École Polytechnique Fédérale de Lausanne
Mathematics Section

ÉCOLE POLYTECHNIQUE
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# AN INTRODUCTION TO STABLE HOMOTOPY THEORY

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Academic year : 2013-2014
Fall Semester

**Abstract**

The aim of this introduction to stable homotopy theory is to present the construction of generalized homology and cohomology theories, using only homotopy-theoretical methods. We show how (co)homology theory is to a large extent a branch of stable homotopy theory.

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# Introduction

In algebraic topology, we define algebraic invariants on topological spaces in order to convert topological problems into algebraic problems. The first examples of invariants stem from homotopy theory and are the *homotopy groups* $\pi_n(X, x_0)$ of a topological based space $(X, x_0)$. The idea of homotopy groups is intuitively simple : we study the shape of the topological spaces by investigating their relations with the simplest topological spaces, the $n$-spheres. However the task is tremendous : even these relations between the $n$-spheres themselves are not known completely nowadays. Fortunately, other algebraic invariants were invented, such as *homology and cohomology groups* of a space. Their definitions are much more complicated than homotopy groups, as they involve a mixture between geometric interpretations of topological spaces, and algebraic methods to output abelian groups. However their computations are easier than homotopy groups.

The founding result of stable homotopy theory is the Freudenthal suspension theorem which states that homotopy groups eventually become isomorphic after sufficiently many iterated suspensions. This lead to the notion of *stable homotopy groups* of a topological space. The aim of this paper is to explain how every homology and cohomology theory can actually be constructed using only homotopy-theoretical methods. We will show that homology and cohomology theories are strongly related to stable objects called *prespectra*.

In Chapter 1, we establish fundamental theorems needed subsequently. We will give a complete homotopic proof of the Freudenthal suspension theorem, as a corollary of the Blakers-Massey theorem, which specifies when excision holds for homotopy groups. We also prove the Hurewicz theorem, which relates homotopy groups and integer homology groups under connectivity assumptions. In Chapter 2, we construct the Eilenberg-MacLane spaces, fundamental topological spaces that will lead to the homotopic construction of ordinary homology and cohomology theories. We will prove that stable homotopy groups define a generalized homology theory and how this result will let us define generalized homology and cohomology theories from prespectra. In Chapter 3, we prove the converse for cohomology theories : every generalized cohomology theory is associated to a particular kind of prepsectra. This will follow from the Brown representability theorem. We also mention briefly the case of generalized homology theories.
In Appendix A, we give the background material needed for the paper. Basic results are recalled and proved. We have included the axiomatic approach of Eilenberg-Steenrod to homology and cohomology theories in Appendix B, as it is needed throughout this paper. We will set our terminology of a generalized (co)homology theory. We will also prove that it is always sufficient to define reduced (co)homology theories on based CW-complexes. In Appendix C, we establish in detail the right categorical language needed for the stable homotopy theory.

# Conventions

Throughout this paper, we will make the following conventions.

- A *space* is a topological space. A *map* is a continuous map.
- Homeomorphism and isomorphism are denoted by the symbol $\cong$. Homotopy equivalence is denoted by $\simeq$, and based homotopy equivalence is denoted by $\simeq_*$.
- If a map $f : X \to Y$ is a weak equivalence, we add the symbol $\sim$ over the map : $f : X \xrightarrow{\sim} Y$.
- We often omit the basepoint when we consider a based space.
- We denote $I = [0, 1] \subseteq \mathbb{R}$ the unit interval endowed with the subspace topology.
- The $(n + 1)$-disk will be denoted $D^{n+1}$ and its boundary, the $n$-sphere, is denoted $S^n$.
- The space $Y^X$ of maps from the spaces $X$ to $Y$ endowed with the compact-open topology is denoted $\mathrm{Map}(X, Y)$.
- The homotopy class of maps from a pair of based spaces $(X, A)$ to a pair $(Y, B)$ is denoted : $[(X, A), (Y, B)]_*$. The homotopy class of a map $f : X \to Y$ is denoted by $[f]$. The based homotopy class is denoted by $[f]_*$.
- For any based space $X$, the reduced cone is denoted $CX$, its based loopspace is denoted $\Omega X$ and its reduced suspension is denoted $\Sigma X$.

# CHAPTER 1

# The Underlying Theorems

We begin this paper by presenting three main results in algebraic topology. These will be fundamental in our work in next chapters. We will rely on Appendix A. Our summary is based on [May, 1999] and [Switzer, 1975].

## 1.1. The Blakers-Massey Theorem (Homotopy Excision)

The major difficulty in computing homotopy groups is due to the failure of the excision axiom. If one has an *excisive triad* $(X; A, B)$, that is spaces $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$, then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ does not necessarly induce an isomorphism of homotopy groups in general. However, the statement is true in some dimensions, and this is called the Blakers-Massey theorem.
Recall that a map $f : X \to Y$ is an *$n$-equivalence* if $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$, and a surjection for $k = n$, when $n \geq 0$. For instance, the inclusion $S^n \hookrightarrow D^{n+1}$ is a $n$-equivalence. One can define the following relative case.

Definition 1.1.1.
A map $f : (A, C) \to (X, B)$ of pairs is an *$n$-equivalence*, where $n \geq 1$, if :

$$f_*^{-1}(\text{im } (\pi_0 B \to \pi_0 X)) = \text{im } (\pi_0 C \to \pi_0 A),$$

which holds automatically when $A$ and $X$ are path-connected, and if for all choices of basepoints in $C$, the map $f_* : \pi_k(A, C) \to \pi_k(X, B)$ is a bijection for $k < n$, and a surjection for $k = n$.

Definition 1.1.2.
A space $X$ is said to be *$n$-connected* if $\pi_k(X, x) = 0$, for $0 \leq k \leq n$, and all points $x$ in $X$. A pair $(X, A)$ is said to be *$n$-connected* if the map $\pi_0(A) \to \pi_0(X)$ induced by the inclusion is surjective, and $\pi_k(X, A, a) = 0$ for $1 \leq k \leq n$, and all $a$ in $A$. This is equivalent[1] to saying that the inclusion $A \hookrightarrow X$ is a $n$-equivalence.

Hence, a path-connected space is 0-connected and a simply connected space is 1-connected. The $n$-sphere $S^n$ is $(n-1)$-connected, for all $n \geq 1$.

Theorem 1.1.3 (Homotopy excision, Blakers-Massey).
*Let $(X; A, B)$ be an excisive triad and let $C = A \cap B$. Assume that $(A, C)$ is $(m-1)$-connected*

[1] Use the long exact sequence of the pair $(X, A)$.

*and $(B,C)$ is $(n-1)$-connected, where $m \geq 2$ and $n \geq 1$. Then the inclusion $(A,C) \hookrightarrow (X,B)$ is an $(m+n-2)$-equivalence.*

To prove the theorem, we must ask ourselves when the inclusion $(A, A\cap B) \hookrightarrow (X,B)$ induces isomorphisms in relative homotopy groups. One way to tackle this problem is to embed the induced map from the inclusion in a long exact sequence, and turn the question into one about the vanishing of certain groups.

Triad Homotopy Groups Recall that for any based map $i : A \to X$, the *homotopy fiber* of $i$, denoted $Fi$, is defined by the following pullback :

$$\begin{array}{ccc} Fi & \longrightarrow & PX \\ \downarrow & & \downarrow \\ A & \xrightarrow{i} & X, \end{array}$$

where $PX \subseteq \mathrm{Map}(I,X)$ is the based path-space of $X$, that is the space of all paths in $X$ that start at the basepoint; and where the right vertical map sends any path to its endpoint. In the particular case where $i : A \hookrightarrow X$ is an inclusion, we get isomorphisms $\pi_n(X,A) \cong \pi_{n-1}(Fi)$, for $n \geq 1$, where we endowed $PX$ with the basepoint the constant path $c_*$ at basepoint $*$ of $A \subseteq X$. It is a bijection for the case $n=1$. One can easily see this by using the exponential law.
We now generalize this argument. Recall that a *triad* $(X;A,B,*)$ is a space $X$ together with two subspaces $A$ and $B$ and basepoint $* \in A\cap B$, such that $A\cup B = X$. Let us name $i : B \hookrightarrow X$ and $j : A\cap B \hookrightarrow A$ the inclusions. Since $A \cap B \subseteq B$ and $PA \subseteq PX$ are subspaces, it follows that $Fj$ is a subspace of $Fi$. For $n \geq 2$, the *n-th triad homotopy group* of $(X;A,B,*)$ is defined by :

$$\begin{array}{rcl} \pi_n(X;A,B,*) & := & \pi_{n-1}(Fi, Fj, (*, c_*)), \\ & := & [(I^{n-1}, \partial I^{n-1}, J^{n-2}), (Fi, Fj, (*, c_*))], \end{array}$$

where $J^{n-2} = \partial I^{n-2} \times I \cup I^{n-2} \times \{0\} \subseteq I^{n-1}$, as usual.

Lemma 1.1.4.
*The n-th triad homotopy group of $(X;A,B,*)$ can be regarded as the set of homotopy classes of maps of tetrads :*

$$(I^n; I^{n-2} \times \{1\} \times I, I^{n-1} \times \{1\}, J^{n-2} \times I \cup I^{n-1} \times \{0\}) \longrightarrow (X;A,B,*).$$

Sketch of the Proof : Name $S$ this set of homotopy classes of maps of tetrads. We construct a bijection between $\pi_{n-1}(Fi,Fj,(*,c_*))$ and $S$. Take a map $f : (I^{n-1}, \partial I^{n-1}, J^{n-2}) \to (Fi, Fj, (*, c_*))$. Let $p : Fi \to PX$ be the projection. Since $I$ is locally compact and Hausdorff, $f$ admits an adjoint $\widehat{f} : I^n = I^{n-1} \times I \longrightarrow X$, where $\widehat{f}(x,t) = p(f(x))(t)$, for any $x$ in $I^{n-1}$ and $t$ in $I$. It is easy to see that $\widehat{f}$ is in $S$.
Conversely, suppose $f$ is in $S$. Let the map $\widetilde{f} : I^{n-1} \to PX$ be its adjoint. Define the map : $\widehat{f} :\ I^{n-1} \to Fi$, by $\widehat{f}(x) = (f(x,1), \widetilde{f}(x))$, for any $x$ in $I^{n-1}$. One can easily check that $[\widehat{f}]_*$ is in $\pi_{n-1}(Fi,Fj,(*,c_*))$. □

From the long exact of the sequence of the pair $(Fi, Fj)$, we get the following long exact sequence :

$$\cdots \longrightarrow \pi_{n+1}(X;A,B) \longrightarrow \pi_n(A, A\cap B) \longrightarrow \pi_n(X,B) \longrightarrow \pi_n(X;A,B) \longrightarrow \cdots.$$

We recognize the homomorphism $\pi_n(A, A \cap B) \to \pi_n(X, B)$ induced by the inclusion $(A, A \cap B) \hookrightarrow (X, B)$, which is the map mentionned in homotopy excision Theorem. Therefore the triad homotopy groups may be considered to measure the amount by which the relative homotopy groups fail to satisfy the excision axiom. So with the conditions of theorem 1.1.3, the goal is to prove that $\pi_k(X; A, B) = 0$, for all $2 \leq k \leq m + n - 2$.

Before proving homotopy excision, we mention another long exact sequence that will be helpful subsequently.

Proposition 1.1.5 (The Exact Sequence of a Triple).
*For a triple $(X, A, B)$, that is spaces $B \subseteq A \subseteq X$, and any basepoint in $B$, the following sequence is exact :*

$$\cdots \longrightarrow \pi_k(A, B) \xrightarrow{i_*} \pi_k(X, B) \xrightarrow{j_*} \pi_k(X, A) \xrightarrow{k_* \circ \partial} \pi_{k-1}(A, B) \longrightarrow \cdots,$$

*where $i : (A, B) \hookrightarrow (X, B)$, $j : (X, B) \hookrightarrow (X, A)$, and $k : (A, *) \hookrightarrow (A, B)$ are the inclusions.*

Sketch of the Proof : The proof consists of a diagram chase of the following diagram, where the rows are the long exact sequences of the pairs $(A, B)$, $(X, B)$ and $(X, A)$, and the unlabeled vertical maps are the induced homomorphisms of the inclusions $A \hookrightarrow X$ and $B \hookrightarrow A$.

$$\begin{tikzcd}
\pi_k(B) \arrow[r] \arrow[d, equal] & \pi_k(A) \arrow[r] \arrow[d] & \pi_k(A,B) \arrow[r] \arrow[d, dashed, "i_*"] & \pi_{k-1}(B) \arrow[r] \arrow[d, equal] & \pi_{k-1}(A) \arrow[d] \\
\pi_k(B) \arrow[r] \arrow[d] & \pi_k(X) \arrow[r] \arrow[d, equal] & \pi_k(X,B) \arrow[r] \arrow[d, dashed, "j_*"] & \pi_{k-1}(B) \arrow[r] \arrow[d] & \pi_{k-1}(X) \arrow[d, equal] \\
\pi_k(A) \arrow[r] & \pi_k(X) \arrow[r] & \pi_k(X,A) \arrow[r, "\partial"] \arrow[dr, dashed, "k_* \circ \partial"'] & \pi_{k-1}(A) \arrow[r] \arrow[d, "k_*"] & \pi_{k-1}(X) \arrow[d] \\
 & & & \pi_{k-1}(A,B) \arrow[r, dashed, "i_*"] & \pi_{k-1}(X,B).
\end{tikzcd}$$

One must prove exactness for the dashed maps, which is a purely algebraic argument. □

We now start proving the Blakers-Massey Theorem. We begin with a special case, where all the hard work is contained. The simplicial approximation Theorem proved in Appendix A is at the heart of the proof.

Lemma 1.1.6.
*Let $C$ be any Hausdorff space, and suppose that $A$ is obtained from $C$ by attaching a $m$-cell $e^m$, where $m \geq 2$, $B$ is obtained from $C$ by attaching a $n$-cell $e^n$, where $n \geq 1$, and $X = A \cup B$, forming a triad $(X; A, B)$. Then $\pi_k(X; A, B) = 0$, for all $2 \leq k \leq n + m - 2$.*

Proof : Take $[f]_*$ in $\pi_k(X; A, B)$, where $2 \leq k \leq n + m - 2$. To see that $[f]_* = 0$, we will show that $[f]_*$ can be seen as an element of $\pi_k(A; A, A - \{q\})$ for some point $q$ in $A$, since $\pi_k(A; A, A - \{q\}) = 0$ by definition. This will stem from the fact $A$ is a strong deformation retract of $X - \{p\}$ for some point $p$ in $\mathring{e}^n \subseteq B$, and $B$ is a strong deformation retract of $X - \{q\}$ for some point $q$ in $\mathring{e}^m \subseteq A$ (using Proposition A.1.2, page 51). We choose the points $p$ and $q$ such that $f$ can be seen as map whose image lies in $(X; A, X - \{p, q\})$. In other words, naming the inclusions $j_1 : (X; A, B) \hookrightarrow (X; A, X - \{q\})$, $j_2 : (X - \{p\}; A, X - \{p, q\}) \hookrightarrow (X; A, X - \{q\})$ and $j_3 : (A; A, A - \{q\}) \hookrightarrow (X - \{p\}; A, X - \{p, q\})$, we get :

$$\pi_k(A; A, A - \{q\}) \xrightarrow{j_{3*}} \pi_k(X - \{p\}; A, X - \{p, q\}) \xrightarrow{j_{2*}} \pi_k(X; A, X - \{q\}) \xleftarrow{j_{1*}} \pi_k(X; A, B),$$

such that $j_{1*}([f]_*)$ is in the image of $j_{2*}$. Since $B$ is a strong deformation retract of $X - \{q\}$, $j_{1*}$ is an isomorphism. Similarly, $A$ is a strong deformation retract of $X - \{p\}$, so $j_{3*}$ is an isomorphism. It follows that $[f]_* = 0$.
In order to find points $p$ and $q$ with such properties, we first replace the map $f$ by a homotopic map $g$. Indeed, by the simplicial approximation Theorem (Theorem A.2.6, page 53), $I^k$ can be triangulated so finely that $f$ is homotopic to a map $g$, where for any simplex $\sigma$ :

$$\begin{cases} g(|\sigma|) \cap e^n_{1/4} \neq \emptyset & \Rightarrow g(|\sigma|) \subseteq \mathring{e}^n_{1/4}, \\ g(|\sigma|) \cap e^m_{1/4} \neq \emptyset & \Rightarrow g(|\sigma|) \subseteq \mathring{e}^m_{1/4}, \end{cases}$$

and $g$ is affine on each $|\sigma|$.

Finding $p$ : For any simplex $\sigma$ in the fine triangulation of $I^k$, if $g(|\sigma|)$ meets $e^n_{1/4}$, then $g(|\sigma|)$ is a convex set, with dimension less or equal to $n$. Define the sets : $C_1 := \{\sigma \in I^k \mid g(|\sigma|) \cap e^n_{1/4} \neq \emptyset, \dim(g(|\sigma|)) < n\}$, and $C_2 := \{\sigma \in I^k \mid g(|\sigma|) \cap e^n_{1/4} \neq \emptyset, \dim(g(|\sigma|)) = n\}$. Then the set $\bigcup_{\sigma \in C_1} g(|\sigma|)$ does not cover $e^n_{1/4}$, so there exists a point $p$ in $e^n_{1/4}$ such that if $p$ is in $g(|\sigma|)$ for some simplex $\sigma$, then $\sigma$ is in $C_2$.

Finding $q$ : The compact space $g^{-1}(p)$ lies in the fine triangulation of $I^k$. It is thus a finite union of polyhedra $P_1, \ldots, P_r$. Name $L_1, \ldots, L_r : \mathbb{R}^k \to \mathbb{R}^n$ the linear surjections, such that $L_{i|_{P_i}} = g_{|_{P_i}}$, for $i = 1, \ldots, r$. Each polyhedron $P_i$ is of dimension at most $k$ since they are contained in $I^k$. Since each $L_i$ is surjective, the kernel $\ker(L_i)$ has dimension at most $k - n$, by the rank theorem. Therefore the space $g^{-1}(p)$ is a polyhedron in $I^k$ of dimension at most $k - n$. Let us denote $\pi : I^k = I^{k-1} \times I \to I^{k-1}$ the projection onto the first factor. Define the space $K := \pi^{-1}(\pi(g^{-1}(p)))$. It is a polyhedron of dimension at most one more than the dimension of $g^{-1}(p)$, i.e., its dimension is at most :

$$k - n + 1 \leq (n + m - 2) - n + 1 \leq m - 1.$$

Thus, $g(|K|)$ does not cover $e^m_{1/4}$, so there exists a point $q$ in $e^m_{1/4}$ such that $q \notin g(|K|)$.

Proving $j_{1*}([g]_*) \in \mathrm{im}(j_{2*})$ : Since $g(\partial I^{k-1} \times I) \subseteq A$, the spaces $\pi(g^{-1}(p))$ and $\pi(g^{-1}(q)) \cup \partial I^{k-1}$ are disjoint closed subset of the normal space $I^{k-1}$. Thus by Uryssohn's Lemma, there is a map $\zeta : I^{k-1} \to I$ such that $\zeta(\pi(g^{-1}(q)) \cup \partial I^{k-1}) = 0$ and $\zeta(\pi(g^{-1}(p))) = 1$. Define the map :

$$\begin{aligned} h : (I^{k-1} \times I) \times I &\longrightarrow I^{k-1} \times I \\ (x, s, t) &\longmapsto (x, s(1 - t\zeta(x))), \end{aligned}$$

which is obviously continuous. We get $h(x, s, 0) = (x, s)$ for any $x$ in $I^{k-1}$ and $s$ in $I$, and $h(I^{k-1} \times I \times \{1\}) \subseteq I^k - g^{-1}(p)$. Moreover, $h(I^{k-1} \times \{1\} \times I) \subseteq I^k - g^{-1}(q)$, and $h(x, s, t) = (x, s)$, if $x$ is in $\partial I^{k-1}$. The composite $j_1 \circ g \circ h$ is therefore a homotopy from $j_1 \circ g$ to a map $f'$ whose image is in $(X - \{p\}; A, X - \{p, q\})$. We have just proved that $j_{1*}([g]) = j_{2*}([f'])$.

It follows that $[g]_* = [f]_* = 0$. □

We can now give the proof of homotopy excision (Blakers-Massey). The goal is to reduce any excisive triad to the case of the previous lemma.

Proof of Homotopy Excision (Theorem 1.1.3) : We want to prove $\pi_k(X; A, B) = 0$ for all $2 \leq k \leq n+m-2$. By theorem A.3.11, the excisive triad $(X; A, B)$ may be approximated by a weakly equivalent CW-triad $(\tilde{X}; \tilde{A}, \tilde{B})$, where for the sake of clarity the CW-triad shall be renamed $(X; A, B)$, and such that $(A, C)$ has no relative $k$-cells for $k < m$, and $(B, C)$ has no relative $k$-cells for $k < n$. Since $I^k$ is compact, any map $I^k \to X$ has its image contained in a finite subcomplex of $X$, and so we may assume that $X$ has finitely many cells. We now prove that inductively, it suffices to show the result when $(A, C)$ has exactly one cell. Suppose that $C \subseteq A' \subseteq A$ where $A$ is obtained from $A'$ by attaching a single cell, and $(A', C)$ has one less cell than $(A, C)$. Define $X'$ as the pushout :

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & A' \arrow[d] \\ B \arrow[r] & X'. \end{tikzcd}$$

If the result holds for the triads $(X'; A', B)$ and $(X; A, X')$ by induction, then by the 5-Lemma, it holds also for the triad $(X; A, B)$, since there is a commutative diagram :

$$\begin{tikzcd} \pi_{k+1}(A, A') \arrow[r] \arrow[d, "\cong"] & \pi_k(A', C) \arrow[r] \arrow[d, "\cong"] & \pi_k(A, C) \arrow[r] \arrow[d] & \pi_k(A, A') \arrow[r] \arrow[d, "\cong"] & \pi_{k-1}(A', C) \arrow[d, "\cong"] \\ \pi_{k+1}(X, X') \arrow[r] & \pi_k(X', B) \arrow[r] & \pi_k(X, B) \arrow[r] & \pi_k(X, X') \arrow[r] & \pi_{k-1}(X', B), \end{tikzcd}$$

where the rows are the exact sequences of the triples $(A, A', C)$ and $(X, X', B)$, by Proposition 1.1.5. For the case $k = 2$, one must be careful, since we are dealing with sets instead of groups. But the result also holds since the pairs of the diagram are all 1-connected, whence one can argue similarly as in the proof of the 5-Lemma. So one can assume that $(A, C)$ has exactly one cell. We can also assume that $(B, C)$ has exactly one cell. Indeed, by induction, suppose that $C \subseteq B' \subseteq B$, where $B$ is obtained from $B'$ by attaching a single cell and $(B', C)$ has one less cell than $(B, C)$. Define again $X'$ as the following pushout :

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & B' \arrow[d] \\ B \arrow[r] & X'. \end{tikzcd}$$

If the result holds for the triads $(X'; A, B)$ and $(X; X', B)$ by induction, then it also holds for $(X; A, B)$, since the inclusion $(A, C) \hookrightarrow (X, B)$ fits into the following commutative diagram :

$$\begin{tikzcd} (A, C) \arrow[rr, hook] \arrow[dr, hook] & & (X, B) \\ & (X', B). \arrow[ur, hook] & \end{tikzcd}$$

Thus we may assume that the triad $(X; A, B)$ is as Lemma 1.1.6, which ends the proof. □

## 1.2. The Freudenthal Suspension Theorem and Stable Homotopy Groups

A useful result for homotopy groups is the isomorphism $\pi_k(X) \cong \pi_{k-1}(\Omega X)$, for $k \geq 1$, which links any based space and its based loop space. However, what about the dual case? In other

words, what is the relationship between a space $X$ and its reduced suspension $\Sigma X$, in terms of homotopy groups?
The Freudenthal suspension Theorem is a fundamental result in homotopy theory that will answer these questions. It leads to the notion of stable homotopy groups.

