A_{2,0}}$$
$$D_{2,1} := \tfrac{C_{2,0}}{A_{2,0}}, \quad D_{1,1} := 2C_{2,0}\left(\tfrac{1}{A_{2,0}}\right)', \quad D_{0,1} := C_{2,0}\left(\tfrac{1}{A_{2,0}}\right)'' + \tfrac{C_{0,0}}{A_{2,0}}.$$

We next reduce the order of w_1 in lower equations with the substitution z_2 for z_1 defined by $z_1 = z_2 + D_{2,1}w_1$. Now we have the system

$$\left\{\begin{array}{lcl} A_{2,0}u_2'' + A_{0,0}u_2 & = & B_{1,1}w_1' + B_{0,1}w_1 \\ C_{2,0}u_2'' + C_{0,0}u_2 + E_{1,1}w_1' + E_{0,1}w_1 & = & z_2'' + \beta_4 z_2 \end{array}\right.$$

where

$$E_{1,1} := D_{1,1} - D_{2,1}', \quad E_{0,1} := D_{0,1} - D_{2,1}'' - \beta_4 D_{2,1}.$$

Next we reduce the order of u_2 in the top equation by substituting w_2 for w_1 with $w_1 = w_2 + \frac{A_{2,0}}{B_{1,1}}u_2'$ resulting in the system

$$\left\{\begin{array}{lcl} A_{1,1}u_2' + A_{0,1}u_2 & = & B_{1,1}w_2' + B_{0,1}w_2 \\ C_{2,1}u_2'' + C_{1,1}u_2' + C_{0,1}u_2 + E_{1,1}w_2' + E_{0,1}w_2 & = & z_2'' + \beta_4 z_2 \end{array}\right.$$

where

$$A_{1,1} := -B_{1,1}\left(\frac{A_{2,0}}{B_{1,1}}\right)' - B_{0,1}\frac{A_{2,0}}{B_{1,1}}, \quad A_{0,1} := A_{0,0}$$
$$C_{2,1} := C_{2,0} + E_{1,1}\frac{A_{2,0}}{B_{1,1}}, \; C_{1,1} := E_{1,1}\left(\frac{A_{2,0}}{B_{1,1}}\right)' + E_{0,1}\frac{A_{2,0}}{B_{1,1}}, \quad C_{0,1} := C_{0,0}.$$

The reduction in order of the top equation continues with the replacement of u_2 with u_3 given by $u_2 = u_3 + \frac{B_{1,1}}{A_{1,1}}w_2$. This results in the system

$$\left\{\begin{array}{lcl} A_{1,1}u_3' + A_{0,1}u_3 & = & B_{0,2}w_2 \\ C_{2,1}u_3'' + C_{1,1}u_3' + C_{0,1}u_3 + D_{2,2}w_2'' + D_{1,2}w_2' + D_{0,2}w_2 & = & z_2'' + \beta_4 z_2 \end{array}\right.$$

where

$$\begin{aligned}
&B_{0,2} := B_{0,1} - A_{1,1}\left(\frac{B_{1,1}}{A_{1,1}}\right)' - A_{0,1}\frac{B_{1,1}}{A_{1,1}}\\
&D_{2,2} := C_{2,1}\frac{B_{1,1}}{A_{1,1}}, \quad D_{1,2} := E_{1,1} + 2C_{2,1}\left(\frac{B_{1,1}}{A_{1,1}}\right)' + C_{1,1}\frac{B_{1,1}}{A_{1,1}},\\
&D_{0,2} := E_{0,1} + C_{2,1}\left(\frac{B_{1,1}}{A_{1,1}}\right)'' + C_{1,1}\left(\frac{B_{1,1}}{A_{1,1}}\right)' + C_{0,1}\frac{B_{1,1}}{A_{1,1}}.
\end{aligned}$$

Next we replace z_2 with z_3 given by $z_2 = z_3 + D_{2,2}w_2$ to arrive at

$$\left\{\begin{array}{lcl} A_{1,1}u_3' + A_{0,1}u_3 & = & B_{0,2}w_2 \\ C_{2,1}u_3'' + C_{1,1}u_3' + C_{0,1}u_3 + E_{1,2}w_2' + E_{0,2}w_2 & = & z_3'' + \beta_4 z_3 \end{array}\right.$$

where

$$E_{1,2} := D_{1,2} - D_{2,2}', \quad E_{0,2} := D_{0,2} - D_{2,2}'' - \beta_4 D_{2,2}.$$

Now we can solve the top equation for w_2 in terms of u_3 and plug the resulting expression into the lower equations:

$$C_{2,2}u_3'' + C_{1,2}u_3' + C_{0,2}u_3 = z_3'' + \beta_4 z_3$$

where

$$\begin{aligned}
&C_{2,2} := C_{2,1} + E_{1,2}\tfrac{A_{1,1}}{B_{0,2}},\\
&C_{1,2} := C_{1,1} + E_{1,2}\left(\tfrac{A_{1,1}}{B_{0,2}}\right)' + E_{0,2}\tfrac{A_{1,1}}{B_{0,2}} + E_{1,2}\tfrac{A_{0,2}}{B_{0,2}},\\
&C_{0,2} := C_{0,1} + E_{1,2}\left(\tfrac{A_{0,1}}{B_{0,2}}\right)' + E_{0,2}\tfrac{A_{0,1}}{B_{0,2}}.
\end{aligned}$$

Next we perform analagous substitutions to eliminate z_3 from the top equation, beginning with substituting u_4 for u_3 defined by $u_3 = u_4 + \frac{1}{C_{2,2}}z_3$, so we have

$$\left\{\begin{array}{lcl} C_{2,2}u_4'' + C_{1,2}u_4' + C_{0,2}u_4 & = & F_{1,1}z_3' + F_{0,1}z_3 \end{array}\right.$$

where

$$F_{1,1} := -2C_{2,2}\left(\frac{1}{C_{2,2}}\right)' - \frac{C_{1,2}}{C_{2,2}}, \quad F_{0,1} := \beta_4 - C_{2,2}\left(\frac{1}{C_{2,2}}\right)'' - C_{1,2}\left(\frac{1}{C_{2,2}}\right)' - \frac{C_{0,2}}{C_{2,2}}.$$

Next, substitute z_4 for z_3 where $z_3 = z_4 + \frac{C_{2,2}}{F_{1,1}}u_4'$. This will result in the equation

$$C_{1,3}u_4' + C_{0,3}u_4 = F_{1,1}z_4' + F_{0,1}z_4$$

where

$$C_{1,3} := C_{1,2} - F_{1,1}\left(\frac{C_{2,2}}{F_{1,1}}\right)' + F_{0,1}\frac{C_{2,2}}{F_{1,1}}, \quad C_{0,3} := C_{0,2}.$$

