Recall that last week we wanted to prove the following theorem. If \( p \) is a prime number, and \( q = p^d \), then:

**Theorem 1 (Quillen).**
\[
K_{2n-1}(\mathbb{F}_q) = \mathbb{Z}/(q^n - 1); \quad K_{2n}(\mathbb{F}_q) = 0, \text{ where } n > 0.
\]

This follows from a more general theorem where we have the diagram:

\[
\begin{array}{ccc}
BGL(\mathbb{F}_q)^+ & \overset{\vartheta}{\longrightarrow} & BU \\
\downarrow \cong & & \downarrow \psi^q - 1 \quad (\ast) \\
F\psi^q, & & BU
\end{array}
\]

where \( F\psi^q \) is the homotopy fiber of \( \psi^q - 1 \), where \( \psi^q \) is the Adams operation. Today we will explain how precisely we get \( \vartheta : BGL(\mathbb{F}_q)^+ \to F\psi^q \) and outline the computation of the homology of \( BGL(\mathbb{F}_q) \).

1. **Adams Operations** \( \psi^k \)

**Definition 2.**
A commutative ring \( R \) is called a \( \lambda \)-ring, if we are given operations \( \lambda^k : R \to R \) for \( k \geq 0 \) such that, for all \( x, y \in R \):

- \( \lambda^0(x) = 1 \) and \( \lambda^1(x) = x \),
- \( \lambda^k(x + y) = \lambda^k(x) + \lambda^{k-1}(x)\lambda^1(y) + \cdots + \lambda(y) \).

For instance, let \( X \) be a topological space and \( E \to X \) a complex vector bundle. Let \( \lambda^k(E) \) be the exterior power bundle \( \bigwedge^k E \). This induces a \( \lambda \)-structure on the ring \( KU^0(X) = [X, BU \times \mathbb{Z}] \). Another example is given by the complex representation ring \( R(G) \) where \( G \) is a finite group: consider the exterior powers of a representation (think of the exterior power of a \( CG \)-module).

**Definition 3.**
Let \( R \) be a \( \lambda \)-ring. The **Adams operations** \( \psi^k : R \to R \), for \( k > 0 \) are defined as follows. The subring of symmetric polynomials in the polynomial algebra \( \mathbb{Z}[x_1, \ldots, x_k] \) is the polynomial algebra \( \mathbb{Z}[[\sigma_1, \ldots, \sigma_k]] \), where \( \sigma_i = \sum_{1 \leq j_1 \leq \cdots \leq j_i \leq k} x_{j_1} \cdots x_{j_i} \) is the \( i \)-th elementary symmetric function. One may write the power sum \( x_1^k + \cdots + x_k^k \) as a polynomial \( Q_k(\sigma_1, \ldots, \sigma_k) \). Then one define:

\[
\psi^k(x) := Q_k(\lambda^1(x), \ldots, \lambda^k(x)),
\]
in other words:
\[
\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) - \lambda^2(x)\psi^{k-2}(x) + \cdots + (-1)^k\lambda^{k-1}(x)\psi^1(x) + (-1)^{k-1}k\lambda^k(x).
\]

For instance, since \(x_1^2 + x_2^2 = 2\sigma_1^2 - 2\sigma_2\), we get \(\psi^2(x) = x^2 - 2\lambda^2(x)\).

Notice that \(\psi^k(\ell) = \ell^k\) for any line element in \(R\) (the definition of line element can make sense for any augmented \(\lambda\)-ring), as \(\lambda^1(\ell) = \ell\) and \(\lambda^k(\ell) = 0\) for any \(k \geq 2\). For a space \(X\), the Adams operations \(\psi^k : KU^0(X) \to KU^0(X)\) are represented by a map \(\psi^k : BU \to BU\) (use Yoneda’s lemma). So if for a line bundle \(\ell\) we have \((\ell - 1)^{\otimes 2} = 0\), then:

\[
\psi^k(\ell - 1) = \psi^k(\ell) - \psi^k(1) = \ell^k - 1 = (\ell - 1 + 1)^k - 1 = k(\ell - 1).
\]

Now recall that in the proof of Bott periodicity, one shows that \(\tilde{K}U^0(S^2)\) is generated as a ring by \([L]\) subject to \((|L| - 1)^2 = 0\), where \(L\) is the tautological line bundle over \(\mathbb{C}P^1 \cong S^2\).