Definition 1.2.1 (The Suspension Homomorphism).
For a based space $X$, an integer $k \geq 0$, define the *suspension homomorphism* by :

$$\begin{array}{rcl} \Sigma : \pi_k(X) & \longrightarrow & \pi_{k+1}(\Sigma X) \\ [f]_* & \longmapsto & [f \wedge \mathrm{id}_{S^1}]_*, \end{array}$$

where for any continuous based map $f : S^k \to X$, the map $\Sigma f := f \wedge \mathrm{id}_{S^1} : S^{k+1} \cong \Sigma S^k \to \Sigma X$ sends any $x \wedge t$ to $f(x) \wedge t$.

The suspension homomorphism is of course a well-defined map, and it is really an homomorphism for $k \geq 1$. An alternative way to introduce the map is to consider the loopspace of a suspension. Namely, define a map $X \to \Omega(\Sigma X)$ which sends any $x$ in $X$, to the loop $t \mapsto x \wedge t$. The induced homomorphism $\pi_k(X) \to \pi_k(\Omega\Sigma X) \cong \pi_{k+1}(\Sigma X)$ *is* the suspension homomorphism.

Lemma 1.2.2.
*The suspension homomorphism is a natural transformation $\Sigma : \pi_n \Rightarrow \pi_{n+1}$, for any $n \geq 0$.*

Proof : We must prove that the following diagram commutes, for any map $f : X \to Y$ :

$$\begin{tikzcd} \pi_n(X) \arrow[r, "\Sigma"] \arrow[d, "f_*"'] & \pi_{n+1}(\Sigma X) \arrow[d, "(\Sigma f)_*"] \\ \pi_n(Y) \arrow[r, "\Sigma"'] & \pi_{n+1}(\Sigma Y). \end{tikzcd}$$

For any $[g]_*$ in $\pi_n(X)$, we have :

$$\begin{array}{rcl} ((\Sigma f)_* \circ \Sigma)([g]_*) & = & (\Sigma f)_*([g \wedge \mathrm{id}_{S^1}]_*) \\ & = & [(f \circ g) \wedge \mathrm{id}_{S^1}]_* \\ & = & \Sigma([f \circ g]_*) \\ & = & (\Sigma \circ f_*)([g]_*), \end{array}$$

which demonstrates the commutativity of the diagram. □

We now establish the Freudenthal suspension theorem which states that the suspension homomorphism is an isomorphism, in a range of dimensions. It can be proved as a corollary of homotopy excision (Theorem 1.1.3).

Theorem 1.2.3 (Freudenthal Suspension).
*Let $X$ be an $(n-1)$-connected space, where $n \geq 1$. Then the suspension homomorphism $\Sigma$ : $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is a bijection for $k < 2n-1$, and a surjection for $k = 2n-1$.*

Proof : We want to apply homotopy excision (Theorem 1.1.3). For this, decompose the reduced suspension $\Sigma X$ as a union of two reduced cones glued together along $X$. One way to describe this is to consider the *reversed reduced cone* $C'X$, which is defined as the following pushout :

$$\begin{tikzcd} X \times \{0\} \cup \{*\} \times I \arrow[r, hook] \arrow[d] & X \times I \arrow[d, "q"] \\ \{*\} \arrow[r] & C'X. \end{tikzcd}$$

Clearly $C'X$ is homeomorphic to the usual reduced cone but only $I$ is given the basepoint 0 rather than 1 in the construction. Thus $\Sigma X$ is homeomorphic to the pushout of $X \to CX$ with $X \to C'X$. We obtain a triad $(\Sigma X; CX, C'X)$, where $CX \cap C'X \cong X$, though it is not excisive, since $CX$ and $C'X$ need not be open in $\Sigma X$. But if one regards the reduced suspension as the usual following pushout :

$$\begin{tikzcd} X \times \partial I \cup \{*\} \times I \arrow[r, hook] \arrow[d] & X \times I \arrow[d] \\ \{*\} \arrow[r] & \Sigma X, \end{tikzcd}$$

and defines the spaces :

$$A := \frac{X \times ]0,1]}{X \times \{1\} \cup \{*\} \times ]0,1]}, \text{ and } B := \frac{X \times [0,1[}{X \times \{0\} \cup \{*\} \times [0,1[},$$

one can see that $A$ and $B$ are open in $\Sigma X$, and there are the based homotopy equivalences $A \simeq_* CX$, $B \simeq_* C'X$, $A \cap B \simeq_* X$. Indeed, the homotopy :

$$\begin{array}{rcl} H : CX \times I & \longrightarrow & CX \\ ([x,t],s) & \longmapsto & [x, s + (1-s)t], \end{array}$$

gives a based homotopy equivalence $CX \simeq_* *$. With the same argument, we have :

$$A \simeq_* *, \text{ and } B \simeq_* * \simeq_* C'X.$$

Hence, the triad $(\Sigma X; A, B)$ is excisive. Moreover, $A$ and $B$ are contractible spaces, so $(A, X)$ and $(B, X)$ are $(n-1)$-connected, by the long exact sequence of the pairs, whence we can apply the excision homotopy Theorem. The inclusion $(B, X) \hookrightarrow (\Sigma X, A)$ is a $(2n-2)$-equivalence, and thus, the inclusion : $i : (C'X, X) \hookrightarrow (\Sigma X, CX)$ is a $(2n-2)$-equivalence.
To end the proof, we need to know the relation between the inclusion $i$ and the suspension homomorphism $\Sigma$. Consider an element $[f]_* \in \pi_k(X) = [(I^k, \partial I^k), (X, *)]_*$. Let us name $q : X \times I \to C'X \cong X \times I/(X \times \{0\} \cup \{*\} \times I)$ the quotient map induced by the definition of $C'X$ as a pushout. Define $g$ to be the composite :

$$I^{k+1} \xrightarrow{f \times \mathrm{id}} X \times I \xrightarrow{q} C'X.$$

It is easy to see that $g(\partial I^{k+1}) \subseteq X$, and $g(J^k) = \{*\}$. Indeed, we have $g(\partial I^k \times I) = [(*, I)] \subseteq X$, $g(I^k \times \{0\}) = * \in X$ and $g(I^k \times \{1\}) \subseteq X$. Hence $g(\partial I^{k+1}) \subseteq X$. It is similar to prove that $g(J^k) = \{*\}$. Therefore $[g]_* \in \pi_{k+1}(C'X, X)$. Moreover, it is clear $g_{|_{I^k \times \{1\}}} = f$. Hence $\partial([g]_*) = [f]_*$, where $\partial$ is the boundary map of the long exact sequence of the pair $(C'X, X)$. We get : $\rho \circ g = \Sigma f$, where the map $\rho : C'X \to \Sigma X$ can be viewed as a quotient map, through the homeomorphism $\Sigma X \cong C'X/(X \times \{1\})$. Thus the following diagram commutes :

$$\begin{tikzcd} \cdots \arrow[r] & \overbrace{\pi_{k+1}(C'X)}^{=0} \arrow[r] & \pi_{k+1}(C'X, X) \arrow[r, "\partial", "\cong"'] \arrow[d, "i_*"'] \arrow[dr, "\rho_*"] & \pi_k(X) \arrow[r] \arrow[d, "\Sigma"] & \overbrace{\pi_k(C'X)}^{=0} \arrow[r] & \cdots \\ \cdots & \underbrace{\pi_k(CX)}_{=0} \arrow[l] & \pi_{k+1}(\Sigma X, CX) \arrow[l] & \pi_{k+1}(\Sigma X) \arrow[l, "\cong"] & \underbrace{\pi_{k+1}(CX)}_{=0} \arrow[l] & \cdots . \arrow[l] \end{tikzcd}$$

The rows of the diagram are the long exact sequences of the pairs $(C'X, X)$ and $(\Sigma X, CX)$. Since $i$ is a $(2n-2)$-equivalence, it follows that $\Sigma$ is a bijection for $k < 2n-1$ and a surjection for $k = 2n-1$. $\square$

Corollary 1.2.4.
*For all $n \geq 1$, $\pi_n(S^n) = \mathbb{Z}$.*

Proof : We already have $\pi_1(S^1) = \mathbb{Z}$. By the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, and the long exact sequence of a fibration, we get $\pi_2(S^2) = \mathbb{Z}$, since $\pi_2(S^3) = 0$. Recall that $S^n$ is a $(n-1)$-connected space and $\Sigma S^n \cong S^{n+1}$. By the Freudenthal suspension theorem, the suspension homomorphism $\Sigma : \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for $n \geq 2$. The result follows. $\square$

Stable Homotopy Groups Let $X$ be a $(n-1)$-connected space, where $n \geq 1$. The suspension homomorphism $\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism for $k < 2n-1$. In particular, it is an isomorphism for $k \leq n-1$. Hence $\pi_k(X) = 0$ implies that $\pi_{k+1}(\Sigma X) = 0$, for $k \leq n-1$, i.e. $\Sigma X$ is a $n$-connected space. Inductively, we get that $\Sigma^r X$ is a $(n+r-1)$-connected space, where $\Sigma^r$ denotes the $r$-th reduced suspension. We get the following homotopy groups sequence :

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} \pi_{k+r}(\Sigma^r X) \xrightarrow{\Sigma} \cdots . \tag{1.1}$$

Since $\Sigma^r X$ is $(n+r-1)$-connected, the suspension homomorphism :

$$\Sigma : \pi_{k+r}(\Sigma^r X) \to \pi_{k+r+1}(\Sigma^{r+1} X),$$

is an isomorphism for $k+r < 2(n+r)-1$, i.e., $r > k-2n+1$. Hence for fixed integers $n$ and $k$, the sequence of homomorphisms in (1.1) are eventually all isomorphisms for a sufficiently large enough $r$. It allows us to define the notion of stable homotopy groups.

Definition 1.2.5 (Stable homotopy groups).
Let $X$ be a $(n-1)$-connected space. Let $k \geq 0$. The *$k$-th stable homotopy group* of $X$, denoted $\pi_k^S(X)$, is defined to be the group $\pi_{k+r}(\Sigma^r X)$, for any $r > k-2n+1$. It is the group for which the sequence (1.1) is stabilized. With the vocabulary of appendix C, the stable group $\pi_k^S(X)$ is the colimit : $\mathrm{colim}_r \pi_{k+r}(\Sigma^r X)$. This definition of stable groups remains valid for any based space (not necessary $(n-1)$-connected).

In particular, we have the stable homotopy group of the spheres, using $\Sigma S^n \cong S^{n+1}$, for each $n \geq 0$. The suspension $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism for $n+k < 2n-1$, i.e., for $n > k+1$. We have $\pi_k^S(S^n) = \pi_{k-n}^S(S^0)$ for $n \leq k$, so the stable homotopy groups of $S^n$ can be expressed in terms of the values of the stable homotopy groups of $S^0$.

Definition 1.2.6 (Stable homotopy groups of the spheres).
The *$k$-th stable homotopy group of the spheres*, denoted $\pi_k^S$, is defined as :

$$\pi_k^S := \pi_k^S(S^0) = \mathrm{colim}_r \pi_{k+r}(S^r) = \pi_{k+n}(S^n), \text{ where } n > k+1.$$

In general, the groups $\pi_{k+n}(S^n)$ are called *stable* if $n > k+1$, and *unstable* if $n \leq k+1$.

## 1.3. The Hurewicz Theorem

The Hurewicz Theorem is a fundamental result in algebraic topology since it relates homotopy groups with integral homology groups. We will again need homotopy excision (Theorem 1.1.3) to prove the theorem. In most of the literature, the Hurewicz Theorem is proved by using a specific definition of the integral homology (such as singular homology, cellular homology, etc). Even if these proofs are very geometric and visually intuitive, it is perhaps more elegant and more algebraic to present the Hurewicz Theorem axiomatically. In other words, what do we really *need* from the integral homology groups to prove the Hurewicz Theorem ? Moreover, anticipating our work for the next chapters, it is more consistent to approach the problem axiomatically. Theorefore we will use the notions of Appendix B.

Let $H_*$ be an ordinary homology theory with coefficents $\mathbb{Z}$; we drop the component $\mathbb{Z}$ in the notation. There is a corresponding ordinary reduced homology theory $\widetilde{H}_*$. Let $i_0$ be a generator of $\widetilde{H}_0(S^0) \cong \mathbb{Z}$. Using the suspension axiom, define inductively the generators $i_n$ of $\widetilde{H}_n(S^n)$ by the suspension homomorphism : $i_{n+1} := \Sigma(i_n)$.

Definition 1.3.1 (The Hurewicz Homomorphism).
For any based space $X$, the *Hurewicz homomorphism* is defined by :

$$\begin{aligned} h : \pi_n(X) &\longrightarrow \widetilde{H}_n(X) \\ [f]_* &\longmapsto \widetilde{H}_n(f)(i_n). \end{aligned}$$

It is well-defined since $\widetilde{H}_n$ is a homotopy invariant functor.

Lemma 1.3.2.
*For any based space $X$, the Hurewicz homomorphism $h$ is indeed a homomorphism for $n \geq 1$.*

Proof : Let us recall that the multiplication on $\pi_n(X) = [(S^n, *), (X, *)]_*$ is given by the natural co-H-structure of $S^n$, seen as the reduced suspension $\Sigma S^{n-1}$. In other words for any classes $[f]_*$ and $[g]_*$ in $\pi_n(X)$, the multiplication $[f]_* \cdot [g]_*$ is the based homotopy class of :

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X,$$

where $p$ is the pinch map, and $\nabla$ is the fold map. Using the additivity axiom ($\widetilde{H}3$) of $\widetilde{H}_*$, we get the following commutative diagram :

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "\widetilde{H}_n(p)"] \arrow[dr, "\Delta"'] & \widetilde{H}_n(S^n \vee S^n) \arrow[r, "\widetilde{H}_n(f \vee g)"] & \widetilde{H}_n(X \vee X) \arrow[r, "\widetilde{H}_n(\nabla)"] & \widetilde{H}_n(X) \\
& \widetilde{H}_n(S^n) \oplus \widetilde{H}_n(S^n) \arrow[u, "\cong"'] \arrow[r, "\widetilde{H}_n(f) + \widetilde{H}_n(g)"] & \widetilde{H}_n(X) \oplus \widetilde{H}_n(X). \arrow[u, "\cong"] \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

Since $\widetilde{H}_n(\nabla \circ f \vee g \circ p) = \widetilde{H}_n(\nabla) \circ \widetilde{H}_n(f \vee g) \circ \widetilde{H}_n(p)$, we get : $h([f]_* \cdot [g]_*) = h([f]_*) + h([g]_*)$. $\square$

Lemma 1.3.3.
*The Hurewicz homomorphism is a natural transformation $h : \pi_n \Rightarrow \widetilde{H}_n$, for all $n \geq 0$, and is compatible with the suspension homomorphism, i.e., the following diagram commutes :*

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h"] \arrow[d, "\Sigma"'] & \widetilde{H}_n(X) \arrow[d, "\Sigma"] \\
\pi_{n+1}(\Sigma X) \arrow[r, "h"'] & \widetilde{H}_{n+1}(\Sigma X).
\end{tikzcd} \tag{1.2}$$

Proof : To prove naturality of $h$, one must check the commutativity of the following diagram, for any map $f : X \to Y$ :

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{h} & \widetilde{H}_n(X) \\ {\scriptstyle \pi_n(f)}\downarrow & & \downarrow{\scriptstyle \widetilde{H}_n(f)} \\ \pi_n(Y) & \xrightarrow{h} & \widetilde{H}_n(Y). \end{array}$$

For any element $[g]_*$ in $\pi_n(X)$, we have :

$$\begin{aligned} (\widetilde{H}_n(f) \circ h)([g]_*) &= \widetilde{H}_n(f)(\widetilde{H}_n(g)(i_n)) \\ &= \widetilde{H}_n(f \circ g)(i_n) \\ &= h([f \circ g]_*) \\ &= (h \circ \pi_n(f))\,([g]_*), \end{aligned}$$

which ends the proof of naturality of $h$.
Now let us prove the commutativity of the diagram (1.2). Recall that the suspension is natural with respect to the ordinary reduced homology $\widetilde{H}_*$ owing to the suspension axiom ($\widetilde{H}2$), and hence the following diagram commutes for any map $f : Y \to Z$ :

$$\begin{array}{ccc} \widetilde{H}_n(Y) & \xrightarrow{\Sigma} & \widetilde{H}_{n+1}(\Sigma Y) \\ {\scriptstyle \widetilde{H}_n(f)}\downarrow & & \downarrow{\scriptstyle \widetilde{H}_{n+1}(\Sigma f)} \\ \widetilde{H}_n(Z) & \xrightarrow[\Sigma]{} & \widetilde{H}_{n+1}(\Sigma Z), \end{array}$$

for any $n$. Recall that $\Sigma i_n = i_{n+1}$, and so, for any $[g]_*$ in $\pi_n(X)$, we get :

$$\begin{aligned} (h \circ \Sigma)([g]_*) &= h([\Sigma f]_*) \\ &= \widetilde{H}_{n+1}(\Sigma f)(i_{n+1}) \\ &= \widetilde{H}_{n+1}(\Sigma f)(\Sigma i_n) \\ &= \Sigma(\widetilde{H}_n(f)(i_n)) \\ &= (\Sigma \circ h)([g]_*), \end{aligned}$$

which proves the commutativity of the diagram (1.2). □

To establish the Hurewicz Theorem, we will need the following theorem, which stems from homotopy excision. The result will be at the heart of the proof of Hurewicz.

Theorem 1.3.4 (A Corollary of Homotopy Excision).
*Let $f : X \to Y$ be a based $(n-1)$-equivalence between $(n-1)$-connected well-pointed spaces (see Appendix B), where $n > 1$, so that $\pi_{n-1}(f)$ is surjective. The quotient map $q : (Mf, X) \to (Cf, *)$ is an $(2n-1)$-equivalence, where $Mf$ denotes the reduced mapping cylinder of $f$, and $Cf$ denotes the reduced mapping cone of $f$. In particular $Cf$ is $(n-1)$-connected.*

Proof : Recall that the reduced mapping cylinder $Mf$ is given by the pushout :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ (X \times I)/(\{*\} \times I) & \longrightarrow & Mf, \end{array}$$

and the reduced mapping cone $Cf$ is given by the pushout :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \\ CX \arrow[r] & Cf, \end{tikzcd}$$

where $CX$ is the reduced cone of $X$. It follows that, for $j : X \to Mf$, which is defined by $x \mapsto [x, 1]$ :

$$Cf = \frac{Mf}{j(X)} = \frac{CX \cup Y}{f(x) \sim (x,0)}.$$

Define $A = \dfrac{(X \times [0,2/3])/(\{*\} \times [0,2/3]) \cup Y}{f(x) \sim (x,0)}$ and $B = \dfrac{(X \times [1/3,1])/(\{*\} \times [1/3,1])}{j(X)}$ subspaces of $Cf$. It follows that :

$$C := A \cap B = \frac{X \times [1/3,2/3]}{\{*\} \times [1/3,2/3]}.$$

We get an excisive triad $(Cf; A, B)$. We now want to apply homotopy excision (Theorem 1.1.3). One can see that $(Mf, j(X)) \simeq_* (A, C)$ and $(Cf, B) \simeq_* (Cf, *)$ (use Lemma B.1.5) : these based homotopy equivalences are visually clear. Also notice that $(B, C) \simeq_* (CX, X)$. Hence the quotient map $q : (Mf, X) \to (Cf, *)$ defined by $Cf = Mf/j(X)$ is homotopic with the composite :

$$(Mf, j(X)) \xrightarrow{\simeq_*} (A, C) \hookrightarrow (Cf, B) \xrightarrow{\simeq_*} (Cf, *).$$

We now argue that $(A, C)$ is $(n-1)$-connected and $(B, C)$ is $n$-connected. From the long exact sequence of the pair $(Mf, X)$ :

$$\cdots \longrightarrow \pi_k(X) \longrightarrow \pi_k(Mf) \longrightarrow \pi_k(Mf, X) \longrightarrow \pi_{k-1}(X) \longrightarrow \cdots,$$

we obtain $\pi_k(Mf, X) = 0$, for $k < n-1$, since $f$ is a $(n-1)$-equivalence so $\pi_k(X) \cong \pi_k(Y)$, and $Mf$ is homotopy equivalent to $Y$ (since the spaces are well-pointed). Thus $\pi_k(A, C) = 0$ for $k < n-1$, owing to $(A, C) \simeq_* (Mf, X)$. So $(A, C)$ is $(n-1)$-connected. From the long exact sequence of the pair $(CX, X)$, since $CX$ is contractible, we obtain isomorphisms $\pi_k(CX, X) \cong \pi_{k-1}(X)$. Now, since $X$ is $(n-1)$-connected, and $(B, C) \simeq_* (CX, X)$, we get that $(B, C)$ is $n$-connected. Thus, homotopy excision states that $(A, C) \hookrightarrow (Cf, B)$ is a $(2n-1)$-equivalence, and hence $q$ is a $(2n-1)$-equivalence. □

The Hurewicz Theorem states that the Hurewicz homomorphism is an isomorphism in the first dimension where the homotopy groups and reduced homology groups are both not trivial. We begin with a particular case of spaces : the wedges of $n$-spheres. Let us compute first the homotopy groups of such spaces.