Replace u_4 with u_5 defined by $u_4 = u_5 + \frac{F_{1,1}}{C_{1,3}}z_4$, giving us the equation

$$C_{1,3}u_5' + C_{0,3}u_5 = F_{0,2}z_4 \tag{3}$$

where

$$F_{0,2} := F_{0,1} - C_{1,3}\left(\frac{F_{1,1}}{C_{1,3}}\right)' + F_{0,1}\frac{F_{1,1}}{C_{1,3}}.$$

Any solution to Equation 3 is determined by the value of u_5, so the solution set is in definable bijeciton with $\mathbb{A}^1$. Each of these linear substitutions gives rise to a definable bijection between systems so long as the substitutions are well-defined, i.e., that the denominators of the coefficients are all non-zero. For this procedure to give a definable bijection from the original system 2 to $\mathbb{A}^1$, we must verify that the following expressions are not zero:

$$\beta_2 - \beta_1, A_{2,0}, B_{1,1}, A_{1,1}, B_{0,2}, C_{2,2}, F_{1,1}, C_{1,3}, \text{ and } F_{0,2}.$$

The expression $\beta_2 - \beta_1$ is non-zero because $\beta_1, \beta_2, \beta_3, \beta_4$ are all distinct. Each of these coefficients can be considered as a differential rational function in terms of $\beta_1, \beta_2, \beta_3, \beta_4$, and so they can be analyzed according to a ranking on $\bar{\beta}$. We will show that these coefficients are nonzero by showing that the initials of each are nonzero in some elimination ranking. It then follows that the coefficients themselves are non-zero because $\beta_1, \beta_2, \beta_3, \beta_4$ are independent differential transcendentals.

Consider the terms of these expressions ordered by some elimination ranking on $\bar{\beta}$ with β_3 ranked highest. The leading term in this ranking of each expression can be calculated using the definitions of previous coefficients. The following table shows that these leading terms are non-zero:

$A_{2,0}$	$\frac{-1}{\beta_2-\beta_1}\beta_3$
$B_{1,1}$	$\frac{-2}{\beta_2-\beta_3}\beta_3'$
$A_{1,1}$	$\frac{-2}{(\beta_2-\beta_1)B_{1,1}}\beta_3''$
$B_{0,2}$	$\frac{-2}{(\beta_2-\beta_1)A_{1,1}}\beta_3^{(3)}$

We turn our attention to an elimination ranking with β_4 ranked highest to prove that the remaining coefficients are nonzero. The following table shows that the leading terms of these coefficients are also non-zero:

$C_{2,2}$	$\frac{-3}{(\beta_2-\beta_1)B_{0,2}}\beta_4''$
$F_{1,1}$	$\frac{-4}{(\beta_2-\beta_1)B_{0,2}C_{2,2}}\beta_4^{(3)}$
$C_{1,3}$	$\frac{-7}{(\beta_2-\beta_1)B_{0,2}F_{1,1}}\beta_4^{(4)}$
$F_{0,2}$	$\frac{-7}{(\beta_2-\beta_1)B_{0,2}C_{1,3}}\beta_4^{(5)}$

We have shown that each substitution is well-defined, and therefore we have constructed a definable bijection between system 2 and $\mathbb{A}^1$. Since $\mathbb{A}^1$ has no infinite rank subspaces, neither does our original system, completing the proof of the proposition. $\square$

We now finish the proof of Theorem 4.1. Since the differential tangent space $T^{\Delta}_{\bar{a}}\left((X^4)_y\right)$ satisfies the conditions of Lemma 4.2, it has no infinite rank subspaces over $\mathbb{Q}\langle\bar{\alpha}, \bar{a}\rangle$. Therefore, X is strongly minimal by Theorem 3.8. $\square$

4.2. **Generic higher order equations.**

The technique used in the previous example can be applied to more general classes of equations. In this section, we use analogous techniques to show that generic equations with high enough degree have differential tangent spaces cut out by generic linear equations and that the generic tangent spaces have no infinite rank subvariety.

Let

$$f(x) = \alpha + \sum_{i=1}^{d} \alpha_{0,i} x^i + \sum_{j \in M_1} \alpha_{1,j} m_j \left(x, x'\right) + \cdots + \sum_{k \in M_h} \alpha_{h,k} m_k \left(x, x', \ldots, x^{(h)}\right)$$

where M_n indexes the set of all order n monomials of degree at most d and the entire collection of coefficients $\alpha, \alpha_{i,j}$ are independent differential transcendentals over $\mathbb{Q}$. Let V be the zero set of $f(x)$ and let m be the degree of nonmininality of f. Following the notation of Section 3, we let $(V^m)_y$ be the following system of equations in $x_1, \ldots, x_m, y$:

$$\begin{cases} \sum_{i=0}^{h} \sum_{j \in M_i} \alpha_{i,j} m_j \left(x_1, \ldots, x_1^{(i)}\right) & = & y \\ \sum_{i=0}^{h} \sum_{j \in M_i} \alpha_{i,j} m_j \left(x_2, \ldots, x_2^{(i)}\right) & = & y \\ & \vdots & \\ \sum_{i=0}^{h} \sum_{j \in M_i} \alpha_{i,j} m_j \left(x_m, \ldots, x_m^{(i)}\right) & = & y \end{cases}$$

Let $\bar{a} = (a_1, \ldots, a_m)$ be an indiscernible sequence in V such that a_m forks over $a_1, \ldots, a_{m-1}$ and $tp(a_m/\mathbb{Q}\langle \alpha, \alpha_{i,j}, a_1, \ldots, a_{m-1}\rangle_{i=0,\ldots,h, j \in M_i}$ has rank between 1 and $h-1$. That is, a_m satisfies a differential equation of order at least 1 but no more than $h-1$ over $\mathbb{Q}\langle \alpha, \alpha_{i,j}, a_1, \ldots, a_{m-1}\rangle_{i=0,\ldots,h,j\in M_i}$. Crucially for this proof, we note that $a_1, \ldots, a_m$ are *algebraically* independent over $\mathbb{Q}\langle \alpha, \alpha_{i,j}\rangle$.