Now since \(S^{2n} = S^2 \wedge \cdots \wedge S^2\), \(\tilde{K}U^0(S^{2n})\) is generated by \((|L| - 1) \otimes \cdots \otimes (|L| - 1)\). Therefore \(\psi^k(x) = k^n x\) for any \(x \in \tilde{K}U^0(S^{2n})\).

**Proof of Theorem 1**: Suppose \(\square\) is true. Then the long exact sequence of the homotopy fiber, together with the fact that \(\psi^q\) acts on \(\tilde{K}U^0(S^{2n}) = \pi_{2n}(BU) \cong \mathbb{Z}\) as multiplication by \(q^n\), and \(\tilde{K}U^0(S^{2n-1}) = \pi_{2n-1}(BU) = 0\) implies directly the Theorem. □

2. **Brauer Lifting**

Let \(G\) be a finite group. The **Borel construction** gives an additive functor:

\[
\mathbb{C}G\text{Mod} \rightarrow \text{Vect}_\mathbb{C}(BG)
\]

\[
V \mapsto \left(\begin{array}{c}
EG \times_G V \\
\downarrow \\
BG
\end{array}\right),
\]

where \(\text{Vect}_\mathbb{R}(BG)\) denotes the category of complex vector bundles over \(BG\). Since it commutes with tensor products and exterior powers, we also get a \(\lambda\)-ring homomorphism:

\[
\Phi : R(G) \rightarrow K^0(BG) = [BG, BU \times \mathbb{Z}].
\]

We will consider the case \(G = \text{GL}_n(\mathbb{F}_q)\). We have the standard representations \(\text{GL}_n(\mathbb{F}_q)\) of \((\mathbb{F}_q)^n\). To obtain representations over \(\mathbb{C}\), we use the following idea.

**Definition 4**.

Let \(\mathbb{F}_q^*\) be the algebraic closure of \(\mathbb{F}_q\). Fix an embedding \(\rho : \mathbb{F}_q^* \hookrightarrow \mathbb{C}^*,\) via the roots of unity. Suppose \(V\) is a representation of \(G\) over \(\mathbb{F}_q\). The **Brauer character** \(\chi_V\) is defined as:

\[
\chi_V(g) := \sum_{\alpha \in S_g} \rho(\alpha)
\]

where \(S_g \subseteq \mathbb{F}_q^*\) is the spectrum of \(\varphi_V(g)\), where \(\varphi_V : G \rightarrow \text{Aut}(V)\), i.e. the set of eigenvalues, with multiplicity, of \(\varphi_V(g)\). Then \(\chi_V\) is the character of a unique virtual complex representation of \(G\) over \(\mathbb{C}\).
We obtain a ring homomorphism $R_{\mathbb{F}_q}(G) \longrightarrow R(G)$. Let us look how the Adams operation $\psi^k$ behave under Brauer lifting.

**Proposition 5.**
Let $\chi$ be a virtual character of $G$. Then $(\psi^k \chi)(g) = \chi(g^k)$ for all $g \in G$.

**Proof:** Let us do by induction on the degree of the representation. If $\chi = \chi_\ell$ is the character of a one dimensional $\ell$ representation of $G$ over $\mathbb{C}$, then we know:

$$(\psi^k \chi_\ell)(g) = \chi_\ell(g) = (\chi_\ell(g))^k = \chi_\ell(g^k),$$

since $\chi_\ell = \varphi_\ell$ as $\ell$ is one-dimensional, i.e., $\chi_\ell$ is a group homomorphism. The same can be said for representation $V$ of direct sum of one-dimensional representation. If $G$ is cyclic, then any $V$ has this form. The general case follows immediately by restricting to the cyclic subgroups $< g >$ for any $g$ in $G$.

**Corollary 6.**
If $\chi$ is the Brauer character of a representation $V$ over $\mathbb{F}_q$, then $(\psi^q \chi) = \chi$.

**Proof:** Fix $g \in G$. Let us show that $S_g = S_{g^q}$. This follows from the fact that the eigenvalues of any linear map lie in a finite Galois extension $K \supset \mathbb{F}_q$, and are permuted by the Galois group $\text{Gal}(K/\mathbb{F}_q)$, which is cyclic and generated by the Frobenius morphism $a \mapsto a^q$. So $S_{g^q} = S_g$.

