Lecture 1.3 - Complex bordism theory

Maximilien Péroux and Jānis Lazovskis
WCATSS 2016 - The University of Oregon

Contents

Goal. The Lazard ring L and its universal formal group law can be described completely homotopically by the complex bordism spectrum MU. In fact, the Hopf algebroid (L, W) can be understood as the pair (MU_*, MU_*MU). We introduce the notion of complex-orientable cohomology theory which will provide a homotopical origin for some formal group laws.

Notation. We shall prefer to write $\pi_*(E)$ instead of E_* for the coefficient ring of a ring spectrum E.

1 Complex-orientable cohomology theories

Complex-orientable cohomology theories are a particular kind of cohomology theory. They generalize the main concepts from ordinary cohomology theory associated to complex vector bundles, such as Chern classes, Thom class, and the Thom isomorphism. For our purposes, each complex orientation of these cohomology theories has an associated formal group law and thus provides a homotopical description of formal group laws.

1.1 Complex orientation

Recall that $\mathbb{C}P^\infty \simeq BU(1) \simeq K(\mathbb{Z}, 2)$ classifies complex line bundles. The homotopy class of the natural inclusion map

$$i : S^2 \cong \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$$

generates the group $\pi_2(\mathbb{C}P^\infty) = \mathbb{Z}$. Stably, this defines a map of spectra

$$i : \Sigma^\infty S^2 \longrightarrow \Sigma^\infty \mathbb{C}P^\infty,$$

i.e., a map $i : \mathbb{S} \longrightarrow \Sigma^{\infty-2}\mathbb{C}P^\infty$, where $\mathbb{S}$ is the sphere spectrum.

We say that a cohomology theory E is *multiplicative* if its representing spectrum is endowed with a multiplication $E \wedge E \to E$ that is associative and unital up to homotopy, i.e. a ring spectrum.

Recall that given a spectrum E, the reduced cohomology of E is defined by

$$\widetilde{E}^k(X) = \mathrm{colim}_n[\Sigma^n X, E_{k+n}]$$

for X a pointed space.

Definition 1. A multiplicative cohomology theory E is *complex-orientable* if the induced map

$$i^* : \widetilde{E}^2(\mathbb{C}P^\infty) \longrightarrow \widetilde{E}^2(S^2) \cong \widetilde{E}^0(S^0) \cong \pi_0(E)$$

is surjective. This means that the image of i^* contains $1 \in \pi_0(E)$, the canonical generator representing the unital map $\eta : \mathbb{S} \to E$. A *complex orientation* is a choice of an element $x^E \in \widetilde{E}^2(\mathbb{C}P^\infty)$ such that $i^*(x^E) = 1$.

Since $\widetilde{E}^k(X) = \mathrm{colim}_n[\Sigma^n X, E_{n+k}]$, one should think of a complex orientation $x^E \in \widetilde{E}^2(\mathbb{C}P^\infty)$ as representing a map of spectra $\Sigma^{\infty-2}\mathbb{C}P^\infty \to E$ such that by precomposing with the map i we get the unital map $\eta : \mathbb{S} \to E$ of the ring spectrum E. That is, the following diagram commutes up to homtopy:

$$\xymatrix{\mathbb{S} \ar[r]^-{i} \ar@/_2pc/[rr]_{\eta} & \Sigma^{\infty-2}\mathbb{C}P^\infty \ar[r]^-{x^E} & E.}$$

Example 1. Let $E = H\mathbb{Z}$ be the ordinary cohomology theory. It is complex-orientable. Indeed, let $x^{H\mathbb{Z}}$ in

$$H^2(\mathbb{C}P^\infty; \mathbb{Z}) \cong H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$$

be the usual generator; it is the first universal Chern class. The same is true for any Eilenberg–MacLane spectrum HR, where R is a commutative ring.

Example 2. Let $E = KU$ be complex K-theory. A complex orientation x^{KU} is given by

$$[\xi^1] - \mathbf{1}_{\mathbb{C}} \in \widetilde{KU}^0(\mathbb{C}P^\infty) \cong \widetilde{KU}^2(\mathbb{C}P^\infty),$$

where ξ^1 is the universal complex line bundle.

