1. Language of operads

Let $S = (S, \otimes, I)$ be a (closed) symmetric monoidal category.

**Definition 1.1.**

An operad $O$ in $S$ is a collection of objects $\{O(j)\}_{j \geq 0}$ in $S$ endowed with:

- a right-action of the symmetric group $\Sigma_j$ on $O(j)$ for each $j$, such that $O(0) = I$;
- a unit map $I \rightarrow O(1)$ in $S$;
- composition operations that are morphisms in $S$:
  \[ \gamma : O(k) \otimes O(j_1) \otimes \cdots \otimes O(j_k) \rightarrow O(j_1 + \cdots + j_k), \]
  defined for each $k \geq 0, j_1, \ldots, j_k \geq 0$, satisfying natural equivariance, unit and associativity relations.

A morphism of operads $\psi : O \rightarrow O'$ is a sequence $\psi_j : O(j) \rightarrow O'(j)$ of $\Sigma_j$-equivariant morphisms in $S$ compatible with the unit map and $\gamma$.

**Example 1.2.**

Let $X$ be an object in $S$. The endomorphism operad $\text{End}_X$ is defined to be $\text{End}_X(j) = \text{Hom}_S(X^{\otimes j}, X)$, with unit $\text{id}_X$, and the $\Sigma_j$-right action is induced by permuting on $X^{\otimes j}$.

**Example 1.3.**

Define $\text{Assoc}(j) = \coprod_{\sigma \in \Sigma_j} I$, the associative operad, where the maps $\gamma$ are defined by equivariance. Let $\text{Com}(j) = I$, the commutative operad, where $\gamma$ are the canonical isomorphisms.

**Definition 1.4.**

Let $O$ be an operad in $S$. An $O$-algebra $(X, \theta)$ in $S$ is an object $X$ together with a morphism of operads $\theta : O \rightarrow \text{End}_X$. Using adjoints, this is equivalent to a sequence of maps $\theta_j : O(j) \otimes X^{\otimes j} \rightarrow X$ such that they are associative, unital and equivariant. A morphism of $O$-algebras $f : (X, \theta) \rightarrow (X', \theta')$ is a morphism $f : X \rightarrow X'$ in $S$ such that the induced morphism on the endomorphism operad is compatible with $\theta$ and $\theta'$.

**Example 1.5.**

An $\text{Assoc}$-algebra is a monoid in $S$. A $\text{Com}$-algebra is a commutative monoid in $S$.

2. Operads as monads

**Definition 2.1.**

A monad $(M, \mu, \eta)$ in $S$ is a monoid in $(\text{Fun}(S, S), \circ, \text{id}_S)$: a functor $M : S \rightarrow S$ together with natural transformations $\mu : M \circ M \Rightarrow M$ and $\eta : \text{id}_S \Rightarrow M$ respecting associativity and unital properties. A morphism of monads $(M, \mu, \eta) \rightarrow (M', \mu', \eta')$ is a morphism of monoids in $\text{Fun}(S, S)$.

**Definition 2.2.**

An algebra $(X, \xi)$ over a monad $(M, \mu, \eta)$ is an object $X$ in $S$ together with a map $\xi : M(X) \rightarrow X$ such that
the following diagrams commute:

\[
\begin{array}{cc}
X & M(X) \\
\downarrow \xi & \downarrow \xi \\
X & M(X)
\end{array}
\quad \quad
\begin{array}{cc}
M(M(X)) & M(X) \\
\downarrow M(\xi) & \downarrow \xi \\
M(X) & X
\end{array}
\]

A morphism of $M$-algebras $(X, \xi) \to (X', \xi')$ is a morphism $f : X \to X'$ in $S$ such that the following diagram commutes:

\[
\begin{array}{cc}
M(X) & X \\
\downarrow M(f) & \downarrow f \\
M(X') & X'
\end{array}
\]

Given an operad $\mathcal{O}$ in $S$ we define a monad $(\mathcal{O}, \mu, \eta)$ in the category $\mathcal{S}_1$ of objects in $S$ under $I$ (objects $X$ in $S$ with a unit morphism $\varepsilon : I \to X$). Subsequently in next section, we will focus when $S = \text{Top}$ with the cartesian product and so objects over the unit is just a choice of a point $* \to X$. For $X$ in $\mathcal{S}_1$, define $O(X)$ to be the coequalizer of:

\[
\bigsqcup_{j \geq 0} \mathcal{O}(j) \otimes X^{\otimes j - 1} \cong \bigsqcup_{j \geq 0} \mathcal{O}(j) \otimes [\Sigma_j] X^{\otimes j},
\]