So the Brauer lifting takes value in $R(G)^{\psi^q}$. So using our previous $\lambda$-ring homomorphism $\Phi : R(G) \longrightarrow K^0(BG)$, and the projection $KU^0(BG) \rightarrow \overline{KU}^0(BG)$, we have constructed a mapping:

$$R_{\mathbb{F}_q}(G) \longrightarrow \overline{KU}^0(BG)^{\psi^q} = [BG, BU]^{\psi^q}.$$

**3. The Homotopy Equivalence $B\text{GL}(\mathbb{F}_q)^+ \longrightarrow F\psi^q$**

We can now construct our map $\overline{\psi} : B\text{GL}(\mathbb{F}_q)^+ \longrightarrow F\psi^q$. The standard action of $\text{GL}_n(\mathbb{F}_q)$ over $(\mathbb{F}_q)^n$ will give maps $\theta_n : B\text{GL}_n(\mathbb{F}_q) \rightarrow BU$ by the Brauer lifting. It is easy to see that these maps are compatible under restriction: $\theta_n|_{B\text{GL}_{n-1}(\mathbb{F}_q)} = \theta_{n-1}$. Using colimit, we obtain a map $\theta : B\text{GL}(\mathbb{F}_q) \rightarrow BU$. Now Corollary 6 shows that $\psi^q \circ \theta_n \simeq \theta_n$, so $(\psi^q - 1) \circ \theta_n$ is nullhomotopic. Now, as $BU$ has abelian fundamental group, the universal property of the $+$-construction gives a unique map up to homotopy $\theta : B\text{GL}(\mathbb{F}_q)^+ \longrightarrow BU$, denoted again $\theta$, such that $(\psi^q - 1) \circ \theta$ is nullhomotopic. Property of the homotopy fiber says that the sequence:

$$[Z, F\psi^q] \longrightarrow [Z, BU]^{(\psi^q - 1)^r} [Z, BU]$$

is exact for any space $Z$. Hence we get a map $\overline{\psi} : B\text{GL}(\mathbb{F}_q) \rightarrow F\psi^q$, which turns out to be unique up to homotopy with some work.

To prove that $\overline{\psi}$ is a homotopy equivalence, we only need to show it is a weak homotopy equivalence, since the spaces are CW-complexes. In fact, the spaces are simple.

**Definition 7.**
A **simple space** $X$ is a space homotopy equivalent to a CW-complex, connected, such that $\pi_1(X)$ is abelian and acts trivially on the higher homotopy groups.
Theorem 8.
Let \( f : X \to Y \) be a map between simple spaces. If \( f \) induces an isomorphism on integer homology \( H_*(\cdot; \mathbb{Z}) \), then \( f \) is a homotopy equivalence.

Sketch of the Proof: For simplicity, let’s omit \( \mathbb{Z} \) in the coefficients of the homology. Consider \( \mathcal{F} \) the homotopy fiber of \( f \). Then \( \mathcal{F} \) is connected, so \( H_0(\mathcal{F}) = \mathbb{Z} \). The Serre spectral sequence:
\[
E_2^{pq} = H_p(Y, H_q(\mathcal{F})) \Rightarrow H_p+q(X)
\]
has no twisted coefficients and depicts as follows:
\[
\begin{array}{cccccc}
H_0(Y, H_1(\mathcal{F})) & H_1(Y, H_1(\mathcal{F})) & H_2(Y, H_1(\mathcal{F})) & \cdots \\
H_0(Y) & H_1(Y) & H_2(Y) & \cdots \\
H_0(X) & H_1(X) & H_2(X) & \cdots \\
\end{array}
\]
where the bottom horizontal maps represent the edge isomorphisms which are exactly the maps \( f_* \). Hence we get:
\[
0 = H_0(Y, H_1(\mathcal{F})) = (H_1(\mathcal{F}))^\pi(Y) = H_1(\mathcal{F}),
\]
and so \( H_1(\mathcal{F}) = \pi_1(\mathcal{F}) = 0 \). It is simply connected, and inductively, one can show that \( H_*(\mathcal{F}) \). Therefore \( f \) is a weak homotopy equivalence.

Therefore, it suffices to prove that \( \overline{\mathcal{F}} : BGL(\mathbb{F}_q) \to F\psi \) induces a homology equivalence over \( \mathbb{Z} \). A standard universal coefficient argument shows that it suffices to prove the isomorphism for homology with coefficients over \( \mathbb{Q} \), and the finite cyclic groups. Since the homology of \( F\psi \) was computed last week, it only remains to look for \( BGL(\mathbb{F}_q) \).