Example 3. If $E = KO$, real K-theory, then E is not complex orientable. Hint: because

$$\mathbb{Z} \cong \widetilde{KO}^2(\mathbb{C}P^\infty) \longrightarrow \widetilde{KO}^2(S^2) \cong \mathbb{Z}$$

corresponds to the multiplication by 2.

The complex orientation x^E will sometimes be simply denoted x if there is no confusion. It can be regarded as the generalization of the universal first Chern class for the spectrum E.

1.2 Formal group laws

The condition that E is complex-orientable determines the structure of $E^*(\mathbb{C}P^\infty)$ as a power series ring.

Proposition 1. *Let E be a complex-orientable cohomology theory with complex orientation x. Then:*

(a) $E^*(\mathbb{C}P^n) \cong \pi_*(E)[i_n^*(x)]/\big(i_n^*(x)^{n+1}\big)$, where $i_n : \mathbb{C}P^n \to \mathbb{C}P^\infty$ is the inclusion map;

(b) $E^*(\mathbb{C}P^\infty) \cong \pi_*(E)[[x]]$;

(c) $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_*(E)[[x_1, x_2]]$, where $x_i = p_i^*(x)$ for $p_i : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ the i-th projection, $i = 1, 2$.

Proof. (a) There is an Atiyah–Hirzebruch spectral sequence for $E^*(\mathbb{C}P^n)$, given by

$$E_2^{p,q} = H^p(\mathbb{C}P^n; \pi_q(E)) \cong H^p(\mathbb{C}P^n; \mathbb{Z}) \otimes \pi_q(E) \Longrightarrow E^{p+q}(\mathbb{C}P^n).$$

Since $H^p(\mathbb{C}P^n) = \mathbb{Z}[X]/(X^{n+1})$ for some X of degree 2, the E_2-page is given by $\pi_*(E)[X]/(X^{n+1})$. The differentials $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ are $\pi_*(E)$-linear and multiplicative, so we get

$$d(a, X^k) = ad(X^k) = akX^{k-1}d(X)$$

for any $a \in \mathbb{Z}$ and $k \geq 1$. The element X belongs to $E_2^{2,0}$. The spectral sequence collapses at the E_2-page means that $X \in H^2(\mathbb{C}P^n, \pi_0(E))$ survives to an element in $E^2(\mathbb{C}P^n)$. This happens when X restricts to a generator of $H^2(\mathbb{C}P^1, \pi_0(E)) = \pi_0(E)$. Since E is complex orientable, the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}P^1 & \xrightarrow{\quad i \quad} & \mathbb{C}P^\infty \\ & \searrow \qquad \nearrow_{i_n} & \\ & \mathbb{C}P^n & \end{array}$$

implies that $i_n^*(x)$ is sent to the generator $1 \in \pi_0(E)$, as in the induced diagram

$$\begin{array}{ccc} \widetilde{E}^2(\mathbb{C}P^\infty) & \xrightarrow{\quad i^* \quad} & \widetilde{E}^2(\mathbb{C}P^1) \\ & {}_{i_n^*}\searrow \qquad \nearrow & \\ & \widetilde{E}^2(\mathbb{C}P^n) & \end{array}$$

the homomorphism i^* is surjective. Thus $d_2(X) = 0$ and X is an infinite cycle which represents $i_n^*(x) \in E^2(\mathbb{C}P^n)$. Thus we get the desired result.

(b) Since $\mathbb{C}P^\infty = \mathrm{colim}_n \mathbb{C}P^n$, the Milnor sequence gives

$$E^*(\mathbb{C}P^\infty) \cong \lim_n E^*(\mathbb{C}P^n) \cong \lim_n \pi_*(E)[i_n^*(x)]/\big(i_n^*(x)^{n+1}\big) \cong \pi_*(E)[[x]],$$

since the induced map $E^*(\mathbb{C}P^m) \to E^*(\mathbb{C}P^n)$ is surjective and hence satisfies the Mittag–Leffler condition, for $n \leq m$.

(c) The proof of (a) and (b) can be carried on to prove (c). Notice that

$$H^*(\mathbb{C}P^n \times \mathbb{C}P^m) \cong H^*(\mathbb{C}P^n) \otimes H^*(\mathbb{C}P^m),$$

so the cohomology $E^*(\mathbb{C}P^n \times \mathbb{C}P^m)$ can be computed again with the Atiyah–Hirzebruch spectral sequence, followed by an application of the Milnor sequence. □

Remark 1. The converse is actually true, if the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty; \pi_q(E)) \Longrightarrow E^{p+q}(\mathbb{C}P^\infty)$$

collapses at the E_2-page, then the ring spectrum E is complex-orientable.

