OPERADS, APPROXIMATION, AND RECOGNITION

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All missing details can be found in [May72].

1. Language of operads

Let $\mathcal{S} = (\mathcal{S}, \otimes, \mathbb{I})$ be a (closed) symmetric monoidal category.

Definition 1.1.
An *operad* $\mathcal{O}$ *in* $\mathcal{S}$ is a collection of object $\{\mathcal{O}(j)\}_{j\geq 0}$ in $\mathcal{S}$ endowed with :

- a right-action of the symmetric group Σ_j on $\mathcal{O}(j)$ for each j, such that $\mathcal{O}(0) = \mathbb{I}$;
- a unit map $\mathbb{I} \to \mathcal{O}(1)$ in $\mathcal{S}$;
- composition operations that are morphisms in $\mathcal{S}$:

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \longrightarrow \mathcal{O}(j_1 + \cdots + j_k),$$

  defined for each $k \geq 0$, $j_1, \ldots, j_k \geq 0$, satisfying natural equivariance, unit and associativity relations.

A morphism of operads $\psi : \mathcal{O} \to \mathcal{O}'$ is a sequence $\psi_j : \mathcal{O}(j) \to \mathcal{O}'(j)$ of Σ_j-equivariant morphisms in $\mathcal{S}$ compatible with the unit map and γ.

Example 1.2.
Let X be an object in $\mathcal{S}$. The endomorphism operad $\mathbf{End}_X$ is defined to be $\mathbf{End}_X(j) = \mathrm{Hom}_{\mathcal{S}}(X^{\otimes^j}, X)$, with unit id_X, and the Σ_j-right action is induced by permuting on $X^{\otimes^j}$.

Example 1.3.
Define $\mathbf{Assoc}(j) = \coprod_{\sigma\in\Sigma_j} \mathbb{I}$, the associative operad, where the maps γ are defined by equivariance. Let $\mathbf{Com}(j) = \mathbb{I}$, the commutative operad, where γ are the canonical isomorphisms.

Definition 1.4.
Let $\mathcal{O}$ be an operad in $\mathcal{S}$. An $\mathcal{O}$-*algebra* (X, θ) *in* $\mathcal{S}$ is an object X together with a morphism of operads $\theta : \mathcal{O} \to \mathbf{End}_X$. Using adjoints, this is equivalent to a sequence of maps $\theta_j : \mathcal{O}(j) \otimes X^{\otimes_j} \to X$ such that they are associative, unital and equivariant. A morphism of $\mathcal{O}$-algebras $f : (X, \theta) \to (X', \theta')$ is a morphism $f : X \to X'$ in $\mathcal{S}$ such that the induced morphism on the endomorphism operad is compatible with θ and θ'.

Example 1.5.
An $\mathbf{Assoc}$-algebra is a monoid in $\mathcal{S}$. A $\mathbf{Com}$-algebra is a commutative monoid in $\mathcal{S}$.

2. Operads as monads

Definition 2.1.
A monad (M, μ, η) in $\mathcal{S}$ is a monoid in $(\mathbf{Fun}(\mathcal{S}, \mathcal{S}), \circ, \mathrm{id}_S)$: a functor $M : \mathcal{S} \to \mathcal{S}$ together with natural transformations $\mu : M \circ M \Rightarrow M$ and $\eta : \mathrm{id}_{\mathcal{S}} \Rightarrow M$ respecting associativity and unital properties. A morphism of monads $(M, \mu, \eta) \to (M', \mu', \eta')$ is a morphism of monoids in $\mathbf{Fun}(\mathcal{S}, \mathcal{S})$.