Lemma 1.3.5.
*Consider the wedge of $n$-spheres : $\bigvee_{j \in \mathscr{J}} S^n$ where $\mathscr{J}$ is any index set. Let $\iota_j^n : S^n \hookrightarrow \bigvee_{j \in \mathscr{J}} S^n$ be the inclusion. If $n = 1$, then its fundamental group is the free group generated by $\{\iota_j^1\}_{j \in \mathscr{J}}$. If $n \geq 2$, then its $n$-th homotopy group is the free abelian group generated by $\{\iota_j^n\}_{j \in \mathscr{J}}$.*

Proof : The statement is obvious if the index set $\mathscr{J}$ consists of only one element. Let us first prove the case $n = 1$. When $\mathscr{J}$ consists of two elements, the statement follows directly from

the Seifert-van Kampen Theorem. The argument can be easily generalized when $\mathscr{J}$ is a finite set. If $\mathscr{J}$ is any set, define $\Theta_{\mathscr{J}} : \ast_{j\in\mathscr{J}} \pi_1(S^1) \to \pi_1(\bigvee_{j\in\mathscr{J}} S^1)$ as the homomorphism induced by the maps $\{\iota_j^1\}_{j\in\mathscr{J}}$. Let us prove that it is an isomorphism. Endow the circle $S^1$ with its usual CW-decomposition. For any map $f : S^1 \to \bigvee_{j\in\mathscr{J}} S^1$, its image $f(S^1)$ is compact, and so there is a finite subset $J \subseteq \mathscr{J}$, such that $[f]_*$ can be seen as an element of $\pi_1(\bigvee_{j\in J} S^1)$. By our previous case, it is in the image of $\Theta_J$. It follows that $\Theta_{\mathscr{J}}$ is surjective. Now suppose there exists $\alpha$ in $\ast_{j\in\mathscr{J}} \pi_1(S^1)$, such that its image $\Theta_{\mathscr{J}}(\alpha)$ equals a class $[f]_*$ where $f : S^1 \to \bigvee_{j\in\mathscr{J}} S^1$ is nullhomotopic, via a homotopy $H$. There exists a finite subset $J \subseteq \mathscr{J}$, such that $\alpha$ is in $\ast_{j\in J} \pi_1(S^1)$. Since the image $H(S^1 \times I)$ is compact, there exists a finite subset $J' \subseteq \mathscr{J}$, such that $f$ is a nullhomotopic map that represents an element in $\pi_1(\bigvee_{j\in J'} S^1)$. Whence we get $\Theta_{J\cup J'}(\alpha) = [f]_* = 0$, and by our previous case, we get that $\alpha = 0$, which proves the injectivity of $\Theta_{\mathscr{J}}$.
Let us prove now the case $n \geq 2$. Let $\mathscr{J}$ be a finite set. Regard $\bigvee_{j\in\mathscr{J}} S^n$ as the $n$-skeleton of the product $\prod_{j\in\mathscr{J}} S^n$, where again the $n$-sphere $S^n$ is endowed with its usual CW-decomposition, and $\prod_{j\in\mathscr{J}} S^n$ has the CW-decomposition induced by the finite product of CW-complexes. Since $\prod_{j\in\mathscr{J}} S^n$ has cells only in dimensions a multiple of $n$, the pair $(\prod_{j\in\mathscr{J}} S^n, \bigvee_{j\in\mathscr{J}} S^n)$ is $(2n-1)$-connected. The long exact sequence of this pair gives the isomorphism :

$$\pi_n(\bigvee_{j\in\mathscr{J}} S^n) \cong \pi_n(\prod_{j\in\mathscr{J}} S^n) \cong \bigoplus_{j\in\mathscr{J}} \pi_n(S^n),$$

induced by the inclusions $\{\iota_j^n\}_{j\in\mathscr{J}}$. The result follows. Let now $\mathscr{J}$ be any index set, let $\Theta_{\mathscr{J}} : \bigoplus_{j\in\mathscr{J}} \pi_n(S^n) \to \pi_n(\bigvee_{j\in\mathscr{J}})$ be the homomorphism induced by the inclusions $\{\iota_j^n\}_{j\in\mathscr{J}}$. Just as the case $n = 1$, one can reduce $\mathscr{J}$ to the case where it is finite to establish that $\Theta_{\mathscr{J}}$ is an isomorphism. $\square$

Lemma 1.3.6.
*Let $n \geq 1$. Consider the wedge of $n$-spheres $\bigvee_{j\in\mathscr{J}} S^n$, where $\mathscr{J}$ is any index set. Then the Hurewicz homomorphism :*

$$h : \pi_n\left(\bigvee_{j\in\mathscr{J}} S^n\right) \to \widetilde{H}_n\left(\bigvee_{j\in\mathscr{J}} S^n\right),$$

*is the abelianization homomorphism if $n = 1$, and is an isomorphism if $n > 1$.*

Proof : For the case of a single $n$-sphere, the result is obvious since $h([\mathrm{id}_{S^n}]_*) = i_n$, i.e. the Hurewicz homomorphism sends the generator $[\mathrm{id}_{S^n}]_*$ of $\pi_n(S^n) \cong \mathbb{Z}$ to the generator $i_n$ of $\widetilde{H}_n(S^n) = \mathbb{Z}$.
For the general case, using previous lemma, the Hurewicz homomorphism $h$ maps the generators of the $n$-th homotopy group of $\bigvee_{j\in\mathscr{J}} S^n$ to the canonical generators of the free abelian group $\widetilde{H}_n(\bigvee_{j\in\mathscr{J}} S^n) \cong \bigoplus_{j\in\mathscr{J}} \widetilde{H}_n(S^n) = \bigoplus_{j\in\mathscr{J}} \mathbb{Z}$, owing to the additivity axiom of $\widetilde{H}_*$. So when $n \geq 2$, it is an isomorphism, and for $n = 1$, it is the abelianization homomorphism since $\widetilde{H}_1(\bigvee_{j\in\mathscr{J}} S^1)$ is abelian. $\square$

We emphasize the following algebraic argument that will be made in the proof of Hurewicz Theorem.

Lemma 1.3.7.

*For any commutative diagram of abelian groups :*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"', "\cong"] & B \arrow[r, "\beta"] \arrow[d, "g"', "\cong"] & C \arrow[r] \arrow[d, "h"] & 0 \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r] & 0, \end{tikzcd}$$

*where the rows are exact sequences, if $f$ and $g$ are isomorphisms, then $h$ is an isomorphism.*

Sketch of the Proof : It is an easy algebraic argument, very similar to the proof of the 5-Lemma. □

Theorem 1.3.8 (Hurewicz).
*Let $X$ be a $(n-1)$-connected based space, where $n \geq 1$. Then the Hurewicz homomorphism*

$$h : \pi_n(X) \to \widetilde{H}_n(X)$$

*is the abelianization homomorphism if $n = 1$ and is an isomorphism if $n > 1$.*

Proof : We can assume that $X$ is a CW-complex, since there is a weakly equivalent CW-complex by Theorem A.3.6, and weak equivalences induce isomorphisms on homotopy groups and reduced homology groups. Since $X$ is $(n-1)$-connected, one may assume that the CW-complex $X$ has a single vertex and no $k$-cells for $1 \leq k \leq n-1$. By Corollary A.3.8, all the attaching maps are based, and the $n$-skeleton $X_n$ is obtained as the following pushout :

$$\begin{tikzcd} \bigvee_{j\in\mathscr{J}_n} S^{n-1} \arrow[r] \arrow[d, tail] & X_0 \arrow[d, tail] \\ \bigvee_{j\in\mathscr{J}_n} D^n \arrow[r] & X_n. \end{tikzcd}$$

Since $X_0 = \{*\}$, we get $X_n = \bigvee_{j\in\mathscr{J}_n} S^n$. Now the $(n+1)$-skeleton $X_{n+1}$ can be regarded as the reduced mapping cone :

$$\begin{tikzcd} & \bigvee\limits_{j\in\mathscr{J}_{n+1}} S^n_j \arrow[r, "f"] \arrow[dl] \arrow[d, tail] & X_n \arrow[d, tail] & \\ C(\bigvee\limits_{j\in\mathscr{J}_{n+1}} S^n_j) \arrow[r, "\cong"] & \bigvee\limits_{j\in\mathscr{J}_{n+1}} D^{n+1}_j \arrow[r] & X_{n+1} \arrow[r, "\cong"] & Cf. \end{tikzcd}$$

We argue that $X$ may be assumed to be a $(n+1)$-dimensional CW-complex, and so one can consider $X = X_{n+1}$. Indeed, the inclusion $X_{n+1} \hookrightarrow X$ is a $(n+1)$-equivalence, and so we have the isomorphism $\pi_n(X_{n+1}) \cong \pi_n(X)$. Let us prove now that $\widetilde{H}_n(X_{n+1}) \cong \widetilde{H}_n(X)$. We first argue that $\widetilde{H}_n(X_{n+1}) \cong \widetilde{H}_n(X_{n+2}) \cong \widetilde{H}_n(X_{n+3}) \cong \ldots$. Recall that $X_{n+2}$ is obtained as the pushout :

$$\begin{tikzcd} \bigvee S^{n+1} \arrow[r] \arrow[d, tail] & X_{n+1} \arrow[d, tail] \\ \bigvee D^{n+2} \arrow[r] & X_{n+2}. \end{tikzcd}$$

We obtain $X_{n+2}/X_{n+1} \cong \bigvee S^{n+2}$. Since $X_{n+1} \rightarrowtail X_{n+2}$ is a cofibration, using theorem B.1.8, there is the following exact sequence :

$$\cdots \longrightarrow \widetilde{H}_{n+1}(\bigvee S^{n+2}) \longrightarrow \widetilde{H}_n(X_{n+1}) \longrightarrow \widetilde{H}_n(X_{n+2}) \longrightarrow \widetilde{H}_n(\bigvee S^{n+2}) \longrightarrow \cdots .$$

But the additivity axiom gives $\widetilde{H}_{n+1}(\bigvee S^{n+2}) = 0$ and $\widetilde{H}_n(\bigvee S^{n+2}) = 0$. Therefore we obtain the isomorphism $\widetilde{H}_n(X_{n+1}) \cong \widetilde{H}_n(X_{n+2})$. Using the same argument, we get the isomorphisms : $\widetilde{H}_n(X_{n+2}) \cong \widetilde{H}_n(X_{n+3})$, $\widetilde{H}_n(X_{n+3}) \cong \widetilde{H}_n(X_{n+4})$, etc. Hence $\mathrm{colim}_j \widetilde{H}_n(X_{n+1+j}) = \widetilde{H}_n(X_{n+1})$. But $\mathrm{colim}_j X_{n+1+j} = X$, thus Theorem C.4.7 gives the desired isomorphism : $\widetilde{H}_n(X_{n+1}) \cong \widetilde{H}_n(X)$. Therefore, we may assume that $X$ is a $(n+1)$-dimensional CW-complex : $X = X_{n+1}$.

For clarity, let us name $A := \bigvee_{j \in \mathscr{J}_n} S^n_j$ which stems from the definition of $X_n$ as a pushout, and $B := \bigvee_{j \in \mathscr{J}_{n+1}} S^n_j = X_n$. The map $f : A \to B$ induces the following commutative diagram :

$$\begin{tikzcd}
\pi_n(A) \arrow[r] \arrow[d] & \pi_n(B) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & 0 \\
\widetilde{H}_n(A) \arrow[r] & \widetilde{H}_n(B) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & 0,
\end{tikzcd} \tag{$\diamond$}$$

where we used that $X \cong Cf$. The vertical homomorphisms are the Hurewicz homomorphisms. From the top row (a priori not exact), since $B \hookrightarrow X$ is a $n$-equivalence, the homomorphism $\pi_n(B) \to \pi_n(X)$ is surjective. Using Theorem B.1.8, the bottom row of the diagram is exact since $\widetilde{H}_{n-1}(A) = 0$. We now argue that the top row is also exact.

When $n = 1$, we use a corollary of the Seifert-van Kampen (a proof can be found in the reference [Munkres, 2000], Corollary 70.4) :

> *Let $X = U \cup V$, where $U$ and $U \cap V$ are path-connected, and $V$ is simply connected. There is an isomorphism $\pi_1(U)/N \cong \pi_1(X)$, where $N$ is the least normal subgroup of $\pi_1(U)$ containing the image of the homomorphism $\pi_1(U \cap V) \to \pi_1(U)$ induced by the inclusion $U \cap V \hookrightarrow U$.*

Here $X$ is built out of $B$ by attaching 2-cells, i.e. $X = B \cup e^2_j$. Since $B \cap e^2_j = A$, and $e^2_j$ is simply connected, we get that $\pi_1(X) \cong \pi_1(B)/N$ where $N$ is the least normal subgroup of $\pi_1(B)$ containing the image of $\pi_1(A) \to \pi_1(B)$, whence the top row of $(\diamond)$ becomes exact by passage to abelianizations. The two left vertical Hurewicz homomorphisms are then isomorphisms by Lemma 1.3.6. Thus the Hurewicz homomorphism $\pi_1(X) \to \widetilde{H}_1(X)$ is the abelianization homomorphism by Lemma 1.3.7.

For $n > 1$, the two left vertical Hurewicz homomorphisms of $(\diamond)$ are isomorphisms by Lemma 1.3.6. We prove the exactness of the top row. Use the mapping cylinder factorisation of $f$ :

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, tail] & & B \\
& Mf, \arrow[ur, "p_f"'] &
\end{tikzcd}$$

where $p_f$ is a homotopy equivalence. We get the commutative diagram :

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(A) \arrow[r] \arrow[d, equal] & \pi_n(Mf) \arrow[r] \arrow[d, "(p_f)_*", "\cong"'] & \pi_n(Mf, A) \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\
& \pi_n(A) \arrow[r] & \pi_n(B) \arrow[r] & \pi_n(X) \arrow[r] & 0, &
\end{tikzcd}$$

where the top row is the long exact sequence of the pair $(Mf, A)$, and where the quotient map $Mf \to Mf/A = Cf = X$ induces the right vertical homomorphism $\pi_n(Mf, A) \to \pi_n(X)$. Since $A$ and $B$ are $(n-1)$-connected and $n > 1$, Theorem 1.3.4 states that the homomorphism $\pi_n(Mf, A) \to \pi_n(X)$ is an isomorphism. Hence, the top row of the diagram $(\diamond)$ is exact. Thus the Hurewicz homomorphism $\pi_n(X) \to \widetilde{H}_n(X)$ is an isomorphism by Lemma 1.3.7. $\square$

# CHAPTER 2

# Generalized (Co)homology Theories

In this chapter, we describe a general homotopy-theoretic method to construct generalized homology and cohomology theories. We begin by the description of Eilenberg-MacLane spaces. These spaces will be fundamental for the homotopy description of ordinary homology and cohomology theories. Afterwards, we introduce prespectra, which are particular sequences of based spaces subsuming stable phenomena, and we will see how they give rise to generalized (co)homology theories.

## 2.1. Eilenberg-MacLane Spaces

Definition 2.1.1 (Eilenberg-MacLane Spaces).
Let $G$ be any group, and $n$ in $\mathbb{N}$. An *Eilenberg-MacLane space of type* $(G, n)$, is a space $X$ of the homotopy type of a based CW-complex such that :

$$\pi_k(X) \cong \begin{cases} G, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

One denotes such a space by $K(G, n)$.

Remark 2.1.2.
Notice that we used the isomorphism symbol $\cong$ instead of the equality $=$. Although subsequently we will fudge the distinction between one another, this means there is a particular choice of a group isomorphism $\pi_n(X) \xrightarrow{\cong} G$ which is called a *structure* of $X$ as an Eilenberg-MacLane space.

We now want to prove that the spaces $K(G, n)$ exist and are unique, up to homotopy, for every group $G$ and every integer $n \geq 0$. For the purpose of this paper, we will only show this statement when $G$ is an abelian group and when $n \geq 1$. The case $n = 0$ is vacuous : one just takes the group $G$ endowed with its discrete topology. The case $n = 1$ but $G$ is not abelian is done in [Hatcher, 2002], chapter 1.B. Notice that when $n \geq 2$, the group $G$ must be abelian. Notice also that when $n \geq 1$, the spaces $K(G, n)$ are path-connected. More generally, the spaces $K(G, n)$ are $(n-1)$-connected.
For the construction of $K(G, n)$, we introduce its analogue in homology.

Definition 2.1.3 (Moore Spaces).
Let $G$ be an abelian group and $n \geq 1$ an integer. A *Moore space of type* $(G,\, n)$ is a path-connected space $X$ of the homotopy type of a based CW-complex such that :

$$\widetilde{H}_k(X;\mathbb{Z}) \cong \begin{cases} G, & \text{if } k = n, \\ 0, & \text{if } k \neq 0, n. \end{cases}$$

One denotes such a space by $M(G, n)$.

Proposition 2.1.4.
*For any abelian group $G$, and any integer $n \geq 1$, there exists a Moore space of type $(G, n)$.*

Proof : Recall that there is a free resolution of $G$, i.e., there exists $F_0$ and $F_1$ free abelian groups such that there is an exact sequence :

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0.$$

Let us recall briefly the algebraic details : define $F_0$ to be the free abelian groups $\bigoplus_{g\in G}\mathbb{Z}$ generated by the elements of $G$. Define the surjective homomorphism $F_0 \to G$ by simply mapping the basis of $F_0$ to the elements of $G$, and let $F_1$ be the kernel of this homomorphism. It is also a free abelian group, and in this case, we have $F_1 \subseteq F_0$. We have obtained the desired exact sequence.
Let $\{e_j\}_{j\in\mathscr{J}}$ be a basis of $F_1$. Let $L_0 := \bigvee_{g\in G} S^n$. By the additivity axiom, we have the isomorphism : $\widetilde{H}_n(L_0;\mathbb{Z}) \cong \bigoplus_{g\in G}\widetilde{H}_n(S^n;\mathbb{Z}) = F_0$. Let $h : \pi_n(L_0) \to \widetilde{H}_n(L_0) \cong F_0$ be the Hurewicz homomorphism. It is an isomorphism by the Hurewicz Theorem (Theorem 1.3.8) since $L_0$ is $(n-1)$-connected. So there exists $[f_j]_*$ in $\pi_n(L_0)$ for each basis element $e_j$ of $F_1 \subseteq F_0$. Let $L_1 := \bigvee_{j\in\mathscr{J}} S^n$. Define the following reduced mapping cone :

$$\begin{tikzcd}
 & L_1 \arrow[r, "f=\sum\limits_{j\in\mathscr{J}} f_j"] \arrow[dl] \arrow[d, tail] & L_0 \arrow[d, tail, "i_f"] \\
C(L_1) \arrow[r, "\cong"] & \bigvee\limits_{j\in\mathscr{J}} D^{n+1} \arrow[r] & Cf.
\end{tikzcd}$$

It is a CW-complex. Using again the additivity axiom, we obtain : $\widetilde{H}_n(L_1;\mathbb{Z}) \cong L_1$ and $\widetilde{H}_m(L_1;\mathbb{Z}) = 0 = \widetilde{H}_m(L_0;\mathbb{Z})$, whenever $m \neq n$. Apply Theorem B.1.8 to the based map $f : L_1 \to L_0$. For $k \in \mathbb{Z}$ such that $k \neq n, n+1$, we obtain the exact sequence :

$$\cdots \longrightarrow \underbrace{\widetilde{H}_k(L_0;\mathbb{Z})}_{=0} \longrightarrow \widetilde{H}_k(Cf;\mathbb{Z}) \longrightarrow \underbrace{\widetilde{H}_{k-1}(L_1;\mathbb{Z})}_{=0} \longrightarrow \cdots,$$

so $\widetilde{H}_k(Cf;\mathbb{Z}) = 0$ for every $k \neq n, n+1$. For the case $k = n+1$, we have the exact sequence :

$$\cdots \longrightarrow \underbrace{\widetilde{H}_{n+1}(L_0;\mathbb{Z})}_{=0} \longrightarrow \widetilde{H}_{n+1}(Cf;\mathbb{Z}) \longrightarrow \underbrace{\widetilde{H}_n(L_1;\mathbb{Z})}_{\cong F_1} \longrightarrow \underbrace{\widetilde{H}_n(L_0;\mathbb{Z})}_{\cong F_0} \longrightarrow \cdots.$$

Since we have the commutativity of the following diagram :

$$\begin{tikzcd}
\widetilde{H}_n(L_1;\mathbb{Z}) \arrow[r] & \widetilde{H}_n(L_0;\mathbb{Z}) \\
F_1 \arrow[u, "\cong"] \arrow[r, hook] & F_0, \arrow[u, "\cong"']
\end{tikzcd}$$

we get that the image of $\widetilde{H}_{n+1}(Cf;\mathbb{Z}) \to \widetilde{H}_n(L_1;\mathbb{Z})$ is trivial since it equals the kernel of $\widetilde{H}_n(L_1;\mathbb{Z}) \to \widetilde{H}_n(L_0;\mathbb{Z})$. Hence $\widetilde{H}_{n+1}(Cf;\mathbb{Z}) = 0$, by exactness. For the case $k = n$, we have the exact sequence :

$$\cdots \longrightarrow 0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \widetilde{H}_n(Cf;\mathbb{Z}) \longrightarrow 0 \longrightarrow \cdots,$$

so $\widetilde{H}_n(Cf;\mathbb{Z}) \cong F_0/F_1 \cong G$. Therefore $Cf$ is a Moore space of type $(G,n)$. □

Theorem 2.1.5 (Existence of $K(G,n)$).
*Let $G$ be an abelian group, and $n \geq 1$ an integer. Then there exists an Eilenberg-MacLane space of type $(G,n)$.*

Proof : Let $M(G,n)$ be a Moore space. We give two constructions of an Eilenberg-MacLane space of type $(G,n)$. The first is a rather explicit construction, whereas the second is more functorial.