Given two vector bundles $p : V_1 \to B$ and $q : V_2 \to B$ over the same base space B, their tensor product $pq : V_1 \otimes V_2 \to B$ is the vector bundle over B whose fiber over any point is the tensor product of modules of the respective fibers of V_1 and V_2. This tensor operation can be classified by a map

$$m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty,$$

which is the multiplication map for the H-structure of $\mathbb{C}P^\infty$. Consider the induced map on cohomology

$$m^* : \pi_*(E)[[x]] \cong E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_*(E)[[x_1, x_2]].$$

It determines a formal group law, denoted $\mu^E(x_1, x_2) = m^*(x)$, as the image of $x = x^E$ under m^* in $\pi_*(E)[[x_1, x_2]]$.

Proposition 2. *The construction $\mu^E(x_1, x_2)$ defined above is indeed a formal group law.*

Proof. This follows from the fact that $\mathbb{C}P^\infty$ is an H-group. In other words, the tensor product of complex line bundles is commutative and associative up to isomorphism. For instance, define, for $i = 1, 2$, the maps $j_i : \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ by $j_1(y) = (y, *)$ and $j_2(y) = (*, y)$. Then we get

$$\mu^E(x_1, 0) = (j_1^* \circ m^*)(x), \qquad \mu^E(0, x_2) = (j_2^* \circ m^*)(x).$$

The maps $m \circ j_i$ are homotopic to the identity, so we get $\mu^E(x_1, 0) = x_1$ and $\mu^E(0, x_2) = x_2$. Commutativity and associativity follows similarly, by noticing that

$$m \circ (m \times 1) \simeq m \circ (1 \times m).$$

□

Example 4. Let $E = H\mathbb{Z}$. Then we get $\mu^{H\mathbb{Z}}(x_1, x_2) = x_1 + x_2$, the usual additive formal product.

Example 5. Let $E = KU$. Then the pullback of a vector bundle p along $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is given by the tensor product $p_1 p_2$, where p_1 and p_2 are obtained by the pullback of p along the projections $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$. Recall $x^{KU} = [\xi^1] - \mathbf{1}_\mathbb{C}$, so

$$m^*(x^{KU} + 1) = (x_1^{KU} + 1)(x_2^{KU} + 1),$$

i.e., $m^*(x^{KU}) = x_1^{KU} + x_2^{KU} + x_1^{KU} x_2^{KU}$, using that m^* is a homomorphism of rings. hence we get that $\mu^{KU}(x_1, x_2) = x_1 + x_2 + x_1 x_2$ is the usual multiplicative formal product.

We have just described some formal group laws homotopically. We ask ourselves, can all the formal group laws be realized homotopically?

2 The complex bordism spectrum MU

The complex bordism spectrum MU is a universal complex-orientable cohomology theory: complex orientations of a spectrum E are in one-to-one correspondence with multiplicative maps $\mathrm{MU} \to E$. The spectrum MU is called a Thom spectrum.

2.1 The Thom spectrum of a real vector bundle

We remind here the construction of a Thom space and some results. Recall that, given a space B, we have

$$\Sigma^r B_+ = \frac{B \times D^r}{B \times S^{r-1}},$$

for $r \geq 1$. The Thom space generalizes the suspension in the following manner. Let $p : E \to B$ be a *real* vector bundle of rank r and suppose it is endowed with a metric (e.g. B is paracompact). Then one can define the unit sphere bundle

$$S(E) = \{v \in E \mid \|x\| = 1\}$$

and the unit disk bundle

$$D(E) = \{v \in E \mid \|v\| \leq 1\}$$

over B, so that we obtain subbundles $S(E) \subseteq D(E) \subseteq E$.

Definition 2. The *Thom space* of the real vector bundle p is $\mathrm{Th}(p) := D(E)/S(E)$.

If p is the trivial bundle $E = B \times \mathbb{R}^r$, then

$$\mathrm{Th}(p) = \frac{B \times D^r}{B \times S^{r-1}} = \Sigma^r B_+.$$

Therefore, for a general p, its Thom space $\mathrm{Th}(p)$ can indeed be regarded as a twisted suspension.