Definition 2.2.
An algebra (X, ξ) over a monad (M, μ, η) is an object X in $\mathcal{S}$ together with a map $\xi : M(X) \to X$ such that

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the following diagrams commute :

$$\begin{array}{ccc} X & \xrightarrow{\eta} & M(X) \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle \xi} \\ & & X, \end{array} \qquad \begin{array}{ccc} M(M(X)) & \xrightarrow{\mu_X} & M(X) \\ \downarrow{\scriptstyle M(\xi)} & & \downarrow{\scriptstyle \xi} \\ M(X) & \xrightarrow{\xi} & X. \end{array}$$

A morphism of M-algebras $(X,\xi) \to (X',\xi')$ is a morphism $f : X \to X'$ in $\mathcal{S}$ such that the following diagram commutes :

$$\begin{array}{ccc} M(X) & \xrightarrow{\xi} & X \\ {\scriptstyle M(f)}\downarrow & & \downarrow{\scriptstyle f} \\ M(X') & \xrightarrow{\xi'} & X'. \end{array}$$

Given an operad $\mathcal{O}$ in $\mathcal{S}$ we define a monad (O,μ,η) in the category $\mathcal{S}_{\mathbb{I}}$ of objects in $\mathcal{S}$ under $\mathbb{I}$ (objects X in $\mathcal{S}$ with a unit morphism $\varepsilon : \mathbb{I} \to X$). Subsequently in next section, we will focus when $\mathcal{S} =$ **Top** with the cartesian product and so objects over the unit is just a choice of a point $* \to X$. For X in $\mathcal{S}_{\mathbb{I}}$, define $O(X)$ to be the coequalizer of :

$$\coprod_{j\geq 0} \mathcal{O}(j) \otimes X^{\otimes^{j-1}} \underset{\coprod \mathrm{id}_{\mathcal{O}(j)} \otimes s_i}{\overset{\coprod(\sigma_i \otimes \mathrm{id}_{X^{\otimes^{j-1}}})}{\rightrightarrows}} \coprod_{j\geq 0} \mathcal{O}(j) \otimes_{\mathbb{I}[\Sigma_j]} X^{\otimes^j},$$

where s_i is the map $\mathrm{id}^{\otimes^{i-1}} \otimes \varepsilon \otimes \mathrm{id}^{\otimes^{j-i}} : X^{\otimes^{j-1}} \to X^{\otimes^j}$ and $\sigma_i : \mathcal{O}(j) \to \mathcal{O}(j-1)$ is induced by :

$$\gamma : \mathcal{O}(j) \otimes \underbrace{(\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1))}_{j-\text{copies with } O(0) \text{ at the } i\text{-th place}} \longrightarrow \mathcal{O}(j-1),$$

for $0 \leq i \leq j$, for each $j \geq 0$. Given a morphism $f : X \to X'$ in $\mathcal{S}_{\mathbb{I}}$, we get a map $O(X) \to O(X')$ induced by the map $f^{\otimes^j} : X^{\otimes^j} \to X'^{\otimes^j}$, such that $O : \mathcal{S}_{\mathbb{I}} \to \mathcal{S}_{\mathbb{I}}$ is a functor. The composition :

$$X \cong \mathbb{I} \otimes X \longrightarrow \mathcal{O}(1) \otimes X \longrightarrow\!\!\!\!\to O(X)$$

defines the natural transformation $\eta : \mathrm{id}_{\mathcal{S}_{\mathbb{I}}} \Rightarrow O$. The natural map $\mu : O(O(X)) \to O(X)$ is defined via γ :

$$\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes X^{\otimes^{j_1}} \otimes \cdots \mathcal{O}(j_k) \otimes X^{\otimes^{j_k}} \underset{\text{shuffle}}{\overset{\cong}{\longrightarrow}} \Big(\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k)\Big) \otimes X^{\otimes^{j_1+\cdots j_k}} \longrightarrow \mathcal{O}(j) \otimes X^{\otimes^j},$$

where $j = j_1 + \ldots + j_k$.

Proposition 2.3.
Given an operad $\mathcal{O}$ in $\mathcal{S}$, an $\mathcal{O}$-algebra (X,θ) satisfying $\varepsilon = \theta_0$ determines and is determined by an O-algebra (X,ξ) in $\mathcal{S}_{\mathbb{I}}$.