First Construction From the Hurewicz homomorphism :

$$h : \pi_k(M(G,n)) \longrightarrow \widetilde{H}_k(M(G,n);\mathbb{Z}),$$

we get that $M(G,n)$ is $(n-1)$-connected and $\pi_n(M(G,n)) \cong G$. Let $M(G,n) = K_n$. By Lemma A.3.5, we build a CW-complex $K_{n+1}$ by attaching $(n+2)$-cells to $K_n$ so that we kill its homotopy only in dimension $n+1$, i.e. : $\pi_{n+1}(K_{n+1}) = 0$ and $\pi_k(K_{n+1}) = \pi_k(K_n)$, for $k \leq n$. Iterating this procedure, name $K$ the colimit $\mathrm{colim}_j K_{n+j}$. It is a CW-complex and by Theorem C.4.5, it is an Eilenberg-MacLane space of type $(G,n)$.

Second Construction There is a homotopical invariant functor : $\mathrm{SP} : \mathbf{Top}_* \to \mathscr{A}$ called the *infinite symmetric product*, where $\mathscr{A}$ is the category of abelian topological monoids (see references [Strom, 2011] and [Aguilar et al., 2002] for details). For any based space $X$, $\mathrm{SP}(X)$ is the free commutative topological monoid generated by $X$. If $X$ is a based CW-complex, then $\mathrm{SP}(X)$ is also a CW-complex. If $X$ is path-connected, then $\widetilde{H}_n(X;\mathbb{Z}) \cong \pi_n(\mathrm{SP}(X))$, for any $n \geq 0$. Hence an Eilenberg-MacLane space of type $(G,n)$ is given by : $\mathrm{SP}(M(G,n))$.

Thus Eilenberg-MacLane spaces exist for any abelian group $G$. □

Lemma 2.1.6.
*Let $Y$ be a space such that $\pi_k(Y) = 0$ for all $k > n$ and any chosen basepoint of $Y$. Let $X$ be a* CW*-complex. Suppose $A \subseteq X$ is a subcomplex with $X_{n+1} \subseteq A \subseteq X$. Then for any map $f : A \to Y$, there exists a map $\phi : X \to Y$ such that the following diagram commutes :*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X. \arrow[ur, dashed, "\exists\phi"'] & \end{tikzcd}$$

Proof : We wish to extend $f$ skelelton by skeleton, and hence cell by cell, by induction. Indeed, suppose we have a map $g : W \to Y$, where $X = W \cup e^{k+1}$. Let $\varphi : S^k \to W$ be the attaching map. Then an extension of $g$ exists if and only if $g \circ \varphi$ is nullhomotopic, by Lemma A.3.4. But this is always the case when $k > n$ for our hypothesis. So we extend $f$ cell by cell until we obtain the CW-complex $X$. To be completely rigorous, we argue using the Zorn's Lemma. Consider the set :

$$\mathscr{P} := \{(U, f_U) \mid U \text{ is a subcomplex of } X \text{ with } A \subseteq U \subseteq X, \text{ and } f_U \text{ an extension of } f\}.$$

We define a partial order $\preceq$ on $\mathscr{P}$ in the obvious way : we say that $(U, f_U) \preceq (V, f_V)$ if $U$ is a subcomplex of $V$ and $f_V$ extends $f_U$. Therefore we apply the Zorn's Lemma to $(\mathscr{P}, \preceq)$ : the set $\mathscr{P}$ contains a maximal element. But noticing that if $U \neq X$ then $(U, f_U)$ is not maximal since we can extend $f_U$ to a cell in $X$ which not in $U$, we have that $X$ is a maximal element of $\mathscr{P}$. Define $\phi$ to be $f_X$. □

Theorem 2.1.7.
*Let $G$ be a group and $n \geq 1$. Let $Y$ be an Eilenberg-MacLane space of type $(G, n)$. Then for any $(n-1)$-connected* CW*-complex $X$, the function :*

$$\begin{aligned} \Phi : [X, Y]_* &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\pi_n(X), \pi_n(Y)) = \operatorname{Hom}_{\mathbb{Z}}(\pi_n(X), G), \\ [f]_* &\longmapsto f_* \end{aligned}$$

*is bijective.*

Proof : We first argue as we did in the proof of Hurewicz (Theorem 1.3.8). Since $X$ is $(n-1)$-connected, we can assume that $X$ has a single vertex and no $k$-cells for $1 \leq k \leq n-1$. It is enough to prove the theorem for the special case where $X$ has dimension at most $n+1$ since the inclusion of the $(n+1)$-skeleton to $X$ is an $(n+1)$-equivalence. Therefore we assume $X = X_{n+1}$. The $n$-skeleton $X_n$ is a wedge of $n$-sphere and $X$ is obtained as a pushout :

$$\begin{tikzcd} \bigvee_{\beta \in \mathscr{B}} S^n_\beta \arrow[r, "\sum \varphi_\beta"] \arrow[d, tail] & X_n \arrow[d, tail, "i"] \\ \bigvee_{\beta \in \mathscr{B}} D^{n+1}_\beta \arrow[r] & X. \end{tikzcd} \tag{$\star$}$$

Since $i$ is a $n$-equivalence, the induced homomorphism $i_* : \pi_n(X_n) \to \pi_n(X)$ is surjective. Let us prove now that $\Phi$ is bijective.

Injectivity of $\Phi$ Let $[f]_*$ and $[g]_*$ be elements of $[X, Y]_*$, such that $f_* = \Phi([f]_*)$ equals $\Phi([g]_*) = g_*$, i.e., $[f \circ h]_* = [g \circ h]_*$ for any map $h : S^n \to X$. By surjectivity of $i_*$, for any element $[h]_*$ in $\pi_n(X)$ there exists $[h']_*$ in $\pi_n(X_n)$ such that $i_*([h']_*) = [h]_*$, and so : $f_*(i_*([h']_*)) = g_*(i_*([h']_*))$. In particular, since by Lemma 1.3.5 the generators of $\pi_n(X_n)$ are represented by the inclusions $S^n \to X_n$, we get $[f \circ i]_* = [g \circ i]_*$. We get then a homotopy $X_n \times I \to Y$ from $f \circ i$ to $g \circ i$. The homotopy and the maps $f$ and $g$ together determine a map $H : (X \times \partial I \cup X_n \times I) \to Y$. The $(n+1)$-skeleton of the $(n+2)$-dimensional CW-complex $X \times I$ is $(X \times I)_n = X \times \partial I \cup X_n \times I$. So by Lemma 2.1.6, we can extend $H$ to $X \times I$, so that we have a homotopy from $f$ to $g$. Thus $[f]_* = [g]_*$.

Surjectivity of $\Phi$ Let $h : \pi_n(X) \to \pi_n(Y)$ be a group homomorphism. Let $X_n = \bigvee_{j \in \mathscr{J}} S^n_j$. The group $\pi_n(X_n)$ is generated by the homotopy classes of the inclusions $\iota_j : S^n_j \hookrightarrow \bigvee_{j \in \mathscr{J}} S^n_j$, by Lemma 1.3.5. From the composite :

$$\pi_n(X_n) \xrightarrow{i_*} \pi_n(X) \xrightarrow{h} \pi_n(Y),$$

we define $f_j : S^n_j \to Y$ as a representative of the image of $[\iota_j]_*$, i.e. : $h(i_*([\iota_j]_*) = [f_j]_*$, for each $j$ in $\mathscr{J}$. The maps $\{f_j\}$ determine a map $f_n : X_n \to Y$ where $f_n \circ \iota_j = f_j$. Let us name $\varphi_\beta : S^n_\beta \to X_n$ the attaching maps from the pushout $(\star)$. For each $\beta$ in $\mathscr{B}$, the map $i \circ \varphi_\beta$ is nullhomotopic by Lemma A.3.4. Hence $(f_n)_*([\varphi_\beta]_*) = h(i_*([\varphi_\beta]_*)) = 0$. Hence $f_n \circ \varphi_\beta$ is nullhomotopic for each $\beta$. Therefore $f_n$ extends to a map $f : X \to Y$, by Lemma A.3.4. From $h \circ i_* = (f_n)_* = f_* \circ i_*$, since $i_*$ is surjective, we obtain that $f_* = h$, i.e. : $\Phi([f]_*) = h$.

Thus the function $\Phi$ is bijective. □

Corollary 2.1.8 (Uniqueness of $K(G,n)$).
*Let $G$ be an abelian group and $n \geq 1$. An Eilenberg-MacLane space of type $(G,n)$ is unique up to homotopy.*

Proof : Let $X$ and $Y$ be Eilenberg-MacLane space of type $(G,n)$. This means that there are isomorphisms $\theta : \pi_n(X) \to G$ and $\rho : \pi_n(Y) \to G$. From the previous theorem, the composite $\rho^{-1} \circ \theta : \pi_n(X) \to \pi_n(Y)$ is induced by a unique homotopy class of $X \to Y$ which is therefore a weak equivalence. Since $X$ and $Y$ are CW-complexes, the Whitehead Theorem implies that $X$ and $Y$ are homotopy equivalent. □

Examples 2.1.9.
$S^1$ is $K(\mathbb{Z},1)$. More generally, $\mathrm{SP}S^n$ is $K(\mathbb{Z},n)$.

Proposition 2.1.10.
*For any abelian group $G$ and $n \geq 1$ : $\Omega(K(G,n)) \simeq_* K(G,n-1)$.*

Proof : This follows directly from the isomorphism $\pi_m(X) \cong \pi_{m-1}(\Omega X)$, for any based space $X$ and $m \geq 1$. However since we require the Eilenberg-MacLane spaces to be of the homotopy type of a (based) CW-complex, we have to prove that $\Omega(K(G,n))$ is indeed of the homotopy type of a CW-complex. But this is given by the following Milnor Theorem. □

Theorem 2.1.11 (Milnor).
*If $X$ is a based CW-complex, then the reduced loop space $\Omega X$ is a based CW-complex.*

Proof : Omitted. A full detailed proof can be found in [Fritsch and Piccinini, 1990]. □

## 2.2. Prespectra and Generalized Homology Theories

In the previous chapter, we saw, via the Freudenthal Suspension Theorem, that after sufficiently many iterated reduced suspensions, the homotopy groups of a based space eventually stabilize. Moreover, in Appendix B, we show that generalized (reduced) homology and cohomology theories are stable invariants, due to the suspension axiom : there exist suspension isomorphisms for (co)homologies that do not alter the (co)homology groups. Thus, to construct a generalized homology or cohomology theory, it seems intuitively natural first consider a family of based spaces endowed with relations among the reduced suspensions of these spaces. As a matter of fact, it will turn out to be sufficient to consider only a sequence $E_0, E_1, E_2, \cdots$ of based spaces, together with maps $\Sigma E_n \to E_{n+1}$. Such a sequence is called generally a *spectrum*.

Throughout the literature, one find various definitions of a spectrum that do not necessarly agree. Without going through the details, the reason for these different definitions is to find a *convenient* category to work with, just as compactly generated spaces form a well-behaved full subcategory of spaces.
Even though this is irrelevant for our goal to describe homotopically homologies and cohomologies, we shall retain the modern terminology, and talk about *prespectra* instead of spectra. Our approach shall gather the different procedures for defining homology and cohomology theories that one can find in most references.

Definition 2.2.1 (Prespectra).
A *prespectrum* $E$ consists of a family of based spaces $\{E_n\}_{n\geq 0}$ together with based maps $\sigma_n$ : $\Sigma E_n \to E_{n+1}$ called the *structure maps*. The space $E_n$ is called the *n-th term* of the prespectrum $E$. If $E$ and $E'$ are prespectra, a *map of prespectra* $f : E \to E'$ is a family of based maps $\{f_n : E_n \to E'_n \mid n \geq 0\}$ such that for each $n \geq 0$, the following diagram commutes :

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma E'_n \arrow[d, "\sigma'_n"] \\ E_{n+1} \arrow[r, "f_{n+1}"] & E'_{n+1}. \end{tikzcd}$$

With the obvious composition of maps of prespectra, we denote $\mathscr{P}$ the *category of prespectra.*

The Suspension Prespectrum To each based space $X$, one can associate a prespectrum. We define $\Sigma^\infty X$ the *suspension prespectrum* of $X$, where the $n$-th term $(\Sigma^\infty X)_n$ is given by the $n$-th reduced suspension of $X$, namely $\Sigma^n X$, and its structure maps $\sigma_n : \Sigma(\Sigma^n X) \to \Sigma^{n+1} X$ are the obvious identity maps. For each based map $f : X \to Y$, we define $\Sigma^\infty f : \Sigma^\infty X \to \Sigma^\infty Y$ as the family of maps $\{\Sigma^n f : \Sigma^n X \to \Sigma^n Y \mid n \geq 0\}$. Obviously, we have the required commutativty of the following diagram for each $n \geq 0$ :

$$\begin{tikzcd} \Sigma(\Sigma^n X) \arrow[r, "\Sigma(\Sigma^n f)"] \arrow[d, equal, "\sigma_n"'] & \Sigma(\Sigma^n Y) \arrow[d, equal, "\sigma_n"] \\ \Sigma^{n+1} X \arrow[r, "\Sigma^{n+1} f"] & \Sigma^{n+1} Y, \end{tikzcd}$$

making $\Sigma^\infty f$ as a map of prespectra. Since compositions and identies are obviously conserved, we have defined a functor $\Sigma^\infty : \mathbf{Top}_* \to \mathscr{P}$, called the *suspension prespectrum functor.*

Example 2.2.2 (The Sphere Prespectrum $\mathbb{S}$).
A particular nice example is given by the 0-sphere $S^0$. Apply the suspension prespectrum functor $\Sigma^\infty S^0$ : its $n$-th term is homeomorphic to $S^n$. Subsequently, we denote the *sphere prespectrum* $\Sigma^\infty S^0$ by $\mathbb{S}$.

Definition 2.2.3 (Homotopy Groups of Prespectra).
Given a prespectrum $E$ and $n \geq 0$, we define its *n-th homotopy group* by :

$$\pi_n(E) = \mathrm{colim}_k \pi_{n+k}(E_k),$$

where the colimit is taken over the maps :

$$\pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma E_k) \xrightarrow{(\sigma_k)_*} \pi_{n+k+1}(E_{k+1}).$$

For convenience, we set $\pi_{-n}(E) := \mathrm{colim}_{k\geq n} \pi_{k-n}(E_k)$, for all $n \geq 0$.

In particular, for the suspension prespectrum $\Sigma^\infty X$ of a based space $X$, we get that $\pi_n(\Sigma^\infty X)$ equals the stable homotopy groups $\pi_n^S(X)$ of $X$ (see definition 1.2.5, page 8). Hence, for the sphere prespectrum $\mathbb{S}$, we have $\pi_n(\mathbb{S}) = \pi_n^S$ (see definition 1.2.6).
More generally, we have the following lemma, which states that homotopy groups of a prespectrum behave somehow as the stable homotopy groups.

Lemma 2.2.4.
*Let $n \geq 0$. For any prespectrum $E$, we have : $\pi_n(E) \cong \mathrm{colim}_k \pi_{n+k}^S(E_k)$.*

Proof : The colimit $\pi_n(E) = \mathrm{colim}_k \pi_{n+k}(E_k)$ is taken over :

$$\pi_n(E_0) \xrightarrow{(\sigma_0)_*\circ\Sigma} \pi_{n+1}(E_1) \xrightarrow{(\sigma_1)_*\circ\Sigma} \pi_{n+2}(E_2) \xrightarrow{(\sigma_2)_*\circ\Sigma} \cdots .$$

For any $k, j \geq 0$ we have the following factorization :

$$\begin{tikzcd} \pi_{n+k}(E_k) \arrow[rr] \arrow[dr] & & \pi_{n+k+j}(E_{k+j}) \\ & \pi_{n+k+j}(\Sigma^j E_k). \arrow[ur] & \end{tikzcd}$$

Hence we get that $\mathrm{colim}_k \pi_{n+k}(E_k) = \mathrm{colim}_k \pi_{n+k+j}(\Sigma^j E_k)$ for any $j \geq 0$. In particular we get that $\pi_n(E) = \mathrm{colim}_j(\mathrm{colim}_k \pi_{n+k+j}(\Sigma^j E_k))$. Using the interchange property of colimits (Theorem C.3.4), we have :

$$\mathrm{colim}_j(\mathrm{colim}_k \pi_{n+k+j}(\Sigma^j E_k)) \cong \mathrm{colim}_k(\mathrm{colim}_j \pi_{n+k+j}(\Sigma^j E_k)).$$

But $\mathrm{colim}_j \pi_{n+k+j}(\Sigma^j E_k) = \pi_{n+k}^S(E_k)$ for any $k \geq 0$, by the definition of stable homotopy groups (Definiton 1.2.5). Therefore we get : $\pi_n(E) \cong \mathrm{colim}_k \pi_{n+k}^S(E_k)$. $\square$

We have defined a functor $\pi_n : \mathscr{P} \to \mathbf{Ab}$, for all $n \in \mathbb{Z}$. Indeed, for a map of prespectra $f : E \to E'$, there is a unique group homomorphism $\pi_n(f) =: f_* : \pi_n(E) \to \pi_n(E')$, given by the universal property of colimits, and obviously $\pi_n$ preserves compositions and identites.

We introduce now well-behaved families of prespectra. The first is the $\Omega$-prespectra that will be fundamental for cohomology theories (see next section).

Definition 2.2.5 ($\Omega$-Prespectra).
An $\Omega$-prespectrum $E$ is a prespectrum such that the adjoint $\widetilde{\sigma}_n : E_n \to \Omega E_{n+1}$ of the structure map $\sigma_n$ is a weak equivalence, for each $n \geq 0$. If $E$ and $E'$ are $\Omega$-prespectra, then a *map of $\Omega$-prespectra* $f : E \to E'$ is a map of the underlying prespectra. We denote by $\Omega\mathscr{P}$ the *category of $\Omega$-prespectra.* It is a subcategory of $\mathscr{P}$.

A fundamental example is the following.

Example 2.2.6 (The Eilenberg-MacLane $\Omega$-prespectrum $HG$).
Let $G$ be an abelian group. Let us consider the family of Eilenberg-MacLane spaces $\{K(G,n)\}_{n\geq 0}$. Proposition 2.1.10 gives weak homotopy equivalences[1] $\widetilde{\sigma}_n : K(G,n) \xrightarrow{\sim} \Omega K(G, n+1)$, for each $n \geq 0$. Hence, we obtained an $\Omega$-prespectrum, denoted $HG$, called the *Eilenberg-MacLane $\Omega$-prespectrum for $G$*, where its $n$-th term is $(HG)_n = K(G,n)$.

[1] Actually homotopy equivalences since we required an Eilenberg-MacLane space to be of the homotopy type of a based CW-complex

Another crucial well-behaved family of prespectra is the family of CW-prespectra.

Definition 2.2.7 (CW-Prespectra).
A CW-*prespectrum* $E$ is a prespectrum such that each $n$-th term is a based CW-complex where its basepoint is a vertex, and such that the structure maps $\sigma_n : \Sigma E_n \hookrightarrow E_{n+1}$ are cellular inclusions. A *map of* CW-*prespectra* $f : E \to E'$ is a map of prespectra such that each $f_n : E_n \to E'_n$ is cellular. We denote by CW-$\mathscr{P}$ the *category of* CW-*prespectra*. It is a subcategory of $\mathscr{P}$.

For instance, for any based CW-complex $X$, its suspension prespectrum $\Sigma^\infty X$ is a CW-prespectrum. Hence we have a functor $\Sigma^\infty : \mathbf{CW}_* \longrightarrow$ CW-$\mathscr{P}$. Since we required the Eilenberg-MacLane spaces to be of the homotopy type of CW-complexes, we get that $HG$ is a CW-prespectrum for any abelian group $G$ (to get the inclusions, we will apply Lemma 2.2.10).

A natural question arises : is it possible to replace prespectra with CW-prespectra without altering their homotopy groups? In other words, is there some kind of a CW-approximation for prespectra? We first introduce the notion of weak equivalence for prespectra.

Definition 2.2.8 (Weak Equivalences).
A map of prespectra $f : E \to E'$ is called a *weak equivalence* if it induces isomorphisms $f_* : \pi_n(E) \xrightarrow{\cong} \pi_n(E')$, for all $n \geq 0$. We say that $E$ is *weakly equivalent* to $E'$.

Definition 2.2.9 (Levelwise Homotopy Equivalences).
A map of prespectra $f : E \to E'$ is called a *levelwise homotopy equivalence* if each map $f_n : E_n \to E'_n$ is a homotopy equivalence.