Remark 2. For motivational matters, we presented the definition of Thom spaces for vector bundles with metrics. More generally, for any rank-r real vector bundle $p : E \to B$, we may also obtain $\mathrm{Th}(p)$ by applying a fiberwise one-point compactification on E and identifying all the added points to a single basepoint. Note these two definitions are homeomorphic. This construction has the advantage of being functorial, as

$$\mathrm{Th} : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Top}_*,$$

where $\mathbf{Vect}_{\mathbb{R}}$ is the category of real vector bundles and $\mathbf{Top}_*$ is the category of pointed topological spaces. Since in this talk all the base spaces of vector bundles will be paracompact, we will allow ourselves to switch between the two constructions.

Proposition 3. *Let $V \xrightarrow{p} X$ and $W \xrightarrow{q} Y$ be two real vector bundles. Then there is a homeomorphism*

$$\mathrm{Th}(p \times q) \cong \mathrm{Th}(p) \wedge \mathrm{Th}(q).$$

Proof. We have the relative homemorphism

$$\begin{aligned} (D(p \times q), S(p \times q)) &\longrightarrow \Big(D(p) \times D(q), S(p) \times D(q) \cup D(p) \times S(q)\Big), \\ (v, w) &\longmapsto \frac{1}{\max(\|v\|, \|w\|)\sqrt{\|v\|^2 + \|w\|^2}}(v, w), \end{aligned}$$

which induces the desired homeomorphism on the quotient spaces. □

Corollary 1. *If $p : E \to B$ is a real vector bundle and $\mathbf{1}^r_{\mathbb{R}}$ is the trivial rank-r real vector bundle over B, then there is an isomorphism $\mathrm{Th}(p \oplus \mathbf{1}^r_{\mathbb{R}}) \cong \Sigma^r \mathrm{Th}(p)$.*

Proof. Apply the previous proposition with $X = Y = B$, and notice that the Whitney sum $p \oplus \mathbf{1}^r_{\mathbb{R}}$ is isomorphic to the bundle $p \times \mathbb{R}^r$, where $\mathbb{R}^r$ is the r-dimensional trivial bundle over a point. As $D^r/S^{r-1} \cong S^r$, we get

$$\mathrm{Th}(p \oplus \mathbf{1}^r_{\mathbb{R}}) \cong \mathrm{Th}(p \times \mathbb{R}^r) \cong \mathrm{Th}(p) \wedge S^r = \Sigma^r \mathrm{Th}(p).$$

□

2.2 The construction of MU

Every complex vector bundle of rank r can be regarded as a real vector bundle of rank $2r$, via $\mathbb{C}^r \cong \mathbb{R}^{2r}$. Let $\xi^n : E(n) \to BU(n)$ be the universal complex bundle of rank n.

Let $n \geq 0$. Let $\mathrm{MU}(n)$ denote the Thom space of ξ^n (regarded as a real vector bundle of rank $2n$). The inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$, by adding a zero at the $(n+1)$-th coordinate, induces a map $U(n) \hookrightarrow U(n+1)$. Regarding $BU(n)$ as the Grassmannian and passing through colimits gives a map $\iota_n : BU(n) \to BU(n+1)$. Notice that ξ^{n+1} pulls back to $\xi^n \oplus \mathbf{1}_{\mathbb{C}}$ along ι_n, where $\mathbf{1}_{\mathbb{C}}$ is the trivial complex line bundle:

$$\begin{array}{ccc} E(n) \oplus \mathbb{C} & \longrightarrow & E(n+1) \\ {\scriptstyle \xi^n \oplus \mathbf{1}_{\mathbb{C}} = \iota_n^*(\xi^{n+1})} \downarrow & & \downarrow {\scriptstyle \xi^{n+1}} \\ BU(n) & \xrightarrow{\iota_n} & BU(n+1). \end{array}$$

Functoriality of Thom spaces gives the map :

$$\Sigma^2 \mathrm{MU}(n) = \mathrm{Th}(\xi^n \oplus \mathbf{1}_{\mathbb{C}}) \longrightarrow \mathrm{Th}(\xi^{n+1}) = \mathrm{MU}(n+1).$$

Definition 3. The *complex bordism spectrum*, denoted MU, is the Thom spectrum whose $2n$-th term is $\mathrm{MU}(n)$ and its $(2n+1)$-th term is $\Sigma\mathrm{MU}(n)$. The non-trivial structure maps are given by :

$$\Sigma^2 \mathrm{MU}(n) \longrightarrow \mathrm{MU}(n+1),$$

defined above.