Proof : Use adjointness to show that a morphism $\mathcal{O}(j) \otimes X^{\otimes_j} \to X$ with the desired properties induces and is induced by a morphism $\xi : O(X) \to X$ with the desired properties. $\square$

The object $O(X)$ can be regarded as the free O-algebra generated by the object X. Indeed, if we denote by $O[\mathcal{S}]$ the category of O-algebras in $\mathcal{S}$, we get the bijection :

$$\mathrm{Hom}_{\mathcal{S}}(X,Y) \longrightarrow \mathrm{Hom}_{O[\mathcal{S}]}((O(X),\mu),(Y,\xi))$$

for any object X and O-algebra (Y,ξ), i.e., we get the following pair of functors are adjoint :

$$\mathcal{S} \rightleftarrows O[\mathcal{S}]$$

$$X \longmapsto (O(X),\mu)$$

$$Y \longleftarrow\!\shortmid (Y,\xi).$$

3. Little cube operads

We define now an operad on the symmetric monoidal cateogry $(\mathbf{Top}, \times, *)$, where by spaces we mean topological weak Hausdorff k-spaces.

Definition 3.1.
Let J^n be the interior of the n-dimensional unit cube $[0,1]^n$. A *little n-cube* is a rectilinear map $c : J^n \hookrightarrow J^n$. Algebraically, this means the map is of the form :

$$(t_1, \dots, t_n) \longmapsto (a_1 + (b_1 - a_1)t_1, \dots, a_n + (b_n - a_n)t_n),$$

with $(a_1, \dots, a_n), (b_1, \dots, b_n) \in [0,1]^n$ such that $0 \leq a_i \leq b_i \leq 1$, for all $1 \leq i \leq n$. The image of c defines a n-dimensional cube in $[0,1]^n$ with a non-empty interior and faces parallel to the faces of the ambiant unit cube.

Definition 3.2.
The little n-cube operad $\mathcal{C}_n$ is defined as follows :

$$\mathcal{C}_n(j) = \{(c_1, \dots, c_j) \mid c_i \text{ are little } n\text{-cubes with disjoint interior}\} \subseteq \mathrm{Map}\left(\coprod_{i=1}^{j} J^n, J^n\right).$$

The identity is defined by the element $\mathrm{id}_{J^n} \in \mathcal{C}_n(1)$. The symmetric group Σ_j acts (freely) by permutation on the indices of the tuple $(c_1, \dots, c_j)$. If we write $\underline{c} = (c_1, \dots, c_j)$, we define the composition operation γ as follows :

$$\begin{array}{rcl} \gamma : \mathcal{C}_n(k) \times \mathcal{C}_n(j_1) \times \cdots \times \mathcal{C}_n(j_k) & \longrightarrow & \mathcal{C}_n(j_1 + \cdots + j_k) \\ (\underline{c}, \underline{d}_1, \dots, \underline{d}_k) & \longmapsto & \underline{c} \circ (\underline{d}_1 + \cdots + \underline{d}_k). \end{array}$$

Notice that there are natural inclusions :

$$\begin{array}{ccc} \mathcal{C}_n(j) & \hookrightarrow & \mathcal{C}_{n+1}(j) \\ \underline{c} & \longmapsto & (c_1 \times \mathrm{id}_J, \dots, c_j \times \mathrm{id}_J), \end{array}$$

allowing to define $\mathcal{C}_\infty(j) = \mathrm{colim}_n \mathcal{C}_n(j)$ for each $j \geq 0$. The composition γ extends naturally so that $\mathcal{C}_\infty$ is an operad.

We can reinterpate the spaces $\mathcal{C}_n(j)$ in terms of configuration space. Let M be a n-manifold, the j-th configuration space of M is :

$$F(M; j) = \left\{(x_1, \dots, x_j) \in M^{\times j} \mid x_r \neq x_s \text{ if } r \neq s\right\} \subseteq M^{\times j}.$$

It is a nj-manifold with Σ_j free-action on coordinates. For $1 \leq n \leq \infty$, the spaces $\mathcal{C}_n(j)$ are Σ_j-equivariantly homotopic to $F(\mathbb{R}^n; j)$ via the map :

$$\begin{array}{rcl} \mathcal{C}_n(j) & \longrightarrow & F(J^n; j) \\ (c_1, \dots, c_j) & \longmapsto & (c_1(p), \dots, c_j(p)), \end{array}$$

where $p = (\frac{1}{2}, \dots, \frac{1}{2})$ in J^n. This makes $\mathcal{C}_1$ an $\mathcal{A}_\infty$-operad, $\mathcal{C}_\infty$ a $\mathcal{E}_\infty$-operad, $\mathcal{C}_n$ a locally $(n-2)$-connected Σ-free operad.