Let $T^{\Delta}_{(\bar{a},\alpha)}\left((V^m)_y\right)$ denote the differential tangent space of $(V^m)_y$ over $(\bar{a}, \alpha)$. Then $T^{\Delta}_{(\bar{a},\alpha)}\left((V^m)_y\right)$ is given by

$$\begin{cases} \sum_{i=0}^{h} \beta_{i,1} z_1^{(i)} & = & y \\ \sum_{i=0}^{h} \beta_{i,2} z_2^{(i)} & = & y \\ & \vdots & \\ \sum_{i=0}^{h} \beta_{i,m} z_m^{(i)} & = & y \end{cases}$$

where $\beta_{i,j} = \frac{\partial f}{\partial x^{(i)}}(a_j)$ as used in previous sections.

Lemma 4.3. *If $\bar{a} = (a_1, \ldots, a_m)$ is an indiscernible sequence such that a_m satisfies an order k equation over $a_1, \ldots, a_m$ and the coefficients of f with $1 < k < h$, then the variety $T^{\Delta}_{(\bar{a},\alpha)}\left((V^m)_y\right)$ has coefficients which are independent differential transcendentals over $\mathbb{Q}$ whenever $d \geq 2m$.*

Proof. We proceed by induction on the order, beginning with $\beta_{0,1}, \beta_{0,2}, \ldots, \beta_{0,m}$. Since $d \geq 2m$, there are $j_1, \ldots, j_m$ such that $\alpha_{0,j_1}, \ldots, \alpha_{0,j_m}$ are independent differential transcendentals over $\mathbb{Q}\left\langle \mathcal{A}_0 a_1 \cdots a_m \right\rangle$ where $\mathcal{A}_0 = \{\alpha_{i,j} : 0 \leq i \leq h, j \in M_i\} \setminus \{\alpha_{0,j_1}, \ldots, \alpha_{0,j_m}\}$. Note that

$$\begin{pmatrix} j_1 a_1^{j_1 - 1} & j_2 a_1^{j_2 - 1} & \cdots & j_m a_1^{j_m - 1} \\ j_1 a_2^{j_1 - 1} & j_2 a_2^{j_2 - 1} & \cdots & j_m a_2^{j_m - 1} \\ \vdots & & & \vdots \\ j_1 a_m^{j_1 - 1} & j_2 a_m^{j_2 - 1} & \cdots & j_m a_m^{j_m - 1} \end{pmatrix} \begin{pmatrix} \alpha_{0,j_1} \\ \alpha_{0,j_2} \\ \vdots \\ \alpha_{0,j_m} \end{pmatrix} = \begin{pmatrix} \beta_{0,1} \\ \beta_{0,2} \\ \vdots \\ \beta_{0,m} \end{pmatrix} + \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{pmatrix},$$

where $l_i \in \mathbb{Q}\left\langle \mathcal{A}_0 a_1 \cdots a_m \right\rangle$.

The above matrix is invertible, as the vanishing of its determinant imposes a nontrivial algebraic relation among $a_1, \ldots, a_m$, which are, by assumption, algebraically independent. It follows that $\beta_{0,1}, \ldots, \beta_{0,m}$ are interdefinable with $\alpha_{0,j_1}, \ldots, \alpha_{0,j_m}$ over $\mathbb{Q}\left\langle \mathcal{A}_0 a_1 \cdots a_m \right\rangle$. Thus, $\beta_{0,1}, \ldots, \beta_{0,m}$ are independent and differentially transcendental over $\mathbb{Q}\left\langle \mathcal{A}_0 a_1 \cdots a_m \right\rangle$. Note that the coefficients $\{\beta_{i,j} : 1 \leq i \leq h, 1 \leq j \leq m\}$ are contained in the field $\mathbb{Q}\left\langle \mathcal{A}_0 a_1 \cdots a_m \right\rangle$, so we've shown that $\beta_{0,1}, \ldots, \beta_{0,m}$ are independent differential transcendentals over $\mathbb{Q}\langle \beta_{i,j} : 1 \leq i \leq h, 1 \leq j \leq m \rangle$.

Suppose we have already shown that $\beta_{n,1}, \ldots, \beta_{n,m}$ are independent differential transcendentals over $\mathbb{Q}\langle \beta_{i,j} : n+1 \leq i \leq h, 1 \leq j \leq m\rangle$ for some $n < h$. Let M^*_{n+1} index the collection of order $n+1$ monomials of order no more than d excluding the monomials of the form $\{x^{(n+1)} x^r : 0 \leq r \leq d-1\}$. Assume the order $n+1$ terms in $f(x)$ are ordered so that

$$\sum_{k \in M_{n+1}} \alpha_{n+1,k} m_k(x, x', \ldots, x^{(n+1)}) = \sum_{k=0}^{d-1} \alpha_{n+1,k} x^{(n+1)} x^k + \sum_{j \in M^*_{n+1}} \alpha_{n+1,j} m_j\left(x, x', \ldots, x^{(n+1)}\right).$$

Since $d \geq 2m$, there are $k_1, \ldots, k_m < d$ such that $\alpha_{n+1,k_1}, \ldots, \alpha_{n+1,k_m}$ are independent and differentially transcendental over $\mathbb{Q}\langle \mathcal{A}_{n+1} a_1 \cdots a_m \rangle$ where $\mathcal{A}_{n+1} = \{\alpha_{i,j} : n+1 \leq i \leq h, j \in M_i\} \setminus \{\alpha_{n+1,k_1}, \ldots, \alpha_{n+1,k_m}\}$. Note that

$$\begin{pmatrix} a_1^{k_1} & a_1^{k_2} & \ldots & a_1^{k_m} \\ a_2^{k_1} & a_2^{k_2} & \ldots & a_2^{k_m} \\ \vdots & & & \vdots \\ a_m^{k_1} & a_m^{k_2} & \ldots & a_m^{k_m} \end{pmatrix} \begin{pmatrix} \alpha_{n+1,k_1} \\ \alpha_{n+1,k_2} \\ \vdots \\ \alpha_{n+1,k_m} \end{pmatrix} = \begin{pmatrix} \beta_{n+1,1} \\ \beta_{n+1,2} \\ \vdots \\ \beta_{n+1,m} \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix},$$

where $r_i \in \mathbb{Q}\langle \mathcal{A}_{n+1} a_1 \cdots a_m \rangle$. The above matrix is invertible, since we are assuming that $a_1, \ldots, a_m$ are algebraically independent. Thus, $\alpha_{n+1,k_1}, \ldots, \alpha_{n+1,k_m}$ are interdefinable with $\beta_{n+1,1}, \ldots, \beta_{n+1,m}$ over $\mathbb{Q}\langle \mathcal{A}_{n+1} a_1 \cdots a_m \rangle$. Since $\{\beta_{i,j} : n+1 < i \leq h, 1 \leq j \leq m\}$ are contained in the field $\mathbb{Q}\langle \mathcal{A}_{n+1} a_1 \cdots a_m \rangle$, it follows that $\beta_{n+1,1}, \ldots, \beta_{n+1,m}$ are independent and differentially transcendental over $\{\beta_{i,j} : n+1 < i \leq h, 1 \leq j \leq m\}$.