where $s_i$ is the map $\text{id}^{\otimes i - 1} \otimes \varepsilon \otimes \text{id}^{\otimes j-i} : X^{\otimes j - i} \to X^{\otimes j}$ and $\sigma_i : \mathcal{O}(j) \to \mathcal{O}(j - 1)$ is induced by:

\[
\gamma : \mathcal{O}(j) \otimes (\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)) \to \mathcal{O}(j - 1),
\]

for $0 \leq i \leq j$, for each $j \geq 0$. Given a morphism $f : X \to X'$ in $\mathcal{S}_1$, we get a map $O(X) \to O(X')$ induced by the map $f^{\otimes j} : X^{\otimes j} \to X'^{\otimes j}$, such that $O : \mathcal{S}_2 \to \mathcal{S}_1$ is a functor. The composition:

\[
X \cong I \otimes X \longrightarrow \mathcal{O}(1) \otimes X \longrightarrow O(X)
\]

defines the natural transformation $\eta : \text{id}_{\mathcal{S}_1} \Rightarrow O$. The natural map $\mu : O(O(X)) \to O(X)$ is defined via $\gamma$:

\[
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes X^{\otimes j_1 + \cdots + j_k} \cong \text{shuffle} \left( \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \mathcal{O}(j_k) \right) \otimes X^{\otimes j_1 + \cdots + j_k} \to \mathcal{O}(j) \otimes X^{\otimes j},
\]

where $j = j_1 + \cdots + j_k$.

**Proposition 2.3.**
Given an operad $\mathcal{O}$ in $S$, an $\mathcal{O}$-algebra $(X, \theta)$ satisfying $\varepsilon = \theta_0$ determines and is determined by an $\mathcal{O}$-algebra $(X, \xi)$ in $\mathcal{S}_1$.

**Proof:** Use adjointness to show that a morphism $\mathcal{O}(j) \otimes X^{\otimes j} \to X$ with the desired properties induces and is induced by a morphism $\xi : O(X) \to X$ with the desired properties. \qed

The object $O(X)$ can be regarded as the free $\mathcal{O}$-algebra generated by the object $X$. Indeed, if we denote by $O[S]$ the category of $\mathcal{O}$-algebras in $S$, we get the bijection:

\[
\text{Hom}_S(X, Y) \longrightarrow \text{Hom}_O[S]((O(X), \mu), (Y, \xi))
\]

for any object $X$ and $\mathcal{O}$-algebra $(Y, \xi)$, i.e., we get the following pair of functors are adjoint:

\[
\begin{array}{ccc}
S & \longrightarrow & O[S] \\
X & \longmapsto & (O(X), \mu) \\
Y & \longmapsto & (Y, \xi).
\end{array}
\]
3. Little cube operads

We define now an operad on the symmetric monoidal cateogry \((\text{Top}, \times, \ast)\), where by spaces we mean topological weak Hausdorff \(k\)-spaces.

**Definition 3.1.**
Let \(J^n\) be the interior of the \(n\)-dimensional unit cube \([0, 1]^n\). A little \(n\)-cube is a rectilinear map \(c : J^n \to J^n\). Algebraically, this means the map is of the form :
\[
(t_1, \ldots, t_n) \mapsto (a_1 + (b_1 - a_1)t_1, \ldots, a_n + (b_n - a_n)t_n),
\]
with \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0, 1]^n\) such that \(0 \leq a_i \leq b_i \leq 1\), for all \(1 \leq i \leq n\). The image of \(c\) defines a \(n\)-dimensional cube in \([0, 1]^n\) with a non-empty interior and faces parallel to the faces of the ambient unit cube.

**Definition 3.2.**
The little \(n\)-cube operad \(\mathcal{C}_n\) is defined as follows :
\[
\mathcal{C}_n(j) = \{(c_1, \ldots, c_j) \mid c_i \text{ are little } n\text{-cubes with disjoint interior} \} \subseteq \text{Map}\left(\prod_{i=1}^j J^n, J^n\right).
\]
The identity is defined by the element \(\text{id}_{J^n} \in \mathcal{C}_n(1)\). The symmetric group \(\Sigma_j\) acts (freely) by permutation on the indices of the tuple \((c_1, \ldots, c_j)\). If we write \(\zeta = (c_1, \ldots, c_j)\), we define the composition operation \(\gamma\) as follows :
\[
\gamma : \mathcal{C}_n(k) \times \mathcal{C}_n(j_1) \times \cdots \times \mathcal{C}_n(j_k) \to \mathcal{C}_n(j_1 + \cdots + j_k)
\]
\[
(c_1, d_1, \ldots, d_k) \mapsto \zeta \circ (d_1 + \cdots + d_k).
\]

Notice that there are natural inclusions :
\[
\mathcal{C}_n(j) \hookrightarrow \mathcal{C}_{n+1}(j)
\]
and allowing to define \(\mathcal{C}_\infty(j) = \text{colim}_n \mathcal{C}_n(j)\) for each \(j \geq 0\). The composition \(\gamma\) extends naturally so that \(\mathcal{C}_\infty\) is an operad.