Remark 3. MU also has a geometrical interpretation, in the context of almost complex manifolds ($2n$-dimensional real manifolds with a tangent bundle endomorphism that squares to $-\mathrm{id}$). For M, N almost complex n-manifolds and any space X, two maps $f : M \to X$ and $g : N \to X$ are *bordant* if there exists an almost complex $(n+1)$-manifold W with $\partial W = M \sqcup N$ and for which $f \sqcup g$ extends to a map $W \to X$. Write $\Omega_n^U(X)$ for the bordism classes of n-manifolds into X, which can be made into a graded ring $\Omega_*^U(X)$, called the *complex bordism ring*. Addition is disjoint union and multiplication is cartesian product. Then $\Omega_*^U(*) = \pi_*(\mathrm{MU})$, a result known as the Thom-Pontryagin theorem.

The spectrum MU is a ring spectrum. The direct sum of complex vector bundles is classified by a multiplication

$$BU(n) \times BU(m) \longrightarrow BU(n+m),$$

which is induced by block-addition of matrices. Using functoriality of Thom spaces, we get maps

$$\mathrm{MU}(n) \wedge \mathrm{MU}(m) \longrightarrow \mathrm{MU}(n+m),$$

which induce $\mathrm{MU} \wedge \mathrm{MU} \longrightarrow \mathrm{MU}$. By convention, we can assume $BU(0) \simeq *$, and as the Thom space over a point is a sphere, we get $\mathrm{MU}(0) = S^0$. This induces a map of spectra $\eta : \mathbb{S} \to \mathrm{MU}$, called the unital map.

Let us describe the Thom space $\mathrm{MU}(n)$ when $n \geq 1$. The unit sphere bundle associated to the universal complex bundle ξ^n of rank n can be understood as follows. The $U(n)$-principal bundle $EU(n) \to BU(n)$ shows that $BU(n) = EU(n)/U(n)$. The group $U(n-1)$ acts also on $EU(n)$ via the inclusion $U(n-1) \hookrightarrow U(n)$, so that we get a fibration

$$BU(n-1) \simeq EU(n)/U(n-1) \longrightarrow EU(n)/U(n) \simeq BU(n)$$

with fiber $U(n)/U(n-1) \simeq S^{2n-1}$. This corresponds to the unit sphere bundle of ξ^n. Since the total space equals $E(n) = EU(n) \times_{U(n)} \mathbb{C}^n$, the space of the unit disk bundle of ξ^n is given by $BU(n)$. Thus we get

$$\mathrm{MU}(n) \simeq BU(n)/BU(n-1).$$

In particular, when $n = 1$, we get a map

$$\mathbb{C}P^\infty \simeq BU(1) \xrightarrow{\simeq} BU(1)/BU(0) \simeq \mathrm{MU}(1),$$

which defines an element $x^{\mathrm{MU}} \in \widetilde{\mathrm{MU}}^2(\mathbb{C}P^\infty)$. We have seen that $\mathrm{MU}(0) = S^0$ and $\mathrm{MU}(1) \simeq \mathbb{C}P^\infty$. The map $\Sigma^2\mathrm{MU}(0) \to \mathrm{MU}(1)$ corresponds to the map $i : S^2 \to \mathbb{C}P^\infty$ that we have defined in the beginning of the previous section. Therefore we see that $i^*(x^{\mathrm{MU}}) = 1$, where 1 is the class of $\eta : \mathbb{S} \to \mathrm{MU}$. Thus MU is a complex-orientable cohomology theory.

Proposition 4. *The complex bordism spectrum* MU *is the universal complex-orientable cohomology theory, in the following sense. Let E be a complex-orientable cohomology theory. Given a complex orientation x^E, there exists a unique, up to homotopy, map of ring spectra $f : \mathrm{MU} \to E$ such that*

$$f_*(x^{\mathrm{MU}}) = x^E \quad \text{and} \quad f_*(\mu^{\mathrm{MU}}) = \mu^E.$$

For instance, given a ring map $\mathrm{MU} \to E$, we get a complex orientation x^E by

$$\Sigma^{\infty-2}\mathbb{C}P^\infty \longrightarrow \mathrm{MU} \longrightarrow E,$$

where the left map is the complex orientation of MU.