4. Approximation and recognition theorems

Proposition 4.1.
Given a pointed space X, its n-th iterated loop space $\Omega^n X$ has a natural $\mathcal{C}_n$-algebra structure in $\mathcal{T}$.

Proof : Regard $\Omega^n X$ as the space $\mathrm{Map}\left(\left(\frac{[0,1]^n}{\partial[0,1]^n}, *\right), (X, *)\right)$. Define the action :

$$\theta : \mathcal{C}_n(j) \times (\Omega^n X)^j \longrightarrow \Omega^n X,$$

as follows : given $(c_1, \dots, c_j)$ in $\mathcal{C}_n(j)$ and $(y_1, \dots, y_j)$ in $(\Omega^n X)^j$ define $\theta(\underline{c}, \underline{y})$ as :

$$\begin{array}{ccl} \dfrac{[0,1]^n}{\partial[0,1]^n} & \longrightarrow & X \\ t & \longmapsto & \begin{cases} y_r \circ c_r^{-1}(t), & \text{if } t \in \operatorname{im}(c_r) \\ *, & \text{if } t \notin \operatorname{im}(c_r) \text{ for any } 1 \leq r \leq j \end{cases} \end{array}$$

One can check that all the desired diagrams commute. □

Recall that given a pointed space X, the associated monad of $\mathcal{C}_n$ is defined as :

$$C_n(X) = \left(\coprod_{j \geq 0} \mathcal{C}_n(j) \times_{\Sigma_j} X^j \right) \Big/ \sim .$$

The above result implies that $\Omega^n X$ is also a C_n-algebra, hence there is a map $C_n(\Omega^n X) \to \Omega^n X$, for any pointed space X. There is a natural map :

$$\alpha_n : C_n(X) \longrightarrow \Omega^n \Sigma^n X,$$

defined as follows. The identity map on $\Sigma^n X$ has an adjoint $X \to \Omega^n \Sigma^n X$. Applying the functor C_n we get the left map in the composite :

$$C_n(X) \longrightarrow C_n(\Omega^n \Sigma^n X) \longrightarrow \Omega^n \Sigma^n X,$$

and the right map is defined by the C_n-algebra structure on $\Omega^n \Sigma^n X$. The above composite defines the map α_n. It is a morphism of monads, where the monad structure on the functor $\Omega^n \Sigma^n : \mathbf{Top}_* \to \mathbf{Top}_*$ is defined for any pointed space Y :

$$\Omega^n \Sigma^n \Omega^n \Sigma^n Y \longrightarrow \Omega^n \Sigma^n Y,$$

by a map $\Sigma^n \Omega^n \Sigma^n Y \to \Sigma^n Y$ which is the adjoint of the identity map $\Omega^n \Sigma^n Y \to \Omega^n \Sigma^n Y$. More concretely, the map $\alpha_n : C_n(X) \to \Omega^n \Sigma^n X$ can be regarded as follows :

$$\begin{array}{ccc} C_n(X) & \longrightarrow & \Omega^n \Sigma^n X = \operatorname{Map}\left((\dfrac{[0,1]^n}{\partial[0,1]^n}, *), (\Sigma^n X, *) \right) \\ ((c_1, \dots, c_j), (x_1, \dots, x_j)) & \longmapsto & \left(\begin{array}{ccl} \frac{[0,1]^n}{\partial[0,1]^n} & \longrightarrow & \Sigma^n X \\ t & \longmapsto & \begin{cases} t \in \frac{[0,1]^n}{\partial[0,1]^n} = S^n = \Sigma^n\{*, x_i\}, \text{if } t \in \operatorname{im}(c_i) \subseteq J^n \\ *, \text{ if } t \notin \operatorname{im}(c_i) \text{ for any } 1 \leq i \leq j \end{cases} \end{array} \right). \end{array}$$

Theorem 4.2 (Approximation).
For any based space X, there is a natural map of $\mathcal{C}_n$-algebras :

$$\alpha_n : \mathcal{C}_n(X) \to \Omega^n \Sigma^n X,$$

for $1 \leq n \leq \infty$, and α_n is a weak homotopy equivalence if X is connected.