Putting together the above analysis, we have proved that the collection of coefficients $\{\beta_{i,j} : 0 \leq i \leq h, 1 \leq j \leq m\}$ are independent and differentially transcendental over $\mathbb{Q}\langle a_1, \ldots, a_m \rangle$ and thus over $\mathbb{Q}$. ☐

Eliminating the variable y from $T^{\Delta}_{(\bar{a},\alpha)}\left((V^m)_y\right)$ results in a system of $m-1$ linear equations in m variables with generic coefficients.

Theorem 4.4. *The solution set of any system of $m-1$ generic linear equations of order h in m variables has no infinite rank subspaces for $h > 1$ and $m > 1$.*

Using the notation from the previous proposition, this system of generic linear equations will be written as follows:

$$(4) \qquad \left\{ \begin{array}{rcl} \sum_{i=0}^{h} \beta_{i,1} z_1^{(i)} & = & \sum_{i=0}^{h} \beta_{i,2} z_2^{(i)} \\ \sum_{i=0}^{h} \beta_{i,1} z_1^{(i)} & = & \sum_{i=0}^{h} \beta_{i,3} z_3^{(i)} \\ & \vdots & \\ \sum_{i=0}^{h} \beta_{i,1} z_1^{(i)} & = & \sum_{i=0}^{h} \beta_{i,m} z_m^{(i)}. \end{array} \right.$$

As in Proposition 4.2, we show that such a system has no infinite rank subspaces by constructing a definable bijection to $\mathbb{A}^1$. This is accomplished by applying the following lemma repeatedly:

Lemma 4.5. *Let $\mathcal{S}$ be the solution set of a linear system of equations*[3]

$$(5)\qquad \begin{cases} \sum_{i=0}^{h} A_{i,0}u_0^{(i)} & = & \sum_{i=0}^{h} B_{i,0}z_0^{(i)} \\ \sum_{i=0}^{h} C_{i,l,0}u_0^{(i)} & = & \sum_{i=0}^{h} \beta_{i,l}v_{l,0}^{(i)} \\ & \vdots & \\ \sum_{i=0}^{h} C_{i,m,0}u_0^{(i)} & = & \sum_{i=0}^{h} \beta_{i,m}v_{m,0}^{(i)} \end{cases}$$

satisfying two properties:

- *For each $l \leq j \leq m$, the $h+1$-tuples $(A_{i,0} : 0 \leq i \leq h)$ and $(C_{i,j,0} : 0 \leq i \leq h)$ are inter-differentially algebraic over $\mathbb{Q}\langle\mathcal{B}\rangle$ where $\mathcal{B} = \{B_{i,0}, \beta_{i,j} : 0 \leq i \leq h, l \leq j \leq m\}$, and*
- *The coefficients $\{A_{i,0} : 0 \leq i \leq h\} \cup \mathcal{B}$ are independent differential transcendentals over $\mathbb{Q}$, and likewise for each $l \leq j \leq m$, $\{C_{i,j,0} : 0 \leq i \leq h\} \cup \mathcal{B}$ are independent differential transendentals.*

Then there is a definable bijection $\mathcal{S} \to \mathcal{T}$ where $\mathcal{T}$ is the solution set of a system of linear equations with one fewer variable and one fewer equation satisfying the same two conditions.

Before proceeding, note that the original system 4 of Theorem 4.4 satisfies the two conditions listed in Lemma 4.5. That means we can iterate the application of this lemma until all variables have been solved in terms of only the u variable, thereby giving a definable bijection to $\mathbb{A}^1$.

Proof. The definable bijection to from $\mathcal{S}$ to $\mathcal{T}$ will be defined by a series of linear substitutions which will eliminate one of the variables in the system. The proof will consist of three parts: First, we will define the necessary substitutions to arrive at $\mathcal{T}$. Second, we must verify that each substitution is well-defined, and hence gives rise to a definable bijection. Third, we verify that the two conditions hold for $\mathcal{T}$.

The first substitution replaces the variable u_0 with u_1 where $u_0 = u_1 + \left(\frac{B_{h,0}}{A_{h,0}}\right) z_0$ which reduces the order of z_0 in the top equation by one. This results in the new system:

$$\begin{cases} \sum_{i=0}^{h} A_{i,0}u_0^{(i)} & = & \sum_{i=0}^{h-1} B_{i,1}z_0^{(i)} \\ & \vdots & \\ \sum_{i=0}^{h} C_{i,j,0}u_1^{(i)} + \sum_{i=0}^{h} D_{i,j,1}z_0^{(i)} & = & \sum_{i=0}^{h} \beta_{i,j}v_{j,0}^{(i)} \end{cases}$$

where

$$D_{h-k,j,1} = \binom{h}{k} C_{h,j,0} \left(\frac{B_{h,0}}{A_{h,0}}\right)^{(k)} + \cdots + C_{h-k,j,0}\left(\frac{B_{h,0}}{A_{h,0}}\right)$$

---

[3] Here we use A and B to refer only to the coefficients in the top equation. The lower equations will be indexed by j with the second equation having index l and the last equation index m. In the case of system 4, $l = 2$ and l will increase as we eliminate variables from the system.

for $0 \leq k \leq h$ and

$$B_{h-k,1} := B_{h-k,0} - \binom{h}{k} A_{h,0} \left(\frac{B_{h,0}}{A_{h,0}}\right)^{(k)} - \cdots - A_{h-k,0} \left(\frac{B_{h,0}}{A_{h,0}}\right)$$

for $1 \leq k \leq h$.