4. \( \widetilde{H}_*(BGL(\mathbb{F}_q); \mathbb{Q}) = 0 \)

This follows from a more general result.

Proposition 9.
If \( G \) is a finite group, then \( \widetilde{H}_*(BG; \mathbb{Q}) = 0 \).

Proof: We have:
\[
H_*(BG; \mathbb{Q}) = H_*(G; \mathbb{Q}) = \text{Tor}^*_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}),
\]
Since for any \( \mathbb{Z}G \)-module \( M \), we clearly have a natural isomorphism:
\[
\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q} \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes_{\mathbb{Z}} M),
\]
then since Tor is a left derived functor, we get:
\[
\text{Tor}^*_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) \cong \text{Tor}^*_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}).
\]
Since the augmentation map \( \varepsilon : \mathbb{Q}G \to \mathbb{Q} \), defined as \( e_g \mapsto 1 \), splits in:
\[
0 \to \ker(\varepsilon) \to \mathbb{Q}G \xrightarrow{\varepsilon} \mathbb{Q} \to 0
\]
via the mapping \( 1 \mapsto \frac{1}{|G|} \sum_{g \in G} e_g \), we get that the trivial module \( \mathbb{Q} \) is a projective \( \mathbb{Q}G \)-module. Thus \( H_0(G; \mathbb{Q}) = \mathbb{Q} \) and \( H_n(G; \mathbb{Q}) = 0 \) for \( n \geq 1 \).

To conclude, use the fact that \( \widetilde{H}_*(BGL(\mathbb{F}_q); M) \cong \text{colim}_n \widetilde{H}_*(BGL_n(\mathbb{F}_q); M) \), for any coefficient \( M \).
5. \( \tilde{H}_*(B\text{GL}_E; \mathbb{Z}/p) = 0 \)

We will make use of the transfer in algebraic \(K\)-theory. Recall that in general, a ring homomorphism \(R \to S\) would induce \(K_nR \to K_nS\) using functoriality. Now if \(R \subseteq S\) is a subring, then we may define in certain cases a homomorphism \(\tau : K_*S \to K_*R\), called the transfer. In our case, it sufficient to think of finite field extensions \(F \leq E\), of degree, say \(d\). Then, recalling the definition of \(K_0\), it is easy to define \(\tau : K_0E \to K_0F\) : let \(V\) be a finite dimensional \(E\)-vector space, set \(\tau([V]) = [V_F]\), where \(V_F\) is \(V\) regarded as a \(F\)-vector space. If \(i : F \to E\), then \(\tau \circ i_* : K_0F \to K_0F\) is multiplication by \(d\). For higher cases, fix a basis of \(E\) over \(F\). It induces maps :

\[
\text{GL}_nE \to \text{GL}_{dn}F,
\]

which are compatible, so that we get : \(\text{GLE} \to \text{GLF}\). Using the universal property of the \(+\)-construction, we get the desired map :

\[
\tau : B\text{GL}(E)^+ \longrightarrow B\text{GL}(F)^+.
\]

It turns out that the above construction is independant of the choice of the basis. And as the case for \(K_0\), we have that the composite :

\[
B\text{GL}(F)^+ \xrightarrow{i} B\text{GL}(E)^+ \xrightarrow{\tau} B\text{GL}(F)^+,
\]

is homotopic to the \(d\)-power map (as a \(H\)-space), so that \(\pi_*(\tau \circ i)\) is multiplication by \(d\). Suppose now we have proved the following.

**Lemma 10.**

Let \(q = p^r\), then \(\tilde{H}_i(B\text{GL}_n(\mathbb{F}_q); \mathbb{Z}/p) = 0\) for \(i < \nu(p - 1)\), for all \(n\).

Choose \(r\) prime to \(p\) and consider now :

\[
B\text{GL}(\mathbb{F}_q)^+ \xrightarrow{i} B\text{GL}(\mathbb{F}_{q^r})^+ \xrightarrow{\tau} B\text{GL}(\mathbb{F}_q)^+.
\]

If we localize our spaces at \(p\), then the map \(\tau \circ i\) become an equivalence (as \(\pi_*(\tau \circ i)\) is multiplication by \(r\) and \(r\) prime to \(p\)). So \(\tau \circ i\) induces an isomorphism on homology with coefficient \(\mathbb{Z}_{(p)}\), and so with coefficient \(\mathbb{Z}/p\). Now, by Lemma 10 applied on \(\mathbb{F}_{q^r}\), (recall that \(q = p^d\)) the map \(\tilde{H}_n(\tau \circ i, \mathbb{Z}/p)\) is the trivial map for \(n < dr(p - 1)\), and so :

\[
\tilde{H}_n(B\text{GL}(\mathbb{F}_q); \mathbb{Z}/p) = 0,
\]

for all \(n < dr(p - 1)\). But \(r\) was arbitrary, so \(\tilde{H}_*(B\text{GL}_E; \mathbb{Z}/p) = 0\).