Lemma 2.2.10.
*Given a prespectrum $E$ where each $n$-th term is a* CW*-complex, and each structure maps $\sigma_n$ is cellular (but not necessarly an inclusion), there is a* CW*-prespectrum $T$ and a levelwise homotopy equivalence of prespectra $r : E \to T$.*

Proof : We proceed by defining each $T_n$ as a double mapping cylinder, as we did for the construction of the homotopy colimit in the proof of Theorem C.4.7. For each $n \geq 0$, define the mapping cylinder :

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\simeq"] \arrow[d, tail, "\sigma_n"'] & \Sigma E_n \times [n, n+1] \arrow[d, tail] \\ E_{n+1} \arrow[r, "\simeq"] & M_{n+1}, \end{tikzcd}$$

which is a CW-complex. Let $T_0 = E_0$, $T_1 = M_1$. Define the double mapping cylinder $T_{n+1}$ inductively as :

$$\begin{tikzcd} & \Sigma E_n \times \{n\} \arrow[r, hook] \arrow[d, tail, "\simeq"] & \Sigma T_n \arrow[d, tail] \\ E_{n+1} \arrow[r, "\simeq"] & M_{n+1} \arrow[r, "\simeq"] & T_{n+1}, \end{tikzcd}$$

which is again a CW-complex. We obtain for each $n \geq 1$ :

$$T_{n+1} = M_1 \coprod_{\Sigma E_1 \times \{1\}} M_2 \coprod_{\Sigma E_2 \times \{2\}} \cdots \coprod_{\Sigma E_{n-1} \times \{n-1\}} M_n,$$

and we obtain by construction cofivrations $\Sigma T_n \rightarrowtail T_{n+1}$ for each $n \geq 0$. Name $r_n : E_n \to T_n$

the homotopy equivalences obtained by construction. For each $n \geq 0$, we get :

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma r_n"] \arrow[d, "\sigma_n"'] & \Sigma T_n \arrow[d, hook] \\ E_{n+1} \arrow[r, "r_{n+1}"] & T_{n+1}, \end{tikzcd}$$

making a homotopy equivalence of prespectra $r : E \to T$. □

Proposition 2.2.11.
*Given a prespectrum $E$, there exists a weakly equivalent* CW*-prespectrum $T$.*

Proof : For each $n$-th term, there is a weakly equivalent CW-complex $E'_n \xrightarrow{\sim} E_n$, such that the structure maps $\sigma_n$ are cellular (we use that if $X$ is a CW-complex, then $\Sigma X$ is a CW-complex). In more details : there is a CW-approximation $E'_0 \xrightarrow{\sim} E_0$. Suppose that given a CW-approximation $E'_n \xrightarrow{\sim} E_n$ for some $n \geq 0$. By Theorem A.3.6, the CW-approximation $E'_{n+1} \xrightarrow{\sim} E_{n+1}$ can be given by the following commutative diagram :

$$\begin{tikzcd} \Sigma E'_n \arrow[r, "\sim"] \arrow[d, dashed, "\exists"'] & \Sigma E_n \arrow[d, "\sigma_n"] \\ E_{n+1} \arrow[r, dashed, "\sim"] & E_{n+1}. \end{tikzcd}$$

We then apply the previous lemma. □

Before introducing the main theorem of this part, we shall make a disgression on smash products.

Smash Products Let $(X, x_0)$ and $(Y, y_0)$ be based spaces. We define the *smash product* of $X$ and $Y$ by :

$$X \wedge Y = \frac{X \times Y}{X \vee Y} = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}.$$

The class $[(x, y)]$ is denoted $x \wedge y$, for any $(x, y)$ in $X \times Y$. The basepoint of $X \wedge Y$ is given by $x_0 \wedge y = x \wedge y_0 = x_0 \wedge y_0$, for any $x$ in $X$ and $y$ in $Y$. If $f : X \to X'$ and $g : Y \to Y'$ are based maps, the map $f \wedge g : X \wedge Y \to X' \wedge Y'$ is the unique map such that the following diagram commutes :

$$\begin{tikzcd} X \times Y \arrow[r, "f \times g"] \arrow[d, two heads] & X' \times Y' \arrow[d, two heads] \\ X \wedge Y \arrow[r, "f \wedge g"] & X' \wedge Y'. \end{tikzcd}$$

Notice that if $x_0 \in A \subseteq X$, then we have by Theorem C.3.4 :

$$(X/A) \wedge Y = \frac{X \times Y}{A \times Y \cup X \times \{y_0\}} \cong \frac{X \wedge Y}{A \wedge Y}. \tag{2.1}$$

Equality (2.1) shows that we already have encountered smash products before in particular cases. Noticing that $X \wedge I$ is the reduced cone of $CX$ and recalling the (based) homeomorphism $S^1 \cong I/\partial I$, we get that $X \wedge S^1 \cong \Sigma X$ from equality (2.1), for any based space $X$. More generally, we get $X \wedge S^n \cong \Sigma^n X$. Hence we have $S^n \wedge S^m \cong S^{n+m}$. In particular we have $X \wedge S^0 \cong X$ for any based space $X$.

We can give nice algebraic and geometric descriptions of the smash product. Algebraically, the smash product behaves as tensor product in the following way. Let us denote by $\mathrm{Fun}_*(X,Y)$ the subspace of $\mathrm{Map}(X,Y)$ consisting of the based maps, endowed with the constant based map as basepoint. There is a natural based homeomorphism (for $Y$ a locally compact Hausdorff space) :

$$\mathrm{Fun}_*(X \wedge Y, Z) \cong \mathrm{Fun}_*(X, \mathrm{Fun}_*(Y,Z)).$$

This is analogue to $\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B,C))$, where $A$, $B$ and $C$ are $R$-modules for $R$ a commutative ring. Geometrically, it is easy to show that if $X$ and $Y$ are based Hausdorff compact spaces, then the Alexandroff compactification (i.e. one-point compactification) of $(X - \{x_0\}) \times (Y - \{y_0\})$ is not given by $X \times Y$ but by $X \wedge Y$. Hence this gives another way to see that $S^n \wedge S^m \cong S^{n+m}$. We state, without a proof, the following properties, which hold generally (except commutativity) only for compactly generated based spaces[2] $X$, $Y$ and $Z$ (see [Whitehead, 1978] for a proof) :

Commutativity $X \wedge Y \cong Y \wedge X$,

Associativity $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$,

Distributivity $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$.

Let $X$ and $Y$ be based CW-complexes which are $r$-connected and $s$-connected respectively. Suppose that $X$ or $Y$ is locally compact. From the relative CW-structure of $(X \times Y, X \vee Y)$, the smash product $X \wedge Y$ is a $(r+s+1)$-connected CW-complex.

Returning to our context of prespectra, for any CW-prespectrum $E$ and based CW-complex $X$, we define $X \wedge E$ to be the prespectum with $n$-th term $(X \wedge E)_n$ given by $X \wedge E_n$, and with structure maps $\mathrm{id}_X \wedge \sigma_n : \Sigma(X \wedge E) = (X \wedge E_n) \wedge S^1 = X \wedge (E_n \wedge S^1) = X \wedge (\Sigma E_n) \longrightarrow X \wedge E_{n+1}$. For any based cellular map $f : X \to Y$, we obtain a family of based maps $\{f \wedge \mathrm{id}_{E_n} : X \wedge E_n \to Y \wedge E_n\}$ forming a map of prepsectra $f \wedge \mathrm{id} : X \wedge E \to Y \wedge E$. We have thus defined a functor :

$$- \wedge E : \mathbf{CW}_* \longrightarrow \mathscr{P}.$$

The following corollary from Blakers-Massey will be at the heart of the proof of our main result.

Theorem 2.2.12 (Another Corollary of Homotopy Excision).
*Let $i : A \rightarrowtail X$ be a cofibration and an $(n-1)$-equivalence between $(n-2)$-connected spaces, where $n \geq 2$. Then the quotient map $(X,A) \to (X/A, *)$ is a $(2n-2)$-equivalence, and it is a $(2n-1)$-equivalence if $A$ and $X$ are $(n-1)$-connected.*

Proof : It is a reformulation of Theorem 1.3.4, page 10. The proof follows from the commutativity of the following diagram :

$$\begin{tikzcd} (Mi, A) \arrow[r] \arrow[d, "\simeq"'] & (Ci, *) \arrow[d, "\simeq"] \\ (X, A) \arrow[r] & (X/A, *), \end{tikzcd}$$

where the top row unlabeled arrow is the quotient map from the reduced mapping cylinder to the reduced mapping cone. $\square$

We are now ready to state and prove the main result.

[2] Compact spaces, metric spaces, locally compact spaces, CW-complexes and manifolds are all examples of compactly generated spaces.

Theorem 2.2.13.
*Given a* CW*-prespectrum $E$, we define a functor for all $n \in \mathbb{Z}$ :*

$$\begin{array}{rcl} \widetilde{E}_n : \mathbf{CW}_* & \longrightarrow & \mathbf{Ab} \\ X & \longmapsto & \pi_n(X \wedge E). \end{array}$$

*Then the functors $\widetilde{E}_*$ define a generalized reduced homology theory on based* CW*-complexes.*

To prove Theorem 2.2.13, we first prove a particular case, where we choose the CW-prespectrum $E$ to be equal to the sphere prespectrum $\mathbb{S}$. This case is particularly interesting, since for any based CW-complex $X$, we get :

$$\widetilde{\mathbb{S}}_n(X) = \pi_n(X \wedge \mathbb{S}) = \pi_n(\Sigma^\infty X) = \pi_n^S(X).$$

In this case, $\widetilde{\mathbb{S}}_n$ is obviously a homotopy invariant functor. Notice that it makes perfect sense to write $\pi_{-n}^S(X)$ for $n \geq 0$. Let us show that $\pi_*^S$ is a reduced homology theory on CW-complexes.

Lemma 2.2.14.
*The stable homotopy groups $\pi_*^S : \mathbf{CW}_* \to \mathbf{Ab}$ defines a generalized reduced homology theory on based* CW*-complexes.*

Proof : Let us show that $\pi_*^S$ respects the Eilenberg-Steenrod axioms for a generalized reduced homology theory on CW-complexes.

Exactness Let $(X, A)$ be a relative CW-complex. We need to show the exactness of the following sequence, for any $n$ :

$$\pi_n^S(A) \longrightarrow \pi_n^S(X) \longrightarrow \pi_n^S(X/A). \qquad (\star)$$

The exact sequence of the pair $(X, A)$ gives the exactness of the following sequence for any $k \geq 0$ :

$$\pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X, A).$$

From Theorem 2.2.12, the isomorphism $\pi_k(X, A) \cong \pi_k(X/A)$ holds under connectivity assumptions that are achieved after sufficiently many suspensions. Recall that stable homotopy groups are obtained as a colimit $\pi_n^S(X) := \mathrm{colim}_k \pi_{n+k}(\Sigma^k X)$. By Theorem C.4.3, it follows that the sequence $(\star)$ is exact.

Suspension For any based CW-complex $X$, by the Freudenthal Theorem (Theorem 1.2.3), we have the isomorphism :

$$\begin{array}{rcl} \pi_n^S(X) & = & \mathrm{colim}_k \pi_{n+k}(\Sigma^k X) \\ & \cong & \mathrm{colim}_{k \geq 1} \pi_{n+k}(\Sigma^k X), \text{ using Proposition C.4.4,} \\ & = & \mathrm{colim}_k \pi_{n+k+1}(\Sigma^{k+1} X) \\ & = & \pi_{n+1}^S(\Sigma X), \end{array}$$

which is natural.

Additivity Let us prove that $\pi_n^S(\bigvee_{j \in \mathscr{J}} X_j) \cong \bigoplus_{j \in \mathscr{J}} \pi_n^S(X_j)$, where $X_j$ are based CW-complexes, for each $j$ in a index set $\mathscr{J}$. Let $X = \bigvee_{j \in \mathscr{J}} X_j$. Recall that $\pi_n^S(X) = \mathrm{colim}_k \pi_{n+k}(\Sigma^k X)$. From the claim of the proof of Theorem C.4.5, it suffices to work with a finite index set $\mathscr{J}$ when dealing with the groups $\pi_{n+k}(\Sigma^k X)$, for each $k \geq 0$, whence,

it suffices to work with a finite index set $\mathscr{J}$ for $\pi_n^S(X)$.
Let now $X$ and $Y$ be any based CW-complexes. By induction, it suffices to prove that $\pi_n^S(X \vee Y) \cong \pi_n^S(X) \bigoplus \pi_n^S(Y)$. Recall that $\Sigma(X \vee Y) \cong (\Sigma X) \vee (\Sigma Y)$, by distributivity of the smash product. For any $k \geq 0$, the CW-complexes $\Sigma^k X$ and $\Sigma^k Y$ are $k$-connected. Let us write $(\Sigma^k X \times \Sigma^k Y)^w$ their products endowed with the weak topology (which is not necessary equivalent to the product topology). Its $(2k-1)$-skeleton is given by $\Sigma^k X \vee \Sigma^k Y$. Hence the relative CW-complex $((\Sigma^k X \times \Sigma^k Y)^w, \Sigma^k X \vee \Sigma^k Y)$ is $(2k-1)$-connected. Therefore we get the isomorphism :

$$\pi_{n+k}(\Sigma^k X \vee \Sigma^k Y) \cong \pi_{n+k}((\Sigma^k X \times \Sigma^k Y)^w),$$

for $n+k < 2k-1$, i.e., $k > n+1$. But by Corollary C.4.6, we have that $\pi_{n+k}((\Sigma^k X \times \Sigma^k Y)^w)$ is isomorphic to $\pi_{n+k}(\Sigma^k X) \oplus \pi_{n+k}(\Sigma^k Y)$, for all $k$. Therefore, for $k > n+1$ we have the isomorphism :

$$\pi_{n+k}(\Sigma^k(X \vee Y)) \cong \pi_{n+k}(\Sigma^k X \vee \Sigma^k Y) \cong \pi_{n+k}(\Sigma^k X) \oplus \pi_{n+k}(\Sigma^k Y).$$

Passing to colimits over $k$, we get the desired isomorphism :

$$\pi_n^S(X \vee Y) \cong \pi_n^S(X) \oplus \pi_n^S(Y).$$

Therefore, $\pi_*^S$ defines a generalized reduced homology theory. $\square$

We now give the proof for Theorem 2.2.13 for any CW-prespectrum. It is a direct consequence of previous lemma and Lemma 2.2.4.

Proof of Theorem 2.2.13 : We start the proof by showing that $\widetilde{E}_n$ is indeed a homotopy invariant functor for all $n$. It is a functor since it is the composite of the functors :

$$\mathbf{CW}_* \xrightarrow{-\wedge E} \mathscr{P} \xrightarrow{\pi_n} \mathbf{Ab}.$$

If $X \to Y$ is a based homotopy equivalence of CW-complexes, then so is $X \wedge E_n \to Y \wedge E_n$, for any $n \geq 0$. Thus we get a homotopy equivalence of prespectra $X \wedge E \to Y \wedge E$ and so $\widetilde{E}_n(X) \cong \widetilde{E}_n(Y)$. Therefore $\widetilde{E}_n$ is homotopy invariant.
We next prove the Eilenberg-Steenrod axioms for reduced homology theories defined on $\mathbf{CW}_*$. Recall from Lemma 2.2.4 that :

$$\pi_n(X \wedge E) \cong \mathrm{colim}_k \pi_{n+k}^S(X \wedge E_k).$$

Exactness Let $(X, A)$ be a relative CW-complex. We need to show that the following sequence is exact, for every $n$ :

$$\widetilde{E}_n(A) \longrightarrow \widetilde{E}_n(X) \longrightarrow \widetilde{E}_n(X/A).$$

Since for all $k \geq 0$ : $(X \wedge E_k)/(A \wedge E_k) \cong (X/A) \wedge E_k$, this follows directly from exactness of $\pi_*^S$ :

$$\pi_{n+k}^S(A \wedge E_k) \longrightarrow \pi_{n+k}^S(X \wedge E_k) \longrightarrow \pi_{n+k}^S((X \wedge E_k)/(A \wedge E_k)),$$

and Theorem C.4.3.

SUSPENSION Let $X$ be any based CW-complex. From the suspension of $\pi_*^S$, we get for any $n$, the isomorphism :

$$\begin{aligned} \widetilde{E}_n(X) &\cong \operatorname{colim}_k \pi^S_{n+k}(X \wedge E_k) \\ &\cong \operatorname{colim}_k \pi^S_{n+1+k}(\Sigma(X \wedge E_k)) \\ &\cong \operatorname{colim}_k \pi^S_{n+1+k}((\Sigma X) \wedge E_k) \\ &\cong \widetilde{E}_{n+1}(\Sigma X), \end{aligned}$$

which is obviously natural.

ADDITIVITY Let $\{X_j\}_{j\in\mathscr{J}}$ be any collection of based CW-complexes. For any $n$, we have :

$$\begin{aligned} \widetilde{E}_n\left(\bigvee_{j\in\mathscr{J}} X_j\right) &\cong \operatorname{colim}_k \pi^S_{n+k}\left(\left(\bigvee_{j\in\mathscr{J}} X_j\right) \wedge E_k\right) \\ &\cong \operatorname{colim}_k \pi^S_{n+k}\left(\bigvee_{j\in\mathscr{J}} (X_j \wedge E_k)\right), \text{ by distributivity of } \wedge, \\ &\cong \operatorname{colim}_k \left(\bigoplus_{j\in\mathscr{J}} \pi^S_{n+k}(X_j \wedge E_k)\right), \text{ by additivity of } \pi^S_*, \\ &\cong \bigoplus_{j\in\mathscr{J}} \left(\operatorname{colim}_k \pi^S_{n+k}(X_j \wedge E_k)\right), \text{ by Theorem C.3.4}, \\ &\cong \bigoplus_{j\in\mathscr{J}} \widetilde{E}_n(X_j). \end{aligned}$$

Therefore, $\widetilde{E}_*$ is a generalized reduced homology theory. □

Let $\mathscr{H}$ denote the category of generalized reduced homology theories on $\mathbf{CW}_*$ where the morphisms are the transformations of reduced homology theories (see Appendix B).

ADDENDUM 2.2.15.
*The previous procedure of associating to each* CW*-prespectrum a generalized reduced homology theory defines a functor* $F : \text{CW-}\mathscr{P} \longrightarrow \mathscr{H}$.

PROOF : Let us first define how $F$ assigns morphisms. Let $f : E \to E'$ be a map of CW-prespectra. We obtain a map of prepsectra $\mathrm{id}_X \wedge f$ which is the collection of the maps $\mathrm{id}_X \wedge f_n : X \wedge E_n \to X \wedge E'_n$. Recall that $\pi_n : \mathscr{P} \to \mathbf{Ab}$ is a functor. We thus obtain an abelian group homomorphism $(\mathrm{id}_X \wedge f)_* : \pi_n(X \wedge E) \to \pi_n(X \wedge E')$. We have determined natural transformations $\{T_n : \widetilde{E}_n = \pi_n(- \wedge E_n) \Rightarrow \pi_n(- \wedge E'_n) = \widetilde{E}'_n\}$. To prove that the induced transformation $T : \widetilde{E}_* \to \widetilde{E}'_*$ is a transformation of reduced homology theories, we must prove that the following diagram commutes :

$$\begin{tikzcd} \pi_n(X \wedge E) \arrow[r, "\Sigma"] \arrow[d, "(\mathrm{id}_X \wedge f)_*"'] & \pi_n(\Sigma X \wedge E_n) \arrow[d, "(\mathrm{id}_{\Sigma X} \wedge f)_*"] \\ \pi_n(X \wedge E') \arrow[r, "\Sigma"] & \pi_n(\Sigma X \wedge E'). \end{tikzcd}$$

But this follows easely from the definition of the suspension homomorphism $\Sigma$ defined in the proof of the previous theorem. Moreover, $F$ obviously preserves compositions and identites. Therefore $F$ is a functor. □

Corollary 2.2.16.
*Let $G$ be an abelian group. The Eilenberg-MacLane $\Omega$-prespectrum $HG$ defines an ordinary reduced homology $\widetilde{HG}_*$.*

Proof : We have to verify the dimension axiom. Recall that for any based space $X$, we have the homeomorphism $S^0 \wedge X \cong X$. For any $n$ in $\mathbb{Z}$ :

$$\widetilde{HG}_n(S^0) = \pi_n(S^0 \wedge HG) = \mathrm{colim}_k \pi_{n+k}(S^0 \wedge K(G,k)) \cong \mathrm{colim}_k \pi_{n+k}(K(G,k)).$$

If $n \geq 0$, then $\pi_{n+k}(K(G,k)) = 0$. If $n = 0$, then $\pi_k(K(G,k)) = G$ for any $k$. Therefore :

$$\widetilde{HG}_n(S^0) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\widetilde{HG}_*$ respects the dimension axiom. □

We now investigate the relations between the homotopic description of the integer homology by Dold-Thom :

$$\widetilde{H}_n(X; \mathbb{Z}) = \pi_{n+1}(\mathrm{SP}(\Sigma X)),$$

for any $n$ and any CW-complex $X$, and our previous homotopy description with the Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$. Recall that the spaces $\mathrm{SP}(S^k)$ are Eilenberg-MacLane spaces of type $(\mathbb{Z}, k)$, whence :

$$\widetilde{H\mathbb{Z}}_n(X) = \mathrm{colim}_k \pi_{n+k}(X \wedge \mathrm{SP}(S^k)),$$

for all $n$ and all $X$. Moreover, there is a map, natural in $X$ :

$$\begin{array}{rcl} X \wedge \mathrm{SP}(S^k) & \longrightarrow & \mathrm{SP}(S^k \wedge X) \\ x \wedge [(s_1, s_2, \ldots)] & \longmapsto & [(x \wedge s_1, x \wedge s_2, \ldots)]. \end{array}$$

It induces a natural homomorphism on $X$ for all $n$ and $k$ :

$$\pi_{n+k}(X \wedge \mathrm{SP}(S^k)) \longrightarrow \pi_{n+k}(\mathrm{SP}(S^k \wedge X)).$$

Using repeatedly the Dold-Thom Theorem, we obtain : $\pi_{n+k}(\mathrm{SP}(S^k \wedge X)) \cong \pi_{n+1}(\mathrm{SP}(\Sigma X))$. Passing to colimit over $k$, we obtain a natural homomorphism on $X$ for all $n$ :

$$\widetilde{H\mathbb{Z}}_n(X) \longrightarrow \pi_{n+1}(\mathrm{SP}(\Sigma X)).$$

This defines a transformation of reduced homology theories, as the suspension homomorphisms are obviously compatible. Applying Theorem 3.2.13, we see that this transformation is in fact an equivalence of reduced homology theories.

We end this section by the following corollary, which tells us how to associate a generalized reduced homology theory from a prespectrum.

Corollary 2.2.17.
*A prespectrum $E$ determines a unique generalized reduced homology theory on based CW-complexes.*

Proof : From Proposition 2.2.11, there exists a weak equivalence $T \to E$, such that $T$ is a CW-prespectrum, though $T$ is not unique. Suppose we have another weak equivalence $T' \to E$ where $T'$ is a CW-prespectrum. Then we obtain a weak equivalence $r_n : T_n \to T'_n$, for each $n$. Therefore we get a homotopy equivalence $T_n \to T'_n$, and so a homotopy equivalence $X \wedge T_n \to X \wedge T'_n$, for all based CW-complexes, for each $n$. Therefore $\widetilde{T}_n(X) \cong \widetilde{T}'_n(X)$ for all $X$ and $n$, and we can set : $\widetilde{E}_n(X) = \widetilde{T}_n(X) \cong \widetilde{T}'_n(X)$. □

## 2.3. Ω-Prespectra and Generalized Cohomology Theories

We have seen how prespectra define generalized reduced homology theories. We are now interested in generalized cohomology theories.