To reduce the order of z_0 in lower equations we replace $v_{j,0}$ with $v_{j,1}$ where $v_{j,0} = v_{j,1} + \left(\frac{D_{h,j,1}}{\beta_{h,j}}\right) z_0$, resulting in the system

$$\begin{cases} \sum_{i=0}^{h} A_{i,0} u_1^{(i)} & = \quad \sum_{i=0}^{h-1} B_{i,1} z_0^{(i)} \\ & \vdots \\ \sum_{i=0}^{h} C_{i,j,0} u_1^{(i)} + \sum_{i=0}^{h-1} E_{i,j,1} z_0^{(i)} & = \quad \sum_{i=0}^{h} \beta_{i,j} v_{j,1}^{(i)} \end{cases}$$

where

$$E_{h-k,j,1} := D_{h-k,j,1} - \binom{h}{k} \beta_{h,j} \left(\frac{D_{h,j,1}}{\beta_{h,j}}\right)^{(k)} - \cdots - \beta_{h-k,j} \left(\frac{D_{h,j,1}}{\beta_{h,j}}\right).$$

Next we substitute z_1 for z_0, defined by $z_0 = z_1 + \left(\frac{A_{h,0}}{B_{h-1,1}}\right) u_1'$. This resulting in the system of equations

$$\begin{cases} \sum_{i=0}^{h-1} A_{i,1} u_1^{(i)} & = \quad \sum_{i=0}^{h-1} B_{i,1} z_1^{(i)} \\ & \vdots \\ \sum_{i=0}^{h} C_{i,j,1} u_1^{(i)} + \sum_{i=0}^{h-1} E_{i,j,1} z_0^{(i)} & = \quad \sum_{i=0}^{h} \beta_{i,j} v_{j,1}^{(i)} \end{cases}$$

where

$$\begin{aligned} C_{h-k,j,1} &:= C_{h-k,j,0} + \binom{h-1}{k} E_{h-1,j,1} \left(\frac{A_{h,0}}{B_{h-1,1}}\right)^{(k)} + \cdots + E_{h-k-1,j,1} \left(\frac{A_{h,0}}{B_{h-1,1}}\right) \\ C_{0,j,1} &:= C_{0,j,0} \end{aligned}$$

for $0 \leq k \leq h-1$ and

$$\begin{aligned} A_{h-k,1} &:= A_{h-k,0} - \binom{h-1}{k} B_{h-1,1} \left(\frac{A_{h,0}}{B_{h-1,1}}\right)^{(k)} - \cdots - B_{h-k-1,1} \left(\frac{A_{h,0}}{B_{h-1,1}}\right) \\ A_{0,1} &:= A_{0,0} \end{aligned}$$

for $1 \leq k \leq h-1$.

Now we have reduced the order of the the top equation by one without increasing the order of the lower equations. In order to eliminate the z variable in the top equation, we apply a trio of analogous substitutions recursively to lower the order in the top equation to zero. After this has already been performed n times, the system of equations will be

$$\begin{cases} \sum_{i=0}^{h-n} A_{i,n} u_n^{(i)} & = \quad \sum_{i=0}^{h-n} B_{i,n} z_n^{(i)} \\ & \vdots \\ \sum_{i=0}^{h} C_{i,j,n} u_n^{(i)} + \sum_{i=0}^{h-1} E_{i,j,n} z_n^{(i)} & = \quad \sum_{i=0}^{h} \beta_{i,j} v_{j,n}^{(i)} \end{cases}$$

First we replace the variable u_n with u_{n+1} where $u_n = u_{n+1} + \left(\frac{B_{h-n,n}}{A_{h-n,n}}\right) z_n$, so we have

$$\begin{cases} \sum_{i=0}^{h-n} A_{i,n} u_{n+1}^{(i)} & = \sum_{i=0}^{h-n-1} B_{i,n+1} z_n^{(i)} \\ & \vdots \\ \sum_{i=0}^{h} C_{i,j,n} u_n^{(i)} + \sum_{i=0}^{h} D_{i,j,n+1} z_n^{(i)} & = \sum_{i=0}^{h} \beta_{i,j} v_{j,n}^{(i)} \end{cases}$$

where

$$D_{h,j,n+1} := C_{h,j,n} \left(\frac{B_{h-n,n}}{A_{h-n,n}}\right),$$
$$D_{h-k,j,n+1} := E_{h-k,j,n} + \binom{h}{k} C_{h,j,n} \left(\frac{B_{h-n,n}}{A_{h-n,n}}\right)^{(k)} + \cdots + C_{h-k,j,n} \left(\frac{B_{h-n,n}}{A_{h-n,n}}\right)$$

for $1 \leq k \leq h$, and

$$B_{h-k,n+1} := B_{h-k,n} - \binom{h-n}{k-n} A_{h-n,n} \left(\frac{B_{h-n,n}}{A_{h-n,n}}\right)^{(k-n)} - \cdots - A_{h-k,n} \left(\frac{B_{h-n,n}}{A_{h-n,n}}\right)$$

for $n+1 \leq k \leq h$.

The next substitution replaces $v_{j,n}$ with $v_{j,n+1}$, defined by $v_{j,n} = v_{j,n+1} + \left(\frac{D_{h,j,n+1}}{\beta_{h,j}}\right) z_n$ for all lower equations j. This results in the system of equations

$$\begin{cases} \sum_{i=0}^{h-n} A_{i,n} u_{n+1}^{(i)} & = \sum_{i=0}^{h-n-1} B_{i,n+1} z_n^{(i)} \\ & \vdots \\ \sum_{i=0}^{h} C_{i,j,n} u_n^{(i)} + \sum_{i=0}^{h-1} E_{i,j,n+1} z_n^{(i)} & = \sum_{i=0}^{h} \beta_{i,j} v_{j,n+1}^{(i)} \end{cases}$$

where

$$E_{h-k,j,n+1} := D_{h-k,j,n+1} - \binom{h}{k} \beta_{h,j} \left(\frac{D_{h,j,n+1}}{\beta_{h,j}}\right)^{(k)} - \cdots - \beta_{h-k,j} \left(\frac{D_{h,j,n+1}}{\beta_{h,j}}\right).$$

for $1 \leq k \leq h$.

To complete the trio, we substitute z_{n+1} for z_n defined by $z_n = z_{n+1} + \left(\frac{A_{h-n,n}}{B_{h-n-1,n+1}}\right) u'_{n+1}$. Now we have the system

$$\begin{cases} \sum_{i=0}^{h-n-1} A_{i,n+1} u_{n+1}^{(i)} & = \sum_{i=0}^{h-n-1} B_{i,n+1} z_{n+1}^{(i)} \\ & \vdots \\ \sum_{i=0}^{h} C_{i,j,n+1} u_{n+1}^{(i)} + \sum_{i=0}^{h-1} E_{i,j,n+1} z_{n+1}^{(i)} & = \sum_{i=0}^{h} \beta_{i,j} v_{j,n+1}^{(i)} \end{cases}$$

where

$$\begin{aligned} C_{h-k,j,n+1} &:= C_{h-k,j,n} + \binom{h-1}{k} E_{h-1,j,n+1} \left(\frac{A_{h-n,n}}{B_{h-n-1,n+1}} \right)^{(k)} - \cdots \\ &\quad - E_{h-k-1,j,n+1} \left(\frac{A_{h-n,n}}{B_{h-n-1,n+1}} \right) \\ C_{0,j,n+1} &:= C_{0,j,n} \end{aligned}$$

for $0 \leq k < h$ and

$$\begin{aligned} A_{h-k,n+1} &:= A_{h-k,n} - \binom{h-n-1}{k-n} B_{h-n-1,n+1} \left(\frac{A_{h-n,n}}{B_{h-n-1,n+1}} \right)^{(k-n)} - \cdots \\ &\quad - B_{h-k-1,n+1} \left(\frac{A_{h-n,n}}{B_{h-n-1,n+1}} \right) \\ A_{0,n+1} &:= A_{0,n} \end{aligned}$$

for $n+1 \leq k \leq h-1$.