3 Quillen's theorem

Recall the Lazard ring $L = \mathbb{Z}[t_1, t_2, \ldots]$, where $|t_i| = 2i$ and the universal formal group law $\mu^L \in L[[x, y]]$. Since L is universal (and by the claim of Proposition 4), there is a unique ring homomorphism $h : L \to \pi_*(\mathrm{MU})$ taking μ^L to μ^{MU}. By using logarithms of formal group laws, Quillen was able to prove the following:

Theorem 1. *(Theorem 2 in [Qui69]) The ring homomorphism $h : L \longrightarrow \pi_*(\mathrm{MU})$ is a ring isomorphism.*

Let X be a topological space and $\mathcal{O}_X$ its structure sheaf (where the stalks are local rings). Recall that $(X, \mathcal{O}_X)$ is an *affine scheme* if $X \cong \mathrm{Spec}(R)$ as topological spaces and $\mathcal{O}_X \cong \mathcal{O}_{\mathrm{Spec}(R)}$ as sheaves, for some ring R. Also recall a *groupoid* is a category whose morphisms are all isomorphisms, and combining these ideas, a *groupoid object* in affine schemes is a pair of affine schemes (S, T) (only the space is indicated, the rest of the data being clear for each scheme) for which there exist maps

$$p, q : S \to T$$

(which may be thought of as the left and right unit maps in a coalgebra structure) making the appropriate diagrams commute.

Definition 4. A pair of commutative rings (A, Γ) such that $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$ is a groupoid object in affine schemes is called a *Hopf algebroid*.

Let E be homotopy commutative ring spectrum such that $E_*(E) = \pi_*(E \wedge E)$ is flat as a left $\pi_*(E)$-module. The multiplication map $E \wedge E \to E$ induces a map

$$E \wedge E \wedge E \wedge E \longrightarrow E \wedge E \wedge E,$$

so that we get an isomorphism

$$\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E) \xrightarrow{\cong} \pi_*(E \wedge E \wedge E).$$

Applying π_* to the composite

$$E \wedge E \simeq E \wedge \mathbb{S} \wedge E \longrightarrow E \wedge E \wedge E,$$

induced by the unital map $\mathbb{S} \to E$, we get the comultiplication

$$\Delta : \pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E).$$

The multiplication $E \wedge E \to E$ induces the augmentation map

$$\pi_*(E \wedge E) \longrightarrow \pi_*(E),$$

and the twist map $E \wedge E \to E \wedge E$ induces the antipode

$$\pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E).$$

The left and right counit maps $\pi_*(E) \to \pi_*(E \wedge E)$ are defined via the maps

$$E \simeq \mathbb{S} \wedge E \longrightarrow E \wedge E, \qquad E \simeq E \wedge \mathbb{S} \longrightarrow E \wedge E.$$

In general, these maps are not homotopy equivalent. Applying the functor Spec, we can see that the pair $\Big(\mathrm{Spec}(\pi_*(E)), \mathrm{Spec}(\pi_*(E \wedge E))\Big)$ is a groupoid object in affine schemes, and so we have proved the following fact.

Proposition 5. *Given E a homotopy commutative ring spectrum such that $\pi_*(E \wedge E)$ is flat as a left $\pi_*(E)$-module, we get that the pair $(\pi_*(E), \pi_*(E \wedge E))$ is a Hopf algebroid.*

We are interested in the case $E = \mathrm{MU}$. Let us now understand $\pi_*(\mathrm{MU} \wedge \mathrm{MU})$.