Suppose we have defined an Ω-prespectrum $E = \{E_n\}$ only for $n \geq 1$. The omitted term $E_0$ can be reconstructed from the remaining terms of the Ω-prespectrum. We need a weak homotopy equivalence $E_0 \xrightarrow{\sim} \Omega E_1$, given for instance by a CW-approximation. Hence we can extend the definition of a Ω-prespectrum where we allow negative terms : $\{E_n\}_{n\in\mathbb{Z}}$ where $E_{-n} = \Omega^n E_0$, for all $n \geq 0$. We have the identites $\widetilde{\sigma}_{-n} : E_{-n} \to \Omega E_{-n+1}$ as the adjoints of the structure maps for the negative terms. For instance, if we extend the Eilenberg-MacLane Ω-prespectrum $HG$, its negative terms are $(HG)_{-n} = *$ for any $n \geq 1$, since $K(G,0)$ is discrete, for any abelian group $G$.

Theorem 2.3.1.
*Given an Ω-prespectrum $E$, we define a contravariant functor for all $n \in \mathbb{Z}$ :*

$$\begin{aligned} \widetilde{E}^n : \mathbf{CW}_* &\longrightarrow \mathbf{Ab} \\ X &\longmapsto [X, E_n]_*. \end{aligned}$$

*Then the functors $\widetilde{E}^*$ define a generalized reduced cohomology theory on based* CW*-complexes.*

Proof : Since $[-,Z]_* : \mathbf{Top}_* \to \mathbf{Set}$ is a contravariant homotopy invariant functor for any based space $Z$, so is $\widetilde{E}^n := [-, E_n]_*$, for each $n$ in $\mathbb{Z}$. We need to argue that $\widetilde{E}^n$ admits an abelian group structure. For each $n$ in $\mathbb{Z}$, there is a weak homotopy equivalence $E_n \xrightarrow{\sim} \Omega^2 E_{n+2}$. Hence from Whitehead Theorem, the bijection between $[X, E_n]_*$ and $[X, \Omega^2 E_{n+2}]_* \cong [\Sigma X, \Omega E_{n+2}]_*$ endows $[X, E_n]_*$ with an abelian group structure. Therefore $\widetilde{E}^n : \mathbf{CW}_* \longrightarrow \mathbf{Ab}$ is indeed a contravariant homotopy invariant functor. We now prove the Eilenberg-Steenrod axioms for a reduced cohomology theory on based CW-complexes.

Exactness Let $(X, A)$ be a relative CW-complex. Since $A$ is closed in $X$, there is a homeomorphism between $X/A$ and the reduced mapping cone $Ci$, where $i : A \hookrightarrow X$ is the inclusion. Hence, for any fixed based space $Z$, from Baratt-Puppe, we obtain an exact sequence (a priori in $\mathbf{Set}_*$) :

$$[X/A, Z]_* \longrightarrow [X, Z]_* \longrightarrow [A, Z]_*.$$

Specifying $Z = E_n$, we have the desired exact sequence for each $n$ in $\mathbb{Z}$ :

$$\widetilde{E}^n(X/A) \longrightarrow \widetilde{E}^n(X) \longrightarrow \widetilde{E}^n(A).$$

Suspension We have the desired natural isomorphism for each $n$ in $\mathbb{Z}$ :

$$\widetilde{E}^n(X) := [X, E_n]_* \xrightarrow[\cong]{(\widetilde{\sigma}_n)_*} [X, \Omega E_{n+1}]_* \cong [\Sigma X, E_{n+1}]_* =: \widetilde{E}^{n+1}(\Sigma X).$$

Additivity A based map $\bigvee_{j\in\mathscr{J}} X_j \to Z$ is equivalent to a collection $\{X_j \to Z\}_{j\in\mathscr{J}}$ of based maps, for any based space $Z$. Therefore we obtain for all $n$ in $\mathbb{Z}$ :

$$\begin{aligned} \widetilde{E}^n\left(\bigvee_{j\in\mathscr{J}} X_j\right) &= [\bigvee_{j\in\mathscr{J}} X_j, E_n]_* \\ &\cong \prod_{j\in\mathscr{J}} [X_j, E_n]_* \\ &= \prod_{j\in\mathscr{J}} \widetilde{E}^n(X_j). \end{aligned}$$

Thus $\widetilde{E}^*$ is a generalized reduced cohomology theory on based CW-complexes. □

Let $co\mathscr{H}$ denotes the category of generalized reduced cohomology theories on $\mathbf{CW}_*$ where the morphisms are the transformations of reduced cohomology theories (see Appendix B).

ADDENDUM 2.3.2.
*The previous procedure of associating to each $\Omega$-prespectrum a generalized reduced cohomology theory defines a functor $G : \Omega\mathscr{P} \longrightarrow co\mathscr{H}$.*

PROOF : We first need to specify how $G$ assigns morphisms of $\Omega$-prespectra. So let $f : E \to E'$ be a map of prepsectra. The following diagram commutes for each $n$ :

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma {E'}_n \arrow[d, "\sigma'_n"] \\ E_{n+1} \arrow[r, "f_{n+1}"] & E'_{n+1}. \end{tikzcd}$$

For each based CW-complex $X$, the map $f_n$ induces a homomorphism natural in $X$ :

$$\widetilde{E}^n(X) = [X, E_n]_* \xrightarrow{(f_n)_*} [X, E'_n]_* = \widetilde{E}'^n(X),$$

i.e., a natural transformation $G(f_n) : \widetilde{E}^n \Rightarrow \widetilde{E}'^n$, for each $n$. Let $G(f) : \widetilde{E}^* \to \widetilde{E}'^*$ be the collection of these natural transformations. In order to prove that $G$ is a functor, we must show that $G(f)$ is a transformation of reduced cohomology theories, i.e., that the following diagram commutes :

$$\begin{tikzcd} \widetilde{E}^n(X) \arrow[r, "\Sigma"] \arrow[d, "G(f_n)_X"'] & \widetilde{E}^n(\Sigma X) \arrow[d, "G(f_{n+1})_{\Sigma X}"] \\ \widetilde{E}'^n(X) \arrow[r, "\Sigma"] & \widetilde{E}'^n(\Sigma X), \end{tikzcd}$$

for every based CW-complex $X$, and each $n$, where $\Sigma$ is the suspension homomorphism defined in the proof of the suspension axiom of Theorem 2.3.1. The desired commutativity follows directly from the commutativity of the following diagram :

$$\begin{tikzcd}
{[X, E_n]_*} \arrow[r, "(\widetilde{\sigma_n})_*"] \arrow[dr, "\Sigma"'] \arrow[ddd, "(f_n)_*"'] & {[X, \Omega E_{n+1}]_*} \arrow[r, "\cong"] & {[\Sigma X, E_{n+1}]_*} \arrow[ddd, "(f_{n+1})_*"] \\
& {[\Sigma X, \Sigma E_n]_*} \arrow[ur, "(\sigma_n)_*"'] \arrow[d, "(\Sigma f_n)_*"] & \\
& {[\Sigma X, \Sigma E'_n]_*} \arrow[dr, "(\sigma'_n)_*"] & \\
{[X, E'_n]_*} \arrow[r, "(\widetilde{\sigma'_n})_*"'] \arrow[ur, "\Sigma"] & {[X, \Omega E'_{n+1}]_*} \arrow[r, "\cong"'] & {[\Sigma X, E'_{n+1}]_*.}
\end{tikzcd}$$

The two triangles of the above diagram commutes by adjoint relation. Moreover, it is clear that this way of defining $G$ preserves compositions and identity. Therefore $G$ is a functor. ☐

Corollary 2.3.3.
*Let $G$ be an abelian group. The Eilenberg-MacLane $\Omega$-prespectrum $HG$ defines an ordinary reduced cohomology $\widetilde{HG}^*$.*

Proof : We have $\widetilde{HG}^n(S^0) = [S^0, K(G,n)]_* = \pi_0(K(G,n))$, for any $n$. Thus :

$$\widetilde{HG}^n(S^0) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\widetilde{HG}^*$ respects the dimension axiom. ☐

We have seen how to define generalized cohomology theories from $\Omega$-prespectra, but one may ask if it possible for to define cohomology from prespectra? Let $E$ be a prespectrum (possibly replaced by weakly equivalent CW-prespectrum). Recall that $E$ defines reduced homology as $\widetilde{E}_n(X) = \pi_n(X \wedge E)$, i.e. :

$$\widetilde{E}_n(X) = \mathrm{colim}_k [S^{n+k}, X \wedge E_k]_*,$$

where the colimit is taken over :

$$[S^{n+k}, X \wedge E_k]_* \xrightarrow{(\sigma_k)_* \circ \Sigma} [S^{n+k+1}, X \wedge E_{k+1}]_*.$$

So one would try to define a reduced cohomology theory as :

$$\widetilde{E}^n(X) = \mathrm{colim}_k [S^k \wedge X, E_{n+k}]_* \cong \mathrm{colim}_k [\Sigma^k X, E_{n+k}]_*,$$

where the colimit is taken over :

$$[\Sigma^k X, E_{n+k}]_* \xrightarrow{(\sigma_{n+k})_* \circ \Sigma} [\Sigma^{k+1} X, E_{n+k+1}]_*.$$

The iterated reduced suspensions give an abelian group structure to $\widetilde{E}^n(X)$, and $\widetilde{E}^n$ is clearly a homotopy invariant functor. As we saw before in the proof of the previous theorem, the general properties of the functor $[-, E_{n+k}]_*$ together with the properties of colimits will prove that $\widetilde{E}^*$ will respect exactness and suspension axiom. However, the additivity need not to be respected : this is mainly due to the fact that products, which are limits, do not interchange with colimits in general. Nevertheless, it is interesting to see the connections with $\Omega$-prespectra. Suppose now that $E$ is a $\Omega$-prespectrum. We get for any based CW-complex $X$ :

$$\begin{aligned}\widetilde{E}^n(X) &= \mathrm{colim}_k[\Sigma^k X, E_{n+k}]_* \\ &\cong \mathrm{colim}_k[X, \Omega^k E_{n+k}]_* \\ &\cong \mathrm{colim}_k[X, E_n]_* \\ &= [X, E_n]_*.\end{aligned}$$

Therefore we recover our previous definition of reduced cohomology theory defined from a $\Omega$-prespectrum.
It is natural now to ask if for any prespectrum, there is a $\Omega$-prespectrum associated to any prespectrum, such that they define the same reduced homology theory. Recall the homotopy colimit $\mathrm{hocolim}_n Z_n$ construction of a telescope $Z_0 \to Z_1 \to Z_2 \to \cdots$ that we have done in the proof of Theorem C.4.7.

Lemma 2.3.4.
*For any telescope $Z_0 \to Z_1 \to Z_2 \to \cdots$ of based spaces, the natural map :*

$$\mathrm{hocolim}_n \Omega Z_n \longrightarrow \Omega(\mathrm{hocolim}_n Z_n),$$

*is a weak equivalence.*

Proof : Omitted, exercise found in [Hatcher, 2002], Section 4.F, exercise 3. □

Proposition 2.3.5.
*For any CW-prespectrum $E$, there is a related $\Omega$-prespectrum $T$, such that $T$ and $E$ share the same homotopy groups.*

Sketch of the Proof : Let $\{\widetilde{\sigma}_n\}$ be the adjoints of the structure maps of $E$. We define the telescope for each $n \geq 0$ :

$$E_n \xrightarrow{\widetilde{\sigma}_n} \Omega E_{n+1} \xrightarrow{\Omega\widetilde{\sigma}_{n+1}} \Omega^2 E_{n+2} \xrightarrow{\Omega^2\widetilde{\sigma}_{n+2}} \cdots .$$

Let $T_n := \mathrm{hocolim}_k \Omega^k E_{n+k}$. Explicitly, we have the mapping cylinder as a pushout for all $k$ :

$$\begin{tikzcd} \Omega^k E_{n+k} \arrow[r] \arrow[d, "\Omega^k\widetilde{\sigma}_{n+k}"'] & \Omega^k E_{n+k} \times [k, k+1] \arrow[d] \\ \Omega^{k+1} E_{n+k+1} \arrow[r] & M_{k+1}. \end{tikzcd}$$

Define the double mapping cylinder of $\widetilde{\sigma}_k$ as : $Y_0 = E_n \times \{0\}$, $Y_1 = M_1$ and for $k \geq 1$ :

$$Y_{k+1} = M_1 \coprod_{\Omega E_{n+1}\times\{1\}} M_2 \coprod_{\Omega^2 E_{n+2}\times\{2\}} \cdots \coprod_{\Omega^{k-1} E_{n+k-1}\times\{n-1\}} M_k.$$

We obtain homotopy equivalences : $Y_k \longrightarrow \Omega^k E_{n+k}$, for each $k$. Therefore :

$$T_n := \mathrm{hocolim}_k \Omega^k E_{n+k} = \mathrm{colim}_k Y^k.$$

Let us prove now that $T = \{T_n\}$ is a $\Omega$-prespectrum. We need to define the adjoints of the its structure maps. Proposition C.4.4 gives a homotopy equivalence

$$\mathrm{hocolim}_k \Omega^k E_{n+k} \xrightarrow{\simeq} \mathrm{hocolim}_k \Omega^{k+1} E_{n+k+1}.$$

Then the adjoints of the structure maps are given by the composite :

$$T_n = \mathrm{hocolim}_k \Omega^k E_{n+k} \xrightarrow{\simeq} \mathrm{hocolim}_k \Omega^{k+1} E_{n+k+1} \xrightarrow{\sim} \Omega(\mathrm{hocolim}_k \Omega^k E_{n+k+1}) = \Omega T_{n+1}$$

where the weak equivalence stems from Lemma 2.3.4. Therefore $T$ is a $\Omega$-prespctrum.
For all $r$ and $n$, we obtain :

$$\begin{aligned} \pi_r(T_n) &= \pi_r(\mathrm{hocolim}_k \Omega^k E_{n+k}) \\ &= \pi_r(\mathrm{colim}_k Y^k) \\ &\cong \mathrm{colim}_k \pi_r(Y_k), \text{ using Theorem C.4.5,} \\ &\cong \mathrm{colim}_k \pi_r(\Omega^k E_{n+k}) \\ &\cong \mathrm{colim}_k \pi_{r+k}(E_{n+k}) \\ &\cong \mathrm{colim}_k \pi_{r-n+k}(E_k), \text{using Propostion C.4.4,} \\ &= \pi_{r-n}(E). \end{aligned}$$

We therefore obtain :

$$\begin{aligned} \pi_n(T) &= \mathrm{colim}_k \pi_{n+k}(T_k) \\ &\cong \mathrm{colim}_k \pi_n(E) \\ &= \pi_n(E). \end{aligned}$$

Therefore $E$ and $T$ share the same homotopy groups. □

# CHAPTER 3

# Brown Representability Theorem

In the previous chapter, we have seen that each $\Omega$-prespectrum determines a reduced cohomology theory. This chapter is devoted to prove the converse : each reduced cohomology theory defines an $\Omega$-prespectrum. This will stem from the fact that every reduced cohomology theory is *representable*, in a sense to be defined. Actually we will carry out our work and prove the Brown Representability Theorem, which identifies which contravariant homotopy functors defined on the category of based CW-complexes $\mathbf{CW}_*$ are representable. We follow [Kochman, 1996], [Hatcher, 2002] and [Aguilar et al., 2002].

## 3.1. Representable Functors

We begin by defining what representable means in the categorical sense. Our result will be stated for the contravariant case. However, all the work in this section can be dualized for covariant functors.

Let $\mathscr{C}$ be a *locally small category*, i.e., a category such that for any object $C$ and $C'$ in $\mathscr{C}$, the class of morphisms $\mathscr{C}(C, C')$ is a set. Let $C_0$ be a fixed object of $\mathscr{C}$. We define the contravariant functor :

$$\begin{aligned} \mathscr{C}(-, C_0) : \mathscr{C} &\longrightarrow \mathbf{Set} \\ C &\longmapsto \mathscr{C}(C, C_0) \\ C \xrightarrow{f} C' &\longmapsto f^* : \mathscr{C}(C', C_0) \to \mathscr{C}(C, C_0), \end{aligned}$$

where $f^*(\varphi) = \varphi \circ f$, for any $\varphi$ in $\mathscr{C}(C', C_0)$.

Definition 3.1.1 (Representable Contravariant Functor).
Let $\mathscr{C}$ be a locally small category. A contravariant functor $F : \mathscr{C} \to \mathbf{Set}$ is said to be *representable* if there is an object $C_0$ in $\mathscr{C}$ and a natural isomorphism :

$$e : \mathscr{C}(-, C_0) \Rightarrow F.$$

We say that $C_0$ *represents* $F$, and $C_0$ is a *classifying object* for $F$.

The following lemma, known as the Yoneda Lemma, relates natural transformations $e : \mathscr{C}(-, C_0) \Rightarrow F$ with elements of $F(C_0)$.

Lemma 3.1.2 (Yoneda).
*Let $\mathscr{C}$ be a locally small category. Let $F : \mathscr{C} \to \mathbf{Set}$ be a contravariant functor. For any object $C_0$ in $\mathscr{C}$, there is a one-to-one correspondance between natural transformation $e : \mathscr{C}(-, C_0) \Rightarrow F$ and elements $u$ in $F(C_0)$, which is given, for any object $C$ in $\mathscr{C}$, by :*

$$\begin{aligned} e_C : \mathscr{C}(C, C_0) &\longrightarrow F(C) \\ \varphi &\longmapsto F(\varphi)(u). \end{aligned}$$

Proof : Suppose we are given a natural transformation $e : \mathscr{C}(-, C_0) \Rightarrow F$. In particular, for any morphism $\varphi$ in $\mathscr{C}(C, C_0)$, the following diagram commutes :

$$\begin{tikzcd} \mathscr{C}(C_0, C_0) \arrow[r, "e_{C_0}"] \arrow[d, "\varphi^*"'] & F(C_0) \arrow[d, "F(\varphi)"] \\ \mathscr{C}(C, C_0) \arrow[r, "e_C"] & F(C). \end{tikzcd}$$

Evaluating with the identity morphism $\mathrm{id}_{C_0}$, we obtain an element $u = e_{C_0}(\mathrm{id}_{C_0})$ in $F(C_0)$; and commutativity of the previous diagram gives : $e_C(\varphi) = F(\varphi)(u)$.
Conversely, if we are given $u$ in $F(C_0)$, define $e_C : \mathscr{C}(C, C_0) \longrightarrow F(C)$ as before, for all objects $C$. Naturality follows directly. $\square$

The Yoneda Lemma leads to the following definition.

Definition 3.1.3 (Universal Elements).
Let $\mathscr{C}$ be a locally small category. If $F : \mathscr{C} \to \mathbf{Set}$ is a representable contravariant functor, given a natural isomorphism $e : \mathscr{C}(-, C_0) \Rightarrow F$, the associated element according to the Yoneda Lemma $u_F := e_{C_0}(\mathrm{id}_{C_0}) \in F(C_0)$ is called the *universal element* of $F$.

Notice that we said «the» universal element. This suggests some kind of relations between universal elements, and so between classifying objects. We start by the following proposition, that will be crucial subsequently.

Proposition 3.1.4.
*Let $\mathscr{C}$ be a locally small category. Let $F, G : \mathscr{C} \to \mathbf{Set}$ be contravariant functors represented by $C_0$ and $C_0'$ with natural isomorphisms $e : \mathscr{C}(-, C_0) \Rightarrow F$ and $e' : \mathscr{C}(-, C_0') \Rightarrow G$. If there is a natural transormation $\kappa : F \Rightarrow G$, then there exists a unique morphism $\rho : C_0 \to C_0'$ such that the following diagram commutes :*

$$\begin{tikzcd} \mathscr{C}(C, C_0) \arrow[r, "\rho_*"] \arrow[d, "e_C"', "\cong"] & \mathscr{C}(C, C_0') \arrow[d, "e'_C", "\cong"'] \\ F(C) \arrow[r, "\kappa_C"] & G(C), \end{tikzcd} \tag{3.1}$$

*for any object $C$ in $\mathscr{C}$. Moreover, if $\kappa$ is a natural isomorphism, then $\rho$ is an isomorphism in $\mathscr{C}$.*

Proof : Let us first define $\rho : C_0 \to C_0'$. The universal element of $F$ is given by : $u_F = e_{C_0}(\mathrm{id}_{C_0}) \in F(C_0)$. Taking its image with $\kappa_{C_0}$, we obtain an element $\kappa_{C_0}(u_F)$ in $G(C_0)$. Since $e'_{C_0} : \mathscr{C}(C_0, C_0') \to G(C_0)$ is a bijection, there is a unique element $\rho$ in $\mathscr{C}(C_0, C_0')$, such that $e'_{C_0}(\rho) = \kappa_{C_0}(u_F)$.
We now prove the commutativity of the diagram (3.1). Let $\varphi$ be any morphism in $\mathscr{C}(C, C_0)$.

On the one hand we have :

$$\begin{array}{rcl} \kappa_C \circ e_c(\varphi) & = & \kappa_C F(\varphi)(u_F), \text{ by Yoneda Lemma,} \\ & = & G(\varphi)\kappa_{C_0}(u_F), \text{ by naturality of } \kappa, \\ & = & G(\varphi)e'_{C_0}(\rho), \text{ by definition of } \rho, \end{array}$$

and on the other hand, we have :

$$\begin{array}{rcl} e'_C \circ \rho_*(\varphi) & = & e'_C(\rho \circ \varphi), \\ & = & G(\rho \circ \varphi)(u_G), \text{ by Yoneda Lemma,} \\ & = & G(\varphi) \circ G(\rho)(u_G) \\ & = & G(\varphi)e'_{C_0}(\rho), \text{ by Yoneda Lemma.} \end{array}$$

We have just proved that the commutativity of diagram (3.1).
Uniqueness of $\rho$ follows immediately from its construction since $\rho$ is the unique morphism making the diagram commute in the case $C = C_0$.