Completing this procedure by performing this trio h times results in the system

$$\begin{cases} A_{0,h} u_h & = & B_{0,h} z_h \\ & \vdots & \\ \sum_{i=0}^{h} C_{i,j,h} u_h^{(i)} + \sum_{i=0}^{h-1} E_{i,j,h} z_h^{(i)} & = & \sum_{i=0}^{h} \beta_{i,j} v_{j,h}^{(i)} \end{cases}$$

Now we can solve for z_h in terms of u_h, and plug the resulting expression in to the lower equations. After eliminating the top equation and simplifying, we have

$$\begin{cases} \sum_{i=0}^{h} C_{i,j,h+1} u_h^{(i)} & = & \sum_{i=0}^{h} \beta_{i,j} v_{j,h}^{(i)} \\ & \vdots & \\ \sum_{i=0}^{h} C_{i,m,h+1} u_h^{(i)} & = & \sum_{i=0}^{h} \beta_{i,m} v_{m,h}^{(i)} \end{cases} \tag{6}$$

where, for $k > 0$,

$$\begin{aligned} C_{h,j,h+1} &:= C_{h,j,h} \\ C_{h-k,j,h+1} &:= C_{h-k,j,h} + \binom{h-1}{k} E_{h-1,j,h} \left(\frac{A_{0,h}}{B_{0,h}} \right)^{(k)} + \cdots + E_{h-k,j,h} \left(\frac{A_{0,h}}{B_{0,h}} \right). \end{aligned}$$

Let $\mathcal{T}$ denote the solution set to system 6. The desired variable has been eliminated, so now we must show that these substitutions are well-defined. It suffices to show that the coefficients appearing in the denominator of each is non-zero.

First, $\beta_{h,j} \neq 0$ for all j by assumption, so the substitutions which decrease the order in the lower equations are all well-defined.

Claim 4.6. *The tuple* $(B_{h-n,n}, A_{h-n,n} : 0 \leq n \leq h)$ *is interdefinable with* $(B_{i,0}, A_{i,0} : 0 \leq i \leq h)$.

Proof. We prove that for each $n < h$, $(A_{h-i,i}, B_{h-i,i}, A_{h-k,n}, B_{h-k,n} : i < n, k \geq n)$ is interdefinable $(A_{h-i,i}, B_{h-i,i}, A_{h-k,n+1}, B_{h-k,n+1} : i \leq n, k > n)$, proving the claim by induction. Fix $n < h$. It is clear from the definition of $B_{i,n+1}$ that $(B_{h-k,n} : n+1 \leq k \leq h)$ is interdefinable with $(B_{h-k,n+1} : n+1 \leq k \leq h)$ over $B_{h-n,n}$ and $\{A_{h-k,n} : n \leq k \leq h\}$. By adding the parameters themselves, we have $(B_{h-k,n}, A_{h-k,n} : n \leq k \leq h)$ is interdefinable with $(B_{h-n,n}, B_{h-k,n+1}, A_{h-i,n} : n+1 \leq k \leq h)$. If $n > 0$, interdefinability is preserved after adding $\{B_{h-i,i}, A_{h-i,i} : 0 \leq i < n\}$ to each tuple. It is also clear from the definition of $A_{i,n+1}$ that $(A_{h-k,n} : n+1 \leq k \leq h)$ is interdefinable with $(A_{h-k,n+1} : n+1 \leq k \leq h)$ over $A_{h-n,n}$ and $\{B_{h-k,n+1} : n+1 \leq k \leq h\}$. Combining these two facts, we conclude that the desired tuples are interdefinable. ☐

It follows from this claim that the coefficients $\{A_{h-n,n}, B_{h-n,n} : 0 \leq n \leq h\}$ are independent differential transcendentals over $\mathbb{Q}\langle \mathcal{B}^* \rangle$ where $\mathcal{B}^* = \mathcal{B} \setminus \{B_{i,0} : 0 \leq i \leq h\}$. Hence, all substitutions used to eliminate the variable in the top equation are well-defined.

Finally, we prove the two conditions that hold for $\mathcal{S}$ also hold for $\mathcal{T}$. In fact, inter-differential algebricity preserves differential transendence degree, so proving the first condition implies the second. Therefore, it suffices to show inductively that, for each n and j, $(C_{i,j,n+1} : 0 \leq i \leq h)$ is inter-differentially algebraic with $(C_{i,j,n} : 0 \leq i \leq h)$ over $\mathbb{Q}\langle \mathcal{B} \rangle$. Assume, $(A_{i,0} : i \leq h)$ is inter-differentially algebraic with $(C_{i,j,n} : 0 \leq i \leq h)$ over $\mathbb{Q}\langle \mathcal{B} \rangle$. By Claim 4.6, this is equivalent to showing that these tuples of C's are inter-differentially algebraic over $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h\} \cup \mathcal{B}^*$.

We begin by proving this for $n = 0$. It is clear from the definitions that $(C_{i,j,1} : 0 \leq i \leq h)$ is differentially algebraic over $(C_{i,j,0} : 0 \leq i \leq h)$. To prove the other direction, observe that in the definitions of $C_{i,j,1}$ for any $i > 0$, $C_{h,j,0}$ appears with order at least one, but any other $C_{k,j,0}$ must be linear and order zero (over $\{A_{h,0}, B_{h,0}, A_{h-1,1}, B_{h-1,1}\}$). We can represent this situation with the following matrix equation:

$$\begin{bmatrix} M_1 & M_0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ M_2 & M_1 & M_0 & \cdots & 0 & 0 & 0 & 0 \\ M_3 & M_2 & M_1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ M_{h-3} & M_{h-4} & M_{h-5} & \cdots & M_1 & M_0 & 0 & 0 \\ M_{h-2} & M_{h-3} & M_{h-4} & \cdots & M_2 & M_1 & M_0 & 0 \\ M_{h-1} & M_{h-2} & M_{h-3} & \cdots & M_3 & M_2 & M_1 & M_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{h-1,j,0} \\ C_{h-2,j,0} \\ C_{h-3,j,0} \\ \vdots \\ C_{3,j,0} \\ C_{2,j,0} \\ C_{1,j,0} \\ C_{0,j,0} \end{bmatrix} = \begin{bmatrix} C_{h-1,j,1} \\ C_{h-2,j,1} \\ C_{h-3,j,1} \\ \vdots \\ C_{3,j,1} \\ C_{2,j,1} \\ C_{1,j,1} \\ C_{0,j,1} \end{bmatrix} - \begin{bmatrix} L_{h-1} \\ L_{h-2} \\ L_{h-3} \\ \vdots \\ L_3 \\ L_2 \\ L_1 \\ 0 \end{bmatrix}$$