Proof : We construct the following commutative diagram :

$$\begin{array}{ccccc} C_n(X) & \hookrightarrow & \widetilde{X}_n & \xrightarrow{\widetilde{p}_n} & C_{n-1}(\Sigma X) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^n \Sigma^n X & \hookrightarrow & P\Omega^{n-1}\Sigma^n X & \xrightarrow{p} & \Omega^{n-1}\Sigma^n X, \end{array}$$

where p is the usual path fibration to a space with fiber its loop space. The space $\widetilde{X}_n$ is constructed such that it is contractible and $\widetilde{p}_n$ is a quasifibration if X is connected. □

Theorem 4.3 (Recognition).
If X is a connected $\mathcal{C}_n$-algebra, there exists a based space Y and a weak equivalence of $\mathcal{C}_n$-algebras between $\Omega^n Y$ and X.

In order to construct this delooping of X, we use the two-sided bar construction in $\mathbf{Top}_*$. Given a monad (M, μ, η) in $\mathcal{S}$ and a category $\mathcal{C}$, a M-functor in $\mathcal{C}$ is a functor $F : \mathcal{S} \to \mathcal{C}$ with a natural transformation $\lambda : FM \Rightarrow F$ such that the following diagram commutes :

$$\begin{array}{ccc} F(M(M(X))) & \xrightarrow{F(\mu_X)} & FM(X), \\ \downarrow{\scriptstyle \lambda_{M(X)}} & & \downarrow{\scriptstyle \lambda_X} \\ FM(X) & \xrightarrow{\lambda_X} & F(X), \end{array} \qquad \begin{array}{ccc} F(X) & \xrightarrow{F(\eta_X)} & F(M(X)) \\ & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \lambda_X} \\ & & F(X). \end{array}$$

For instance, (M, μ) is itself a M-functor in $\mathcal{S}$.

Definition 4.4.
Given a monad (M, μ, η) in $\mathcal{S}$, a M-functor (F, λ) in $\mathcal{C}$, and a M-algebra (X, ξ) in $\mathcal{S}$, define the *two-sided bar construction of* (F, M, X) by :

$$B_q(F, M, X) = F(M^q(X)).$$

The object is simplicial in $\mathcal{C}$:

$$F(X) \leftleftarrows F(M(X)) \quad F(M(M(X))) \quad F(M(M(M(X)))) \qquad \cdots$$

where the blue arrows are induced by $\xi : M(X) \to X$, the red arrows by $\lambda : F(M(X)) \to F(X)$, the green arrows by $\mu : M(M(X)) \to M(X)$, and the black arrows by $\eta : X \to M(X)$. We denote its geometric realization by $B(F, M, X) =\mid B_*(F, M, X) \mid$.

Proof : The operad $\mathcal{C}_n$ is replaced by a "nicer" equivalent operad $\mathcal{D}$ so that $B_*(F, D, X)$ is a strictly proper simplicial space. We construct a zig-zag of maps :

$$X \longleftarrow B(D, D, X) \longrightarrow B(\Omega^n \Sigma^n, D, X) \longrightarrow \Omega B(\Sigma^n, D, X).$$

The map $B(D, D, X) \to X$ is induced by $D(X) \to X$ as X is a D-algebra and $B(D, D, X)$ should be regarded as the usual simplicial resolution of X. The map $B(D, D, X) \to B(\Omega^n \Sigma^n, D, X)$ is induced by $\alpha_n : D \to \Omega^n \Sigma^n$ (and should now be regarded as a morphism of D-functors). It is a weak equivalence when X is connected (not obvious on the simplicial resolution). The last map $B(\Omega^n \Sigma^n, D, X) \to \Omega^n B(\Sigma^n, D, X)$ should be regarded as the non-trivial weak equivalence $\mid \Omega X_* \mid \to \Omega \mid X_* \mid$, true only when X is connected. Thus let Y be $B(\Sigma^n, D, X)$. □

References

[May72] J. Peter May. *The Geometry of Iterated loop spaces.* Springer-Verlag, 1972.