Let $\kappa$ be a natural isomorphism. This means that for any object $C$ in $\mathscr{C}$, the morphism $\kappa_C : F(C) \to G(C)$ is bijective. So there exists an inverse $\kappa_C^{-1} : G(C) \to F(C)$, for each object $C$. This obviously defines a natural transformation $\overline{\kappa} : G \Rightarrow F$, where $\overline{\kappa_C} = \kappa_C^{-1}$. Applying the first part of this proof, there is a unique morphism $\overline{\rho} : C'_0 \to C_0$ corresponding to $\overline{\kappa}$. Moreover, we have $\overline{\kappa_C} \circ \kappa_C = \mathrm{id}_{F(C)}$ and $\kappa_C \circ \overline{\kappa_C} = \mathrm{id}_{G(C)}$, for every object $C$. But these composites of natural transformations correspond respectively to $\overline{\rho} \circ \rho$ and $\rho \circ \overline{\rho}$. By uniqueness, we obtain $\overline{\rho} \circ \rho = \mathrm{id}_{C_0}$ and $\rho \circ \overline{\rho} = \mathrm{id}_{C'_0}$, and so $\rho$ is an isomorphism. □

We can now prove that classyfing objects are unique up to isomorphism.

Corollary 3.1.5.
*Let $\mathscr{C}$ be a locally small category. Let $F : \mathscr{C} \to \mathbf{Set}$ be a representable contravariant functor. If $C_0$ and $C'_0$ are representing objects of $F$ with universal elements $u_F$ and $u'_F$ resepctively, then there is an isomorphism $\rho : C_0 \to C'_0$ in $\mathscr{C}$ such that $F(\rho)(u'_F) = u_F$.*

Proof : There are natural isomorphisms $e : \mathscr{C}(-, C_0) \Rightarrow F$ and $e' : \mathscr{C}(-, C'_0) \Rightarrow F$. Taking the composites, we obtain another natural transformation : $\lambda := e'^{-1} \circ e : \mathscr{C}(-, C_0) \Rightarrow \mathscr{C}(-, C'_0)$, which is obviously a natural isomorphism. By Proposition 3.1.4, $\lambda$ determines a unique isomorphism $\rho : C_0 \to C'_0$, such that $\lambda_C(f) = \rho \circ f$, for any object $C$ and morphism $f : C \to C_0$. In particular $\lambda_{C_0}(\mathrm{id}_{C_0}) = \rho$.
The universal elements $u_F$ and $u'_F$ are given respectively by $e_{C_0}(\mathrm{id}_{C_0})$ and $e'_{C'_0}(\mathrm{id}_{C'_0})$. We get :

$$\begin{array}{rcl} F(\rho)(u'_F) & = & F(\rho) \circ e'_{C_0}(\mathrm{id}_{C'_0}) \\ & = & e'_{C_0}(\rho), \text{ by naturality of } e', \\ & = & e'_{C_0} \circ \lambda_{C_0}(\mathrm{id}_{C_0}) \\ & = & e_{C_0}(\mathrm{id}_{C_0}), \text{ since } e' \circ \lambda = e, \\ & = & u_F, \end{array}$$

and so $F(\rho)(u'_F) = u_F$ as desired. □

## 3.2. Brown Functors and The Representability Theorem

Let us write $h\mathbf{Top}_*$ the homotopy category with based spaces as objects, and based homotopy classes of based maps as morphisms. For any based spaces $X$ and $Y$, we have written $h\mathbf{Top}_*(X,Y)$ as $[X,Y]_*$. Therefore, a homotopical invariant (contravariant) functor $\mathbf{Top}_* \to \mathscr{C}$ is equivalent (i.e. there is a natural isomorphism) to a (contravariant) functor $h\mathbf{Top}_* \to \mathscr{C}$, for any category $\mathscr{C}$.

We introduce the following useful notation.

Notation 3.2.1 ($x|_A$).
Let $\mathscr{C}$ be any category. Let $h : \mathbf{Top}_* \to \mathscr{C}$ be a contravariant functor. For any spaces $* \in A \subseteq X$, we denote by $x|_A$ the image $h(j)(x) \in h(A)$ of an element $x$ in $h(X)$, where $j : A \hookrightarrow X$ denotes the inclusion.

Definition 3.2.2 (Brown Functor).
Let $\mathscr{T}$ be a full subcategory of $\mathbf{Top}_*$. A *Brown functor* $h : \mathscr{T} \to \mathbf{Set}$ is a contravariant homotopy functor, which respects the following axioms.

Additivity [1] For any collection $\{X_j \mid j \in \mathscr{J}\}$ of based spaces in $\mathscr{T}$, the inclusion maps $i_j : X_j \hookrightarrow \bigvee_{j\in\mathscr{J}} X_j$ induce an isomorphism on $\mathbf{Set}$ :

$$(h(i_j))_{j\in\mathscr{J}} : h\left(\bigvee_{j\in\mathscr{J}} X_j\right) \xrightarrow{\cong} \prod_{j\in\mathscr{J}} h(X_j).$$

Mayer-Vietoris For any excisive triad $(X; A, B)$ in $\mathscr{T}$, if $a$ is in $h(A)$, and $b$ is in $h(B)$, such that $a|_{A\cap B} = b|_{A\cap B}$, then there exists $x$ in $h(X)$, such that $x|_A = a$ and $x|_B = b$.

We first state and prove some properties of Brown functors.

Lemma 3.2.3.
*Let $\mathscr{T}$ be a full subcategory of $\mathbf{Top}_*$. Let $h : \mathscr{T} \to \mathbf{Set}$ be a Brown functor. Then $h(*)$ is a set that consists of a single element, and $h$ corestricts to the category of based sets : $h : \mathscr{T} \to \mathbf{Set}_*$.*

Proof : By the additivity axiom, there is an isomorphism of sets : $h(* \vee *) \cong h(*) \times h(*)$. Since $* \vee * \cong *$, the isomorphism becomes the diagonal function $h(*) \to h(*) \times h(*)$, which is an isomorphism only if $h(*)$ has a single element. Therefore, for any based space $(X, x_0)$, we obtain a based set $(h(X), h(x_0))$. For any based map $(X, x_0) \to (Y, y_0)$, we get the commutative diagram :

$$\begin{tikzcd} h(Y) \arrow[r, "h(f)"] & h(X) \\ h(y_0) \arrow[u, hook] \arrow[r] & h(x_0). \arrow[u, hook] \end{tikzcd}$$

Thus $h(f) : (h(Y), h(y_0)) \to (h(X), h(x_0))$ is a morphism of based sets. □

The second axiom of a Brown functor is called Mayer-Vietoris, since it is a particular case of the exactness of the Mayer-Vietoris sequence of a cohomology theory, as we prove in the following lemma.

Lemma 3.2.4.
*Any generalized reduced cohomology theory on $\mathbf{CW}_*$ defines a Brown functor in each dimension.*

[1] In most of the literature, this axiom is named the Wedge axiom.

PROOF : Let $\widetilde{E}^*$ be a reduced cohomology theory. Each contravariant functor $\widetilde{E}^n$, composed with the forgetful functor $\mathbf{Ab} \to \mathbf{Set}_*$ that we shall omit from the notation, is a Brown functor. The additivity is apparent, since Brown functors and reduced cohomology share the same axiom. The Mayer-Vietoris axiom also holds. Indeed, take $(X; A, B)$ an excisive triad. Then the Mayer-Vietoris long exact sequence of $\widetilde{E}^*$ :

$$\cdots \longrightarrow \widetilde{E}^{n-1}(A \cap B) \longrightarrow \widetilde{E}^n(X) \longrightarrow \widetilde{E}^n(A) \oplus \widetilde{E}^n(B) \longrightarrow \widetilde{E}^n(A \cap B) \longrightarrow \cdots,$$

implies the Mayer-Vietoris axiom, as a particular case of the exactness of the sequence. □

Brown functors respect some kind of an exactness axiom, as we show in the following lemma.

LEMMA 3.2.5.
*Let $h : \mathbf{Top}_* \to \mathbf{Set}_*$ be a Brown functor. Let $f : X \to Y$ be a based map. Then the following sequence is exact in $\mathbf{Set}_*$ :*

$$h(Cf) \xrightarrow{h(i_f)} h(Y) \xrightarrow{h(f)} h(X),$$

*where $i_f : Y \to Cf$ is the canonical map.*

PROOF : Let us prove that im $h(i_f) = \ker h(f)$. Recall that the mapping cone $Cf$ is given by the following pushout :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, "i_f"] \\ CX \arrow[r] & Cf. \end{tikzcd}$$

Therefore the composite $i_f \circ f : X \to C_f$ factors through the reduced cone $CX$, which is contractible. Hence $i_f \circ f$ is nullhomotopic. Thus the composite $h(i_f \circ f) = h(f) \circ h(i_f)$ factors through $h(*)$ which is a singleton by Lemma 3.2.3. We have just proved the inclusion im $h(i_f) \subseteq \ker h(f)$.[2]
Let $y$ be in $h(Y)$ such that $h(f)(y)$ is the trivial element of $h(X)$. Define $A = ([1/4, 1] \times X)/\sim$ and $B = (([0, 3/4] \times X) \cup Y)/\sim$ with the appropriate equivalence relation $\sim$ making $A$ and $B$ based subspaces of $Cf$. We obtain the following homotopy equivalences $A \simeq_* *$, $B \simeq Y$ and $A \cap B \simeq X$. We emphazise that the previous homotopy equivalences are such that the following diagram commutes :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\simeq"'] & Y \arrow[r, "i_f"] \arrow[d, "\simeq"'] & Cf \arrow[d, equal] \\ A \cap B \arrow[r, hook] & B \arrow[r, hook] & Cf. \end{tikzcd}$$

We have obtained an excisive triad $(Cf; A, B)$. Since $y|_{A\cap B} = h(f)(y)$ is the trivial element, the Mayer-Vietoris axiom implies that there is $z$ in $h(Cf)$, such that $z|_A$ is the trivial element and $z|_B = y$, i.e., $h(i_f)(z) = y$. Therefore $\ker h(f) \subseteq$ im $h(i_f)$. □

The last general property of Brown functors that we wish to emphasize tells us when this contravariant functor takes values in the category of groups.

LEMMA 3.2.6.
*Let $h$ be a Brown functor. If $X$ is a co-H-group, then $h(X)$ is a group.*

[2] We recall that the kernel in $\mathbf{Set}_*$ of a function $f$ is defined as the preimage of the based element of the codomain of $f$.

*In addition suppose that $h$ corestricts to the category* **Ab** *of abelian groups. For any based map $f, g : \Sigma Y \to Z$, we have : $h(f+g) = h(f) + h(g)$.*

Sketch of the Proof : Let $\psi : X \to X \vee X$ be the comultiplication map. The additivity axiom gives a map : $h(\psi) : h(X) \times h(X) \cong h(X \vee X) \to h(X)$ which gives a multiplication on $h(X)$. The co-H-group structure on $X$ induces a group structure on $h(X)$, since $h$ is homotopy invariant.
Suppose now that we have based maps $f, g : \Sigma Y \to Z$. The map $f+g$ is the composite :

$$\Sigma Y \xrightarrow{\ p\ } \Sigma Y \vee \Sigma Y \xrightarrow{f \vee g} Z \vee Z \xrightarrow{\ \nabla\ } Z,$$

where $p : \Sigma Y \to \Sigma Y \vee \Sigma Y$ is the pinch map. Let $i_1, i_2 : \Sigma Y \hookrightarrow \Sigma Y \vee \Sigma Y$ be the two inclusions and $q_1, q_2 : \Sigma Y \vee \Sigma Y \to \Sigma Y$ be the quotient maps restricting to the identity on the summand indicated by the subscript and collapsing the other summand to a point. From the wedge axiom we have the isomorphisms of abelian groups :

$$(h(i_1), h(i_2)) : h(\Sigma Y \vee \Sigma Y) \xrightarrow{\ \cong\ } h(\Sigma Y) \times h(\Sigma Y).$$

We have the commutative diagram for any $j = 1, 2$ :

$$\begin{tikzcd}
h(Z) \arrow[r, "h(\nabla)"] \arrow[dr, "\Delta"'] & h(Z \vee Z) \arrow[r, "h(f \vee g)"] \arrow[d, "\cong"] & h(\Sigma Y \vee \Sigma Y) \arrow[r, "h(p)"] & h(\Sigma Y) \arrow[d, "h(q_j)"] \arrow[dl, dashed] \\
& h(Z) \times h(Z) \arrow[r, "h(f \times g)"] & h(\Sigma Y) \times h(\Sigma Y) \arrow[u, "(h(i_1),h(i_2))^{-1}", "\cong"'] & h(\Sigma Y \vee \Sigma Y) \arrow[l, "\cong"', "(h(i_1),h(i_2))"]
\end{tikzcd}$$

where the dashed map is the map $y \mapsto (y, 0)$ if $j = 1$, and the map $y \mapsto (0, y)$ if $j = 2$. Indeed, take $z$ in $h(Z)$. We have : $h(f \times g) \circ \Delta(z) = (h(f)(z), h(g)(z))$. But we also have $h(i_1, i_2) \circ h(\nabla \circ f \vee g)(z) = (h(f)(z), h(g)(z))$ since $(f \vee g) \circ i_1 = f$ and $(f \vee g) \circ i_2 = g$. An element $(y, 0)$ in $h(\Sigma Y) \times h(\Sigma Y)$ is sent to $y$ in $h(\Sigma Y)$ by the composite $h(p) \circ (h(i_1), h(i_2))^{-1}$ since $q_1 \circ p$ is homotopic to the identity. Similarly, an element $(0, y)$ in $h(\Sigma Y \times \Sigma Y)$ is sent to $y$ in $h(\Sigma Y)$. This proves the commutativity of the diagram. Therefore the previous element $(h(f)(z), h(g)(z))$ in $h(\Sigma Y) \times h(\Sigma Y)$ is sent to $h(f)(z) + h(g)(z)$, where the addition law comes from the functor $h$. But this composition is precisely $h(f+g)(z)$ where the addition law on maps comes from the co-H-structure of $\Sigma Y$. Since addition is defined component wise in $h(\Sigma Y) \times h(\Sigma Y)$, this proves : $h(f+g) = h(f) + h(g)$. □

We now restrict our attention to the full subcategory $\mathbf{CW}_*$ of based CW-complexes. The Brown Representability Theorem says that a Brown functor $h$ on $\mathbf{CW}_*$ is representable. More specifically there exists a based CW-complex $E$ such that there is a natural isomorphism :

$$e : [-, E]_* \Rightarrow h,$$

which is given by $e_X([f]_*) = h(f)(u)$, for any based CW-complex $X$ and any based map $f : X \to E$, where $u \in h(E)$ is the universal element of $h$.
Applying Corollary 3.1.5, the previous CW-complex $E$ and universal element $u \in h(E)$ are unique up to homotopy, in the sense that if $E'$ and $u' \in h(E')$ are another based CW-complex and universal element that represent $h$, then there is a homotopy equivalence, unique up to homotopy, $\rho : E \to E'$ such that $h(\rho)(u') = u$.

The idea to prove the Brown Representability Theorem is to represent $h$ first on spheres, and then to construct the desired based CW-complex $E$ skeleton-by-skeleton, with elements $u_n \in h(E_n)$ that represent $h$ on spheres of dimension at most $n$.

Definition 3.2.7 ($n$-Universal Elements, $\infty$-Universal Elements).
Let $n \geq -1$. Given a Brown functor $h : \mathbf{CW}_* \to \mathbf{Set}_*$ and a based CW-complex $K$, we say that an element $u$ in $h(K)$ is an *$n$-universal element*, if the function :

$$\begin{aligned} \varphi_u : \pi_k(K) = [S^k, K]_* &\longrightarrow h(S^k) \\ [f]_* &\longmapsto h(f)(u), \end{aligned}$$

is an isomorphism for $k < n$, and a surjection for $k \leq n$. By convention any element of $h(K)$ is $(-1)$-universal. An element $u$ in $h(K)$ is an *$\infty$-universal element* if it is $n$-universal for all $n \geq -1$.

The first step of the proof is to construct $n$-universal elements of $h$ inductively. This will lead to a construction of an $\infty$-universal element of $h$.

Lemma 3.2.8.
*Let $h : \mathbf{CW}_* \to \mathbf{Set}_*$ be a Brown functor. Let $K$ be any based* CW*-complex. If $u_n$ in $h(K)$ is $n$-universal, then there is a based space $L$ constructed from $K$ by attaching $(n+1)$-cells to $K$, and a $(n+1)$-universal element $u_{n+1}$ in $h(L)$ such that $u_{n+1}|_K = u_n$.*

Proof : If $n = -1$, let $B = \emptyset$, while if $n \geq 0$, let $B$ denote a set of representative based maps of the kernel of $\varphi_{u_n} : \pi_n(K) \to h(S^n)$. Define the spaces :

$$X := \bigvee_{\beta \in B} S^n, \quad \text{and,} \quad Y := K \vee \bigvee_{\alpha \in h(S^{n+1})} S^{n+1},$$

and let $f : X \to Y$ be the composite :

$$X = \bigvee_{\beta \in B} S^n \xrightarrow{\bigvee_{\beta \in B} \beta} K \hookrightarrow Y = K \vee \bigvee_{\alpha \in h(S^{n+1})} S^{n+1}.$$

Consider the pushout :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, "i_f"] \\ CX \arrow[r] & Cf =: L. \end{tikzcd}$$

Notice that for the particular case $n = -1$, we have $X = \emptyset$, and $L \cong Y$.
By the wedge axiom, there is a bijection :

$$h(Y) \xrightarrow{\cong} h(K) \times \prod_{\alpha \in h(S^{n+1})} h(S^{n+1}).$$

Let $u_n'$ in $h(Y)$ be the element corresponding to $(u_n, (\alpha))$ under the bijection above. We have the following commutative diagram :

$$\begin{tikzcd} h(Y) \arrow[rr, "h(f)"] \arrow[d, "\cong"] & & h(X) \arrow[d, equal] \\ h(K) \times \prod_{\alpha \in h(S^{n+1})} h(S^{n+1}) \arrow[rr, "h(\bigvee \beta)\circ \mathrm{proj}_{h(K)}"] & & h(X) \end{tikzcd}$$

Therefore $h(f)(u_n')$ is the trivial element, since $\beta \in B$. By Lemma 3.2.5, there is $u_{n+1}$ in $h(L)$ such that $h(i_f)(u_{n+1}) = u_n'$. Let $j : K \hookrightarrow L$ be the inclusion map. We have the following commutative diagram for all $k$ :

$$\begin{tikzcd}
\pi_k(K) \arrow[rr, "j_*"] \arrow[dr, "\varphi_{u_n}"'] & & \pi_k(L) \arrow[dl, "\varphi_{u_{n+1}}"] \\
& h(S^k). &
\end{tikzcd}$$

Since $u_n$ is $n$-universal, the function $\varphi_{u_{n+1}}$ is an isomorphism for $k < n$ and a surjection for $k = n$.

For $n \geq 0$, let us prove that $\varphi_{u_{n+1}}$ is an isomorphism for $k = n$. Let $x$ be an element of the kernel of $\varphi_{u_{n+1}} : \pi_n(L) \to h(S^n)$. Since $j$ is an $n$-equivalence, there is $y$ in $\pi_n(K)$ such that $j_*(y) = x$. So $\varphi_{u_n}(y)$ is the trivial element by commutativity of the diagram, and so $y$ lies in the kernel $B$ of $\varphi_{u_n}$. So there exists a representative in $B$, say $\beta_0 : S^n \to K$, which corresponds to the class $y$. Let us name $\iota_{\beta_0} : S^n \to \bigvee_{\beta\in B} S^n =: X$ the inclusion of $S^n$ in the $\beta_0$ component. By definiton of $f$, we have $f_*([\iota_{\beta_0}]_*) = y$. We get :

$$\begin{aligned}
x &= j_*(y) \\
&= \left(i_{f_{|K}}\right)_*(y) \\
&= \left(i_{f_{|K}} \circ f\right)_*([\iota_{\beta_0}]_*).
\end{aligned}$$

Since $i_f \circ f$ is nullhomotopic, we get that $x$ is the trivial element. Therefore $\varphi_{u_{n+1}}$ is an isomorphism for $k = n$.

Let us prove now that $\varphi_{u_{n+1}}$ is surjective for $k = n+1$, where $n \geq -1$. For any element $\alpha$ in $h(S^{n+1})$, let us name $\iota_\alpha : S^{n+1}_\alpha \to Y$ the inclusion of the sphere to its $\alpha$ component. We get :

$$\begin{tikzcd}
h(Y) \arrow[r, "h(\iota_\alpha)"] \arrow[d, "\cong"'] & h(S^{n+1}_\alpha) \arrow[d, equal] \\
h(K) \times \prod_{\alpha\in h(S^{n+1})} h(S^{n+1}) \arrow[r, "\mathrm{proj}_\alpha"] & h(S^{n+1}_\alpha),
\end{tikzcd}$$

where $\mathrm{proj}_\alpha$ is the projection onto the $\alpha$ component. We get :

$$\begin{aligned}
\varphi_{u_{n+1}}([i_f \circ \iota_\alpha]_*) &= h(i_f \circ \iota_\alpha)(u_{n+1}) \\
&= h(\iota_\alpha) \circ h(i_f)(u_{n+1}) \\
&= h(\iota_\alpha)(u'_n) \\
&= \mathrm{proj}_\alpha(u_n, (\alpha)) \\
&= \alpha.
\end{aligned}$$

Therefore $\varphi_{u_{n+1}}$ is surjective for $k = n+1$. Thus $u_{n+1}$ is $(n+1)$-universal. □

Proposition 3.2.9.
*Let $h : \mathbf{CW}_* \to \mathbf{Set}_*$ be a Brown functor, let $K$ be a based path-connected* CW*-complex, and $v$ an element of $h(K)$. Then there is a based space $L$ obtained from $K$ by attaching cells and an $\infty$-universal element $u$ in $h(L)$ such that $u|_K = v$.*

Proof : Let $L_{-1} = K$ and $u_{-1} = v$. By induction on $n \geq 0$, from Lemma 3.2.8, there are CW-complexes $L_n$ and $n$-universal element $u_n \in h(L_n)$, such that $u_n|_{L_{n-1}} = u_{n-1}$. Define $L$ as the colimit :

$$L_{-1} \hookrightarrow L_0 \hookrightarrow L_1 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow L_{n+1} \hookrightarrow \cdots,$$

that is $L = \bigcup_{n \geq -1} L_n$ endowed with its weak topology. Let us prove that there is an element $u$ in $h(L)$ such that $u|_{L_n} = u_n$, for all $n \geq -1$. We first take the homotopy colimit of the sequence (see the proof of Theorem C.4.7) :

$$L' = \bigcup_{n \geq -1} L_n \times [n, n+1]/\sim,$$

where we identified elements $(x_n, n+1)$ in $L_n \times [n, n+1]$, with the same elements $(x_n, n+1)$ but seen in $L_{n+1} \times [n+1, n+2]$. We have proved (in Theorem C.4.7) that there is a weak equivalence $r : L' \to L$. Since $L$ and $L'$ are both CW-complexes, Whitehead Theorem implies that $r : L' \to L$ is actually a homotopy equivalence. Define :

$$\begin{aligned} A &= \bigcup_{n \geq -1} L_{2n+1} \times [2n+1, 2n+2]/\sim \subseteq L', \\ B &= \bigcup_{n \geq 0} L_{2n} \times [2n, 2n+1]/\sim \subseteq L'. \end{aligned}$$

Since $L_{-1}$ is path-connected, we get : $A \simeq \bigvee_{n \geq -1} L_{2n+1}$, $B \simeq \bigvee_{n \geq 0} L_{2n}$, and $A \cap B \simeq \bigvee_{n \geq -1} L_n$. Now let us apply the additivity axiom. Define $a \in h(A)$, with $a|_{L_{2n+1}} = u_{2n+1}$, for $n \geq -1$. Define $b \in h(B)$, with $b|_{L_{2n}} = u_{2n}$, for $n \geq 0$. Define $c \in h(A \cap B)$, with $c|_{L_n} = u_n$, for $n \geq -1$. Then we obtain :

$$a|_{A \cap B} = b|_{A \cap B} = c.$$

The triad $(L'; A, B)$ is excisive. By Mayer-Vietoris axiom, there is an element $u \in h(L) \cong h(L')$ such that $u|_A = a$ and $u|_B = b$. Therefore, we get $u|_{L_n} = u_n$ for all $n \geq -1$. In particular $u|_K = v$.