where $L_i \in \mathbb{Q}\langle C_{h,0}, A_{h,0}, B_{h,0}, A_{h-1,1}, B_{h-1,1}, \mathcal{B}^*\rangle$ and the matrix entries can be computed as follows:

$$\begin{aligned}
M_0 &= \frac{B_{h,0}}{B_{h-1,1}} \\
M_1 &= h\left(\frac{B_{h,0}}{A_{h,0}}\right)'\left(\frac{A_{h,0}}{B_{h-1,1}}\right) + (h-1)\left(\frac{B_{h,0}}{A_{h,0}}\right)\left(\frac{A_{h,0}}{B_{h-1,1}}\right)' \\
M_i &= \binom{h-1}{0}\binom{h}{i}\left(\frac{B_{h,0}}{A_{h,0}}\right)^{(i)}\left(\frac{A_{h,0}}{B_{h-1,1}}\right) + \binom{h-1}{1}\binom{h-1}{i-1}\left(\frac{B_{h,0}}{A_{h,0}}\right)^{(i-1)}\left(\frac{A_{h,0}}{B_{h-1,1}}\right)' + \cdots \\
&\quad + \binom{h-1}{i}\binom{h-i}{i-i}\left(\frac{B_{h,0}}{A_{h,0}}\right)\left(\frac{A_{h,0}}{B_{h-1,1}}\right)^{(i)}.
\end{aligned}$$

Using Gaussian elimination, we can make this matrix lower triangular:

$$M^* = \begin{bmatrix}
M_{1,h-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
M_{2,h-2} & M_{1,h-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
M_{3,h-3} & M_{2,h-3} & M_{1,h-3} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
M_{h-3,3} & M_{h-4,3} & M_{h-5,3} & \cdots & M_{1,3} & 0 & 0 & 0 \\
M_{h-2,2} & M_{h-3,2} & M_{h-4,2} & \cdots & M_{2,2} & M_{1,2} & 0 & 0 \\
M_{h-1,1} & M_{h-2,1} & M_{h-3,1} & \cdots & M_{3,1} & M_{2,1} & M_{1,1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix}$$

where the new matrix entries are defined recursively for $2 \le j \le h-1$ and $1 \le i \le h-j$

$$M_{i,1} := M_i, \quad M_{i,j} := M_i - \left(\frac{M_0}{M_{1,j-1}}\right) M_{i+1,j-1}.$$

Claim 4.7. *For all $1 \le j \le h-1$ and $1 \le i \le h-j$, $M_{i,j} \neq 0$, and therefore, the determinant of M^* is non-zero.*

Proof. We prove this by showing that $M_{i,j}$ is order $i+j-1$ in $A_{h,0}$ using induction on j. It follows from the independence of the A's and B's that $M_{i,1}$ is order i in $A_{h,0}$ for each i. Now suppose the claim holds for j. Since $M_{i,j+1} = M_i - \left(\frac{M_0}{M_{1,j}}\right) M_{i+1,j}$, we can see that $M_{i,j+1}$ is order $i+j$ since $M_{i+1,j}$ is order $i+j$ by assumption, and all other terms in the definition have strictly smaller order. Thus the determinant is non-zero. □

As a consequence of the claim, we see that $(C_{i,j,0} : 0 \le i \le h-1)$ and $(C_{i,j,1} : 0 \le i \le h-1)$ are interdefinable over $\{C_{h,j,0}, A_{h,0}, B_{h,0}, A_{h-1,1}, B_{h-1,1}\} \cup \mathcal{B}^*$. Now we can solve the equation resulting from the top row of M^* for $C_{h-1,j,0}$ in terms of $(C_{i,j,1} : 0 \le i \le h-1)$ and $C_{h,j,0}$. We can plug the resulting expression in for $C_{h-1,j,0}$ in the following definition of $C_{h,j,1}$, resulting in a differential relation between $C_{h,0}$ and $(C_{i,j,1} : 0 \leq i \leq h)$:

$$C_{h,j,1} = \left(\frac{B_{h,0}}{B_{h-1,1}}\right) C_{h-1,j,0} - \left(\frac{hB_{h,0}}{B_{h-1,1}}\right) C'_{h,j,0} + \left[1 + \frac{hB_{h,0}}{B_{h-1,1}}\left(\frac{B_{h,0}}{A_{h,0}}\right)' - \frac{B'_{h,0}}{B_{h-1,1}} + \frac{hB_{h,0}(A_{h_0}\beta_{h,j})'}{B_{h-1,1}A_{h,0}\beta_{h,j}} - \frac{\beta_{h-1,j}B_{h,0}}{\beta_{h,j}B_{h-1,1}}\right] C_{h,j,0}.$$

Thus, $C_{h,j,0}$ is differentially algebraic over $(C_{i,j,1} : 0 \leq i \leq h)$. It follows that $(C_{i,j,0} : 0 \leq i \leq h-1)$ is differentially algebraic over $\{A_{h,0}, B_{h,0}, A_{h-1,1}, B_{h-1,1}\} \cup \mathcal{B}^*$, and $(C_{i,j,1} : 0 \leq i \leq h)$. Hence, $(C_{i,j,0} : 0 \leq i \leq h)$ and $(C_{i,j,1} : 0 \leq i \leq h)$ are inter-differentially algebraic over $\mathcal{B}$, proving the desired result for $n = 0$.

Proving that $(C_{i,j,n+1} : 0 \leq i \leq h)$ is inter-differentially algebraic with $(C_{i,j,n} : 0 \leq i \leq h)$ over $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h\} \cup \mathcal{B}^*$ for positive n is similar to the case where $n = 0$. As before, one direction is clear from the definitions. The only difference with the previous case is the inclusion of the $E_{h-k,j,n}$ term in the definition of $D_{h-k,j,n+1}$ for $k \geq 1$.

Claim 4.8. *For each* n, $(E_{i,j,n} : 0 \leq i \leq h-1)$ *is differentially algebraic over* $(C_{i,j,n-1} : 0 \leq i \leq h)$ *and* $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h\} \cup \mathcal{B}^*$.