Proposition 6. *Let E be a complex-orientable cohomology theory with complex orientation x. Let $\{b_i\} \subseteq E_*(\mathrm{MU}(1))$ be dual to the topological basis $\{x^{i+1}\}$ of $E^*(\mathrm{MU}(1)) \cong \pi_*(E)[[x]]$. Then the images of the b_i in $E_*(\mathrm{MU})$ determine a ring isomorphism*

$$(\pi_*(E))[b_1, b_2, \dots] \cong E_*(\mathrm{MU}) = \pi_*(\mathrm{MU} \wedge E).$$

Proof. See Proposition 6.2 of [Hop99]. □

Let E be any complex-orientable cohomology theory. Then $\mathrm{MU} \wedge E$ is another complex-orientable cohomology theory. It has two canonical complex orientations: one can define two classes, denoted again x^E and x^{MU} in $\widetilde{MU \wedge E}^2(\mathbb{C}P^\infty)$, represented by the map of spectra

$$\Sigma^{\infty-2}\mathbb{C}P^\infty \longrightarrow E \simeq E \wedge \mathbb{S} \longrightarrow E \wedge \mathrm{MU},$$

and

$$\Sigma^{\infty-2}\mathbb{C}P^\infty \longrightarrow \mathrm{MU} \simeq \mathbb{S} \wedge \mathrm{MU} \longrightarrow E \wedge \mathrm{MU}.$$

In particular we get isomorphisms

$$(\pi_*(E))[b_1, b_2, \dots][[x^E]] \cong \widetilde{E \wedge \mathrm{MU}}^*(\mathbb{C}P^\infty) \cong (\pi_*(E))[b_1, b_2, \dots][[x^{\mathrm{MU}}]].$$

Thus x^{MU} can be written as a power series in $(\pi_*(E))[b_1, b_2, \dots][[x^E]]$. It turns out that

$$x^{\mathrm{MU}} = x^E + b_1(x^E)^2 + b_2(x^E)^3 + \cdots$$

(See Lecture 7 of [Lur10] for a proof). Now if we let R be the commutative ring $\pi_*(\mathrm{MU} \wedge E)$, the complex orientations x^E and x^{MU} give formal group laws μ^E and μ^{MU} on $R[[x, y]]$, which are defined as

$$m^*(x^E) = \mu^E(x_1^E, x_2^E), \qquad m^*(x^{\mathrm{MU}}) = \mu^{\mathrm{MU}}(x_1^{\mathrm{MU}}, x_2^{\mathrm{MU}}).$$

If we set $g(t) = t + b_1t^2 + b_2t^3 + \cdots \in R[[t]]$, we get the formal identity $x^{\mathrm{MU}} = g(x^E)$ and the formula

$$\mu^{\mathrm{MU}}(x, y) = g \circ \mu^E(g^{-1}(x), g^{-1}(y)).$$

Shifting our perspective, define $G = \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \dots])$. Let E be a complex-orientable cohomology theory. Let $R = \pi_*(E)$. Then $\pi_*(E \wedge \mathrm{MU})$ is the ring of functions on the affine scheme $G \times \mathrm{Spec}(\pi_*(E))$. In particular, applying Quillen's theorem we get

$$\mathrm{Spec}(\pi_*(\mathrm{MU} \wedge \mathrm{MU})) \cong G \times \mathrm{Spec}(L).$$

The Hopf algebroid structure of $(\pi_*(\mathrm{MU}), \pi_*(\mathrm{MU} \wedge \mathrm{MU}))$ induces a pair of maps

$$G \times \mathrm{Spec}(L) \longrightarrow \mathrm{Spec}(L)$$

Given a formal group law $f(x, y) \in R[[x, y]]$ and a power series $g(t) = t + b_1t^2 + b_2t^3 + \cdots$, we can naturally construct two formal group laws over R : the first is given by f itself and the second is given by the formula

$$gf(g^{-1}(x), g^{-1}(y)).$$

The group G of coordinate changes acts on $\mathrm{Spec}(L)$, which parametrizes formal group laws. In the language of Algebraic Geometry, we can reformulate our claims as follows.

Theorem 2. *Let E be any spectrum. Then $\mathrm{MU}_*(E)$ is a module over the commutative ring $L \cong \pi_*(\mathrm{MU})$ and therefore, using the functor-of-points perspective, may be understood as a quasi-coherent sheaf on the affine scheme $\mathrm{Spec}(L)$, which parametrizes formal group laws. This sheaf admits an action of the affine group scheme $G = \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \ldots])$, which assigns to each commutative ring R, the group $\{g \in R[[t]] \mid g(t) = t + b_1t^2 + b_2t^3 + \cdots\}$ compatible with the action of G on $\mathrm{Spec}(L)$ by the construction: to each $g \in G(R)$ and $f(x, y)$ a formal group law on R, we get another formal group law*

$$gf(g^{-1}(x), g^{-1}(y))$$

on R.

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