We can reinterpret the spaces \(\mathcal{C}_n(j)\) in terms of configuration space. Let \(M\) be a \(n\)-manifold, the \(j\)-th configuration space of \(M\) is :
\[
F(M; j) = \{(x_1, \ldots, x_j) \in M^\times j \mid x_r \neq x_s \text{ if } r \neq s\} \subseteq M^\times j.
\]
It is a \(nj\)-manifold with \(\Sigma_j\) free-action on coordinates. For \(1 \leq n \leq \infty\), the spaces \(\mathcal{C}_n(j)\) are \(\Sigma_j\)-equivariantly homotopic to \(F(\mathbb{R}^n; j)\) via the map :
\[
\mathcal{C}_n(j) \to F(J^n; j)
\]
\[
(c_1, \ldots, c_j) \mapsto (c_1(p), \ldots, c_j(p)),
\]
where \(p = (\frac{1}{j}, \ldots, \frac{1}{j})\) in \(J^n\). This makes \(\mathcal{C}_1\) an \(A_\infty\)-operad, \(\mathcal{C}_\infty\) a \(\mathcal{E}_\infty\)-operad, \(\mathcal{C}_n\) a locally \((n-2)\)-connected \(\Sigma\)-free operad.

4. Approximation and recognition theorems

**Proposition 4.1.**

*Given a pointed space \(X\), its \(n\)-th iterated loop space \(\Omega^n X\) has a natural \(\mathcal{C}_n\)-algebra structure in \(\mathcal{I}\).*

**Proof:** Regard \(\Omega^n X\) as the space \(\text{Map}\left(\left(\prod_{j \in [0,1]^n}, \ast\right), (X, \ast)\right)\). Define the action :
\[
\theta : \mathcal{C}_n(j) \times (\Omega^n X)^j \to \Omega^n X,
\]
We construct the following commutative diagram:

Proof: for \(1 \leq \infty\) define \(\tilde{\text{c}}_n(y)\) as:

\[
\begin{align*}
[0, 1]^n & \to X \\
\frac{\partial[0, 1]^n}{\partial[0, 1]^n} & \to \frac{\partial[0, 1]^n}{\partial[0, 1]^n}
\end{align*}
\]

\[
t \mapsto \begin{cases} 
y_r \circ c_r^{-1}(t), & \text{if } t \in \text{im}(c_r) \\
* & \text{if } t \notin \text{im}(c_r) \text{ for any } 1 \leq r \leq j
\end{cases}
\]

One can check that all the desired diagrams commute.

Recall that given a pointed space \(X\), the associated monad of \(\mathbb{C}_n\) is defined as:

\[
\mathbb{C}_n(X) = \left( \bigcup_{j \geq 0} \mathbb{C}_n(j) \times_{\mathcal{S}_j} X^j \right) / \sim.
\]

The above result implies that \(\Omega^n X\) is also a \(\mathbb{C}_n\)-algebra, hence there is a map \(\mathbb{C}_n(\Omega^n X) \to \Omega^n X\), for any pointed space \(X\). There is a natural map:

\[
\alpha_n : \mathbb{C}_n(X) \to \Omega^n \Sigma^n X,
\]

defined as follows. The identity map on \(\Sigma^n X\) has an adjoint \(X \to \Omega^n \Sigma^n X\). Applying the functor \(\mathbb{C}_n\) we get the left map in the composite:

\[
\mathbb{C}_n(X) \to \mathbb{C}_n(\Omega^n \Sigma^n X) \to \Omega^n \Sigma^n X,
\]

and the right map is defined by the \(\mathbb{C}_n\)-algebra structure on \(\Omega^n \Sigma^n X\). The above composite defines the map \(\alpha_n\). It is a morphism of monads, where the monad structure on the functor \(\Omega^n \Sigma^n : \text{Top}_* \to \text{Top}_*\) is defined for any pointed space \(Y\):

\[
\Omega^n \Sigma^n \Omega^n \Sigma^n Y \to \Omega^n \Sigma^n Y,
\]

by a map \(\Sigma^n \Omega^n \Sigma^n Y \to \Sigma^n Y\) which is the adjoint of the identity map \(\Omega^n \Sigma^n Y \to \Omega^n \Sigma^n Y\). More concretely, the map \(\alpha_n : \mathbb{C}_n(X) \to \Omega^n \Sigma^n X\) can be regarded as follows:

\[
\begin{align*}
\mathbb{C}_n(X) & \to \Omega^n \Sigma^n X = \text{Map}\left(\left(\frac{[0, 1]^n}{\partial[0, 1]^n}, *, (\Sigma^n X, *)\right) \to \Sigma^n X, \begin{cases} t \in \frac{[0, 1]^n}{\partial[0, 1]^n} = S^n = \Sigma^n \{*, x_i\}, & \text{if } t \in \text{im}(c_i) \subseteq J^n \\
* & \text{if } t \notin \text{im}(c_i) \text{ for any } 1 \leq i \leq j \end{cases}\right)
\end{align*}
\]