Proof. The claim holds for $n = 1$ by examination of the definitions. Now suppose the claim holds for n. $E_{i,j,n+1}$ is defined by $(C_{i,j,n} : 0 \leq i \leq h)$, $(E_{i,j,n} : 0 \leq i \leq h-1)$, and $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h\} \cup \mathcal{B}^*$. It follows from the inductive hypothesis that $(E_{i,j,n} : 0 \leq i \leq h-1)$ is differentially algebraic over $(C_{i,j,n-1} : 0 \leq i \leq h)$, $(C_{i,j,n} : 0 \leq i \leq h)$, and $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h\} \cup \mathcal{B}^*$. By assumption, $(C_{i,j,n-1} : 0 \leq i \leq h)$ and $(C_{i,j,n} : 0 \leq i \leq h)$ are inter-differentially algebraic, so $(E_{i,j,n} : 0 \leq i \leq h-1)$ is differentially algebraic over $(C_{i,j,n} : 0 \leq i \leq h)$, and $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h\} \cup \mathcal{B}^*$. ☐

By this claim, it suffices to show that $(C_{i,j,n+1} : 0 \leq i \leq h)$ is inter-differentially algebraic with $(C_{i,j,n} : 0 \leq i \leq h)$ over $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h\} \cup \mathcal{B}^* \cup \{E_{i,j,n+1} : 0 \leq i \leq h-1\}$. This can be shown using the argument from the $n = 0$ case, although we will not present the details here.

It follows from the definition that $(C_{i,j,h} : 0 \leq i \leq h)$ is interdefinable with $(C_{i,j,h+1} : 0 \leq i \leq h)$ over $\{A_{h-i,i}, B_{h-i,i} : 0 \leq i \leq h, \} \cup \mathcal{B}^* \cup \{E_{i,j,h} : 0 \leq i \leq h-1\}$. By Claim 4.8, the desired differential algebricity follows.

We have shown that for all j, $(C_{i,j,h+1} : 0 \leq i \leq h)$ is inter-differentially algebraic with $(C_{i,j,0} : 0 \leq i \leq h)$ over $\mathcal{B}^*$ and thus with $(A_{i,0} : 0 \leq i \leq h)$. It follows that for distinct j_0 and j_1, both $(C_{i,j_0,h+1} : 0 \leq i \leq h)$ and $(C_{i,j_1,h+1} : 0 \leq i \leq h)$ are inter-differentially algebraic over $\mathcal{B}^*$. Finally, since each $(C_{i,j,0} : 0 \leq i \leq h)$ is independent over $\mathcal{B}^*$, it follows from inter-differential algebricity that $\{C_{i,j,h+1} : 0 \leq i \leq h\} \cup \mathcal{B}^*$ is differentially independent. ☐

Combining the results of Lemma 4.3, Theorem 4.4, and Theorem 3.8 results in a proof of the main result, Theorem 1.3.

5. ORTHOGONALITY TO THE CONSTANTS

We've seen that bounds of [**?**] on the degree of nonminimality can be used in conjunction with linearization techniques to establish the strong minimality of general classes of differential equations. There are two main obstacles to the wide application of these techniques for establishing strong minimality of many classical nonlinear equations.

1. The methods developed in the previous subsection seems to require at least one coefficient of the equation in question to be differentially transcendental.
2. For a given equation, even of small order, the computations required to verify strong minimality are quite involved.

In this section, we will show how the computational demands can be significantly reduced if a weaker condition than strong minimality is the goal. Consider the following (weaker) condition: that V *is either strongly minimal or almost internal to the constants*. In [**?**], it is shown that if $\mathrm{nmdeg}(p) > 1$, then p is an isolated type which is almost internal to a non-locally modular type. In differentially closed fields, this means that p is almost internal to the constant field whenever $\mathrm{nmdeg}(p) > 1$. Thus, the computations required to show our weaker condition will be much simpler, for instance, involving only two variables.

Example 5.1. We will show the equation

$$x'' + x^2 - \alpha = 0,$$

where α is a differential transcendental, is either strongly minimal or internal to the constants. If the equation is not internal to the constants and not strongly minimal, then by the results of [**?**] there is an indiscernible sequence of length two (x_1, x_2), which would satisfy the system

$$\begin{cases} x_1'' + x_1^2 & = & \alpha \\ x_2'' + x_2^2 & = & \alpha \end{cases}$$

such that x_2 satisfies an order one equation over x_1. Using the same strategy as in the previous sections, we replace α with a variable y in both equations, and then compute the differential tangent space:

$$\begin{cases} u'' + 2x_1 u & = & y \\ v'' + 2x_2 v & = & y. \end{cases}$$

Eliminating y, we are left with the single equation

$$u'' + 2x_1 u = v'' + 2x_2 v.$$

Consider the definable bijection given by the substitution $(u, v) \mapsto (w, v)$ where $u = w + v$. This transforms the above equation into

$$w'' + 2x_1 w = 2(x_2 - x_1)v$$

which can be solved for v since $x_1 \neq x_2$. Therefore the differential tangent space has no infinite rank subvarieties, a contradiction, so $x'' + x^2 - \alpha = 0$ is either strongly minimal or almost internal to the constants.

5.1. **Questions and conjectures.**

Question 5.2. Varieties which are internal to the constants have certain stronger properties that may, in general, allow one show via some additional argument the strong minimality of specific equations. For instance, by [**?**], if X is nonorthogonal to the constants, then there are infinitely many co-order one subvarieties of X. Thus, in this setting, showing strong minimality (after an argument like that of the example above) is equivalent to ruling out co-order one subvarieties. Are there interesting classes of equations in which one can successfully employ this strategy?

Question 5.3. Can the techniques of this paper be adapted to situations with non-generic coefficients?

Conjecture 5.4. *Generic differential equations of fixed order and degree greater than one are strongly minimal.*

Conjecture 5.5. *Any two solutions of a generic differential equation of fixed order and degree greater than one are (differentially) algebraically independent.*

After the completion of this work, the authors, together with Guy Casale and Joel Nagloo, were able to give affirmative answers to questions above, which we describe next. We have left the questions as stated above, since we feel pursuing these directions for other classes of equations is an important direction for future research. In the forthcoming work, joint with Casale and Nagloo, the techniques of this paper are part of a new proof of the main theorem of [**?**]: the differential equation satisfied by the j-function,

$$\left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 + (y')^2 \cdot \frac{y^2 - 1968y + 2654208}{y^2(y-1728)^2} = 0$$

is strongly minimal. We also employ the strategy to establish the strong minimality of several new equations.

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