**Sketch of the Proof of Lemma 10:** Let \(B_n \subseteq \text{GL}_n(\mathbb{F}_q)\) denote the subgroup of upper triangular matrices. Since the inclusion induces an injection onto group cohomology, it turns out that it is sufficient to prove that in group homology :

\[
\tilde{H}_i(B_n; \mathbb{Z}/p) = 0,
\]

for \(i < \nu(p - 1)\). One proceeds by induction on \(n\). For \(n = 1\), we have \(B_1 = \mathbb{F}_q^*\), so \(p\) does not divide the order of \(B_1\), and so \(\tilde{H}_i(B_1; \mathbb{Z}/p) = 0\) for all \(i\). For the inductive step, we have the usual group extension :

\[
A_n \to B_n \to B_{n-1},
\]
where $A_n$ is the top row subgroup. Using some Hochschild-Serre spectral sequence argument, it is sufficient to prove that the homology of $A_n$ vanishes in dimensions $i$, for $i < d(p - 1)$, and this is done by group cohomology, via the semidirect product structure of $A_n$:

$$V \to A_n \to \mathbb{F}_q^*,$$

where $V$ is the additive group of a $(n - 1)$-dimensional vector space over $\mathbb{F}_q$. □

6. **The isomorphism** $\overline{\theta}_s : H_*(BGL(\mathbb{F}_q); \mathbb{Z}/\ell) \longrightarrow H_*(F\psi^q; \mathbb{Z}/\ell)$

We only briefly outline the proof. Recall $\ell$ is a prime different than $p$. We will now omit the $\mathbb{Z}/\ell$ coefficients. Let $\mu = \mu_\ell$ denote a $\ell$-th root of unity different than $1$, and $G = \text{Gal}(\mathbb{F}_q(\mu)/\mathbb{F}_q)$. Let $r$ be the smallest integer such that $\ell$ divides $q^r - 1$. So $G$ is a group of order $r$ and generated by the Frobenius automorphism., which acts on $C = \mathbb{F}_q(\mu)^*$ as multiplication by $q$. We have an embedding:

$$C \hookrightarrow \text{GL}_r(\mathbb{F}_q),$$

induced by the representation $L$ of $\mathbb{F}_q(\mu)$ of $C$. We obtain an homomorphism:

$$H_*(BC)_G \longrightarrow H_*(BGL_r(\mathbb{F}_q)),
$$

where the subscript denote the invariants by the group action. The former has a basis consisting of elements $\xi_j'$ of degree $2jr$ for each $j \geq 0$ and $\eta_j'$ of degree $2jr - 1$ for each $j \geq 1$. Their images under the homomorphism are denoted:

$$\xi_j \in H_{2jr}(BGL_r(\mathbb{F}_q)),
\eta_j \in H_{2jr-1}(BGL_r(\mathbb{F}_q)).$$

Let $\varepsilon$ denote the distinguished generator of $H_0(\text{GL}_1 \mathbb{F}_q)$. Clearly, $\xi_0 = \varepsilon^r$. The key point is to realize that the representation of $C$ lifts by Brauer to the representation $W$ of last week:

$$W = \zeta \oplus \zeta^2 \oplus \cdots \oplus \zeta^{q^r-1},$$

where $\zeta : \mathbb{Z}/(q^r - 1) \to \mathbb{C}^*$ is done via root of unity. So the ring homomorphism:

$$\bigoplus_n H_*(BGL_n(\mathbb{F}_q)) \longrightarrow H_*(F\psi^q),$$

sends the $\varepsilon$ to 1 and sends $\eta_j$ and $\xi_j$ to the same letters element we have defined last week. Now one can prove again that:

$$\bigoplus_n H_*(BGL_n(\mathbb{F}_q)) \cong P[\varepsilon, \xi_1, \xi_2, \ldots] \otimes \Lambda[\eta_1, \eta_2, \ldots].$$

Thus $\overline{\theta}_s$ is an isomorphism on homology with coefficients $\mathbb{Z}/\ell$.  

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