**Theorem 4.2 (Approximation).**

*For any based space \(X\), there is a natural map of \(\mathbb{C}_n\)-algebras:

\[
\alpha_n : \mathbb{C}_n(X) \to \Omega^n \Sigma^n X,
\]

for \(1 \leq n \leq \infty\), and \(\alpha_n\) is a weak homotopy equivalence if \(X\) is connected.*

**Proof:** We construct the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}_n(X) & \xrightarrow{\tilde{\text{X}}_n} & \mathbb{C}_{n-1}(\Sigma X) \\
\mathbb{C}_n(X) & \xrightarrow{\tilde{\text{p}}_n} & \mathbb{C}_{n-1}(\Sigma X) \\
\Omega^n \Sigma^n X & \xrightarrow{\text{p}} & \Omega^{n-1} \Sigma^n X,
\end{array}
\]

where \(p\) is the usual path fibration to a space with fiber its loop space. The space \(\tilde{\text{X}}_n\) is constructed such that it is contractible and \(\tilde{p}_n\) is a quasifibration if \(X\) is connected.

**Theorem 4.3 (Recognition).**

*If \(X\) is a connected \(\mathbb{C}_n\)-algebra, there exists a based space \(Y\) and a weak equivalence of \(\mathbb{C}_n\)-algebras between \(\Omega^n Y\) and \(X\).*
In order to construct this delooping of $X$, we use the two-sided bar construction in $\text{Top}_\ast$. Given a monad $(M, \mu, \eta)$ in $\mathcal{S}$ and a category $\mathcal{C}$, a $M$-functor in $\mathcal{C}$ is a functor $F : \mathcal{S} \to \mathcal{C}$ with a natural transformation $\lambda : FM \Rightarrow F$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F(M(M(X))) & \xrightarrow{F(\mu_X)} & FM(X) \\
\downarrow{\lambda_{M(X)}} & & \downarrow{\lambda_X} \\
FM(X) & \xrightarrow{\lambda_X} & F(X),
\end{array}
$$

For instance, $(M, \mu)$ is itself a $M$-functor in $\mathcal{S}$.

**Definition 4.4.**

Given a monad $(M, \mu, \eta)$ in $\mathcal{S}$, a $M$-functor $(F, \lambda)$ in $\mathcal{C}$, and a $M$-algebra $(X, \xi)$ in $\mathcal{S}$, define the two-sided bar construction of $(F, M, X)$ by:

$$B_q(F, M, X) = F(M(M(X))).$$

The object is simplicial in $\mathcal{C}$:

$$F(X) \xleftarrow{} F(M(X)) \xleftarrow{} F(M(M(X))) \xleftarrow{} F(M(M(M(X)))) \cdots$$

where the blue arrows are induced by $\xi : M(X) \to X$, the red arrows by $\lambda : F(M(X)) \to F(X)$, the green arrows by $\mu : M(M(X)) \to M(X)$, and the black arrows by $\eta : X \to M(X)$. We denote its geometric realization by $B(F, M, X) = |B_q(F, M, X)|$.

**Proof:** The operad $\mathcal{C}_n$ is replaced by a "nicer" equivalent operad $\mathcal{D}$ so that $B_*(F, D, X)$ is a strictly proper simplicial space. We construct a zig-zag of maps:

$$X \xleftarrow{} B(D, D, X) \xrightarrow{} B(\Omega^n \Sigma^n, D, X) \xrightarrow{} \Omega B(\Sigma^n, D, X).$$

The map $B(D, D, X) \to X$ is induced by $D(X) \to X$ as $X$ is a $D$-algebra and $B(D, D, X)$ should be regarded as the usual simplicial resolution of $X$. The map $B(D, D, X) \to B(\Omega^n \Sigma^n, D, X)$ is induced by $\alpha_n : D \to \Omega^n \Sigma^n$ (and should now be regarded as a morphism of $D$-functors). It is a weak equivalence when $X$ is connected (not obvious on the simplicial resolution). The last map $B(\Omega^n \Sigma^n, D, X) \to \Omega^n B(\Sigma^n, D, X)$ should be regarded as the non-trivial weak equivalence $|\Omega^a X_\ast| \to |\Omega| |X_\ast|$, true only when $X$ is connected.

Thus let $Y$ be $B(\Sigma^n, D, X)$.

**References**