Let us now prove that $u$ is an $\infty$-universal element. Let $i_n : L_n \hookrightarrow L$ denote the inclusion map. We have the following commutative diagram :

$$\begin{tikzcd} \pi_k(L_n) \arrow[rr, "(i_n)_*"] \arrow[dr, "\varphi_{u_n}"'] & & \pi_k(L) \arrow[dl, "\varphi_u"] \\ & h(S^k). & \end{tikzcd}$$

Since $i_n$ is an $n$-equivalence ($L_n$ has no relative cell of dimension greater than $n$), we get that $\varphi_u$ is an isomorphism for $k < n$. Since this is true for any $n \geq -1$, it follows that $u$ is $\infty$-universal. □

Proposition 3.2.10.

*Let $Y$ be a* CW*-complex with $u \in h(Y)$ an $\infty$-universal element. Let $(X, A)$ be a relative* CW*-complex, with basepoint in $A$, and let $g : A \to Y$ be a based map. If $w \in h(X)$ is such that $w|_A = h(g)(u)$, then there is a based map $G : X \to Y$ extending $g$ such that $h(G)(u) = w$ :*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook] & Y \\ X. \arrow[ur, "G"'] & \end{tikzcd}$$

Proof : Use the reduced mapping cylinder factorisation of $g$ :

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, tail, "i"'] & & Y \\ & Mg. \arrow[ur, "\simeq_*"'] & \end{tikzcd}$$

Let us define :

$$Z = X \cup Mg/(a \sim i(a)).$$

We now define :

$$\begin{aligned} B &= X \cup A \times [0, 3/4]/\sim, \\ C &= A \times [1/4, 1] \cup Y/\sim, \end{aligned}$$

with the appropriate equivalence relations $\sim$ such that $B$ and $C$ are based subspaces of $Z$. The triad $(Z; B, C)$ is excisive. We have the homotopy equivalences : $B \simeq_* X$ and $C \simeq_* Y$, and $B \cap C \simeq_* A$. Since $Mg \simeq_* Y$, we get $h(Mg) \cong h(Y)$, and the assumption $w|_A = h(g)(u)$ becomes :

$$w|_{B\cap C} = u|_{B\cap C}.$$

By the Mayer-Vietoris axiom, there is $z \in h(Z)$ such that $z|_X = w$ and $z|_Y = u$. Since $Z$ is path-connected, by Proposition 3.2.9 we can embed $Z$ in a CW-complex $Y'$ with $\infty$-universal element $u' \in h(Y')$ such that $u'|_Z = z$. Let $j : Y \to Y'$ be the inclusion map obtained as the composite :

$$Y \overset{\simeq_*}{\hookrightarrow} Mg \hookrightarrow Z \hookrightarrow Y'.$$

We have $h(j)(u') = u$. Since $u$ and $u'$ are both $\infty$-universal, the map $j$ is a weak equivalence. Since $Y$ and $Y'$ are CW-complexes, the Whitehead Theorem implies that $j$ is actually an homotopy equivalence. The structure of the mapping cylinder $Mg$ gives a based map $H : A \times I \to Y'$, defined as the composite :

$$A \times I \longrightarrow Mg \hookrightarrow Z \hookrightarrow Y',$$

which can be regarded as a homotopy from the restriction of the inclusion $g' : X \hookrightarrow Z \hookrightarrow Y'$ to $A$ given by the composite (recall that $g(a) \sim (a, 0)$ for any $a$ in $A$) :

$$A \cong A \times \{0\} \hookrightarrow A \times I \longrightarrow Mg \hookrightarrow Z \hookrightarrow Y',$$

to the composite $j \circ g$ :

$$A \xrightarrow{g} Y \overset{j}{\hookrightarrow} Y'.$$

The inclusion $A \hookrightarrow X$ is a cofibration since $(X, A)$ is a relative CW-complex. The homotopy extension property of $(X, A)$ gives a homotopy $\widehat{H} : X \times I \to Y'$ :

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dd, tail] & & A \times I \arrow[dl, "H"'] \arrow[dd, tail] \\ & Y' & \\ X \arrow[rr, hook] \arrow[ur, hook, "g'"] & & X \times I, \arrow[ul, dashed, "\widehat{H}"'] \end{tikzcd}$$

which extends $H$, and therefore gives an extension $G' : X \to Y'$ of the composite $j \circ g$, such that $G' \simeq_* g'$ through $\widehat{H}$. Since $j : Y \hookrightarrow Y'$ is a homotopy equivalence, there exists a homotopy inverse, say $r : Y' \to Y$. Composing $G'$ with $r$, we obtain a map $G : X \to Y$. We have $j \circ G = j \circ r \circ G' \simeq_* G'$. We obtain :

$$\begin{array}{rcl} h(G)(u) & = & h(G)(h(j)(u')) \\ & = & h(j \circ G)(u') \\ & = & h(G')(u') \\ & = & h(g')(u') \\ & = & u'|_X \\ & = & (u'|_Z)|_X \\ & = & z|_X \\ & = & w, \end{array}$$

and so we proved : $h(G)(u) = w$. □

Theorem 3.2.11 (Brown Representability Theorem).
*Let $h : \mathbf{CW}_* \to \mathbf{Set}_*$ be a Brown functor. Then $h$ is representable.*

Proof : From Lemma 3.2.3, $h(*)$ contains a single element, say $v$. By Proposition 3.2.9, there exists a CW-complex $E$ and an $\infty$-universal element $u$ in $h(E)$ such that $u|_{\{*\}} = v$. To prove that $E$ represents $h$, it suffices to prove, by the Yoneda Lemma, that $u$ is an universal element of $F$, i.e., we need to show that the function :

$$\begin{array}{rcl} e_X : [X, E]_* & \longrightarrow & h(X) \\ {[f]_*} & \longmapsto & h(f)(u), \end{array}$$

is a bijection, for all based CW-complexes $X$.

Surjectivity of $e_X$ Let $w$ be in $h(X)$. Let us apply Proposition 3.2.10 for the relative CW-complex $(X, *)$ and the obvious map $g : \{*\} \to E$. There exists a based map $G : X \to E$ such that $w = h(G)(u) = e_X([G]_*)$. Thus $e_X$ is surjective for all $X$.

injectivity of $e_X$ Let $g_0, g_1 : X \to E$ be based maps such that $e_X([g_0]_*) = e_X([g_1]_*)$. For convenience, we introduce the following notation :

$$V \rtimes W = \frac{V \times W}{\{*\} \times W},$$

for any based spaces $V$ and $W$.[3] Returning to our task, consider the based space $X \rtimes I$, this is the reduced cylinder of $X$. It is a based CW-complex with $k$-skeleton $X_k \rtimes I \cup X_k \rtimes \partial I$, where $X_k$ denotes the $k$-skeleton of $X$. Let $A = X \rtimes \partial I$, so we get a relative CW-complex $(X \rtimes I, A)$. Let :

$$\begin{array}{rcl} g : A & \longrightarrow & E \\ (x, 0) & \longmapsto & g_0(x) \\ (x, 1) & \longmapsto & g_1(x). \end{array}$$

[3]This is called the *half-smash product* of $V$ with $W$.

and let :

$$p : X \rtimes I \longrightarrow X,$$

be the projection. It is obviously a based homotopy equivalence, with homotopy inverse the inclusion $X \hookrightarrow X \rtimes I$. Thus we obtain the equality :

$$h(p) \circ h(g_0)(u) = h(p) \circ h(g_1)(u),$$

from $e_X([g_0]_*) = e_X([g_1]_*)$. Define $w = h(p) \circ h(g_0)(u) = h(p) \circ h(g_1)(u)$ the element in $h(X \rtimes I)$. Notice that $A$ is homeomorphic to $X \vee X$, whence by the wedge axiom : $h(A) \cong h(X) \times h(X) \cong h(X \rtimes I) \times h(X \rtimes I)$. Moreover $w|_A = h(g)(u)$. By Proposition 3.2.10, there exists a based map $G : X \rtimes I \to E$ that extends $g$ and such that $h(G)(u) = w$. The map $G$ can be regarded as a based homotopy from $g_0$ to $g_1$. Therefore : $g_0 \simeq_* g_1$, and so $e_X$ is injective for all $X$.

Thus $e_X$ is a bijection for all based CW-complex $X$. □

We can now present the most important application of the Brown Representability Theorem which is to represent reduced cohomology theories by Ω-prespectra.

Corollary 3.2.12.
*Let $\widetilde{h}^*$ be a reduced cohomology theory on $\mathbf{CW}_*$. Then there is an Ω-prespectrum $E$ and an equivalence of cohomology theories $T : \widetilde{E}^* \to \widetilde{h}^*$.*

Proof : From Lemma 3.2.4, the contravariant functors $\widetilde{h}^n : \mathbf{CW}_* \to \mathbf{Ab}$ are Brown functors, for every $n$. Therefore, by the Brown Representability Theorem, there are CW-complexes $E_n$ that represent $\widetilde{h}^n$, and universal elements $u_n$ in $h^n(E_n)$, i.e., we have natural isomorphisms :

$$\begin{aligned} e^n_X : [X, E_n]_* &\xrightarrow{\cong} \widetilde{h}^n(X) \\ [f]_* &\longmapsto \widetilde{h}^n(f)(u_n) \end{aligned}$$

for any based CW-complex $X$, and every $n \in \mathbb{Z}$. By Proposition 3.1.4, there exists a unique based homotopy equivalence $\widetilde{\sigma}_n : E_n \to \Omega E_{n+1}$ such that the following diagram commute :

$$\begin{tikzcd} {[X, E_n]_*} \arrow[r, dashed, "(\widetilde{\sigma}_n)_*"] \arrow[dd, "e^n_X"', "\cong"] & {[X, \Omega E_{n+1}]_*} \arrow[d, "\cong"'] \\ & {[\Sigma X, E_{n+1}]_*} \arrow[d, "\cong"', "e^{n+1}_{\Sigma X}"] \\ \widetilde{h}^n(X) \arrow[r, "\Sigma", "\cong"'] & \widetilde{h}^{n+1}(\Sigma X), \end{tikzcd} \qquad (\star)$$

for any based CW-complex $X$ and for each $n$ in $\mathbb{Z}$. We have defined (up to homotopy) the adjoints $\widetilde{\sigma}_n$ of the structure maps of the Ω-prespectrum $E$ obtained.[4]
We must prove that the isomorphism $[X, E_n]_* \cong \widetilde{h}^n(X)$ on $\mathbf{Set}_*$ is actually an isomorphism of abelian groups, where the group structure of $[X, E_n]_*$ is given by the bijection with $[X, \Omega E_{n+1}]_* \cong [\Sigma X, E_{n+1}]_*$. Therefore, it suffices to prove that $[\Sigma X, E_{n+1}]_* \cong \widetilde{h}^{n+1}(X)$ is an isomorphism of groups, for any $n$ and any based CW-complex $X$. We have to prove that :

$$\begin{aligned} e^{n+1}_{\Sigma X} : [\Sigma X, E_{n+1}]_* &\longrightarrow \widetilde{h}^{n+1}(\Sigma X) \\ [f]_* &\longmapsto h(f)(u_{n+1}), \end{aligned}$$

[4]We implicitly used the Milnor Theorem (Theorem 2.1.11) which proves that $\Omega E_{n+1}$ is a CW-complex

is an isomorphism of group. Recall that addition on $[\Sigma X, E_{n+1}]_*$ is given in the following way. Let $[f]_*$ and $[g]_*$ be elements of $[\Sigma X, E_{n+1}]_*$. The addition $[f]_* + [g]_*$ is represented by the composite :

$$\Sigma X \xrightarrow{p} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} E_{n+1} \vee E_{n+1} \xrightarrow{\nabla} E_{n+1}.$$

Therefore, we have to prove that :

$$h(f+g)(u_{n+1}) = h(f)(u_{n+1}) + h(g)(u_{n+1}).$$

But this follows directly from Lemma 3.2.6.
From Theorem 2.3.1, the $\Omega$-presepctrum $E$ defines a reduced cohomology theory $\widetilde{E}^*$ on $\mathbf{CW}_*$ exactly by $\widetilde{E}^n(X) = [X, E_n]_*$ for any $X$ and $n$, and the suspension homomorphism is given precisely by :

$$\widetilde{E}^n(X) = [X, E_n]_* \xrightarrow[\cong]{(\widetilde{\sigma}_n)_*} [X, \Omega E_{n+1}]_* \xrightarrow{\cong} [\Sigma X, E_{n+1}]_* = \widetilde{E}^{n+1}(\Sigma X).$$

Therefore, defining $T : \widetilde{E}^* \to \widetilde{h}^*$ as the collection of natural isomorphisms $\{e^n : \widetilde{E}^n \Rightarrow \widetilde{h}^n\}$, the diagram $(\star)$ gives the compatibility of the natural suspension homomorphisms of $\widetilde{h}^*$ and $\widetilde{E}^*$, making $T : \widetilde{E}^* \to \widetilde{h}^*$ an equivalence of reduced cohomology theories, as desired. $\square$

We wish that the previous procedure of associating to each generalized reduced cohomology theory an $\Omega$-prespectrum defined a functor $F : co\mathscr{H} \longrightarrow \Omega\mathscr{P}$, i.e., a converse of Addendum 2.3.2, but this is not possible since $\Omega$-prepsectra are defined only up to homotopy. In more detail, let $T : \widetilde{h}^* \to \widetilde{h}'^*$ is a transformation on cohomology theories on $\mathbf{CW}_*$, let $E$ and $E'$ be the corresponding $\Omega$-prespectra. Let us define $f : E \to E'$ to be the corresponding map of $\Omega$-prespectra. From Proposition 3.1.4, there exists a unique based map $f_n : E_n \to E'_n$ such that the following diagram commutes :

$$\begin{tikzcd}
{[X, E_n]_*} \arrow[r, dashed, "(f_n)_*"] \arrow[d, "e^n_X"', "\cong"] & {[X, E'_n]_*} \arrow[d, "e'^n_X"', "\cong"] \\
\widetilde{h}^n(X) \arrow[r, "T^n_X"] & \widetilde{h}'^n(X),
\end{tikzcd}$$

for any based CW-complex $X$. Then by applying again repeatedly Proposition 3.1.4, we see that $\Sigma f_n : \Sigma E_n \to \Sigma E'_n$ is the unique map, up to homotopy, such that the following diagram commutes for any $n$ and based CW-complex $X$ :

$$\begin{tikzcd}
{[X, E_n]_*} \arrow[r, equal] \arrow[dd, "(\widetilde{\sigma}_n)_*"] \arrow[ddddr, bend right, "e^n_X"'] & {[X, E_n]_*} \arrow[r, "(f_n)_*"] \arrow[d, "\Sigma"] & {[X, E'_n]_*} \arrow[r, equal] \arrow[d, "\Sigma"] & {[X, E_n]_*} \arrow[dd, "(\widetilde{\sigma}'_n)_*"'] \arrow[ddddl, bend left, "e'^n_X"] \\
 & {[\Sigma X, \Sigma E_n]_*} \arrow[r, dashed, "(\Sigma f_n)_*"] \arrow[d, "(\sigma_n)_*"'] & {[\Sigma X, \Sigma E'_n]_*} \arrow[d, "(\sigma'_n)_*"] & \\
{[X, \Omega E_{n+1}]_*} \arrow[r, "\cong"] & {[\Sigma X, E_{n+1}]_*} \arrow[r, "(f_{n+1})_*"] \arrow[d, "e^{n+1}_{\Sigma X}"', "\cong"] & {[\Sigma X, E'_{n+1}]_*} \arrow[d, "\cong"', "e'^{n+1}_{\Sigma X}"] & {[X, \Omega E'_{n+1}]_*} \arrow[l, "\cong"'] \\
 & \widetilde{h}^{n+1}(\Sigma X) \arrow[r, "T^{n+1}_{\Sigma X}"] \arrow[d, "\cong"', "\Sigma^{-1}"] & \widetilde{h}'^{n+1}(\Sigma X) \arrow[d, "\cong"', "\Sigma^{-1}"] & \\
 & \widetilde{h}^n(X) \arrow[r, "T^n_X"] & \widetilde{h}'^n(X). &
\end{tikzcd}$$

Let us take the particular case where $X = E_n$ and we evaluate with the element $[\mathrm{id}_{E_n}]_*$ in $[E_n, E_n]_*$. We get that $[\sigma'_n \circ \Sigma f_n]_* = [f_{n+1} \circ \sigma_n]_*$. Therefore, the following diagram is homotopy commutative only :

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma E'_n \arrow[d, "\sigma'_n"] \\ E_{n+1} \arrow[r, "f_{n+1}"] & E'_{n+1}. \end{tikzcd}$$

We can make this diagram commutative by applying the HELP theorem (suppose that $\sigma_n$ is a cofibration, if it isn't, use the reduced mapping cylinder factorization). So $f : E \to E'$ is a map of prepspectra. The problem of functoriality is that composition are preserved only up to homotopy. Indeed, let $\widetilde{h}^* \xrightarrow{T} \widetilde{k}^* \xrightarrow{U} \widetilde{r}^*$ be a composition of transformations of reduced cohomology theories. Let $E$, $P$ and $L$ be the corresponding $\Omega$-prespectra of the reduced cohomology theories $\widetilde{h}^*$, $\widetilde{k}^*$ and $\widetilde{r}^*$. Let $F(T) =: f : E \to P$ and $F(U) =: g : P \to L$ be the corresponding map of prespectra. Using Proposition 3.1.4, we get that :

$$\begin{tikzcd} {[X, E_n]_*} \arrow[r, "(f_n)_*"] \arrow[d, "\cong"] \arrow[rr, bend left, "F(U^n \circ T^n)"] & {[X, P_n]_*} \arrow[r, "(g_n)_*"] \arrow[d, "\cong"] & {[X, L_n]_*} \arrow[d, "\cong"] \\ \widetilde{h}^n(X) \arrow[r, "T^n_X"] & \widetilde{k}^n(X) \arrow[r, "U^n_X"] & \widetilde{r}^n(X). \end{tikzcd}$$

Therefore the map $F(U^n \circ T^n) : E_n \to L_n$ is homotopic to $g_n \circ f_n$, for each $n$, but they cannot be equal strictly in general.

We now give a nice result of this homotopy description of reduced cohomology theories.

Theorem 3.2.13 (Eilenberg-Steenrod).
*Let $\widetilde{h}^*$ and $\widetilde{k}^*$ be generalized reduced cohomology theories on $\mathbf{CW}_*$. Let $T : \widetilde{h}^* \to \widetilde{k}^*$ be a transformation of reduced cohomology theories. If $T^n_{S^0} : \widetilde{h}^n(S^0) \to \widetilde{k}^n(S^0)$ is an isomorphism for all $n$, then $T$ is an equivalence of cohomology theories.*

Proof : For any $k$ in $\mathbb{Z}$, the following diagram commutes :

$$\begin{tikzcd} \widetilde{h}^{n-k}(S^0) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "T^{n-k}_{S^0}"', "\cong"] & \widetilde{h}^n(S^0) \arrow[d, "T^n_{S^k}"] \\ \widetilde{k}^{n-k}(S^0) \arrow[r, "\Sigma", "\cong"'] & \widetilde{k}^n(S^k). \end{tikzcd}$$

Therefore $T^n_{S^k} : \widetilde{h}^n(S^k) \to \widetilde{k}^n(S^k)$ is an isomorphism, for all $k$ and $n$. Let $E$ be the associated $\Omega$-prespectrum of $\widetilde{h}^*$ and $P$ the associated $\Omega$-prespectrum of $\widetilde{k}^*$. Recall that $E_n$ and $P_n$ are based CW-complexes for all $n$ in this case. Using Proposition 3.1.4, there exists a unique based map $\rho_n : E_n \to P_n$ up to homotopy, such that the following diagram commutes for all based CW-complex $X$ :

$$\begin{tikzcd} {[X, E_n]_*} \arrow[r, "(\rho_n)_*"] \arrow[d, "\cong"] & {[X, P_n]_*} \arrow[d, "\cong"] \\ \widetilde{h}^n(X) \arrow[r, "T_X"] & \widetilde{k}^n(X). \end{tikzcd}$$

In particular, for $X = S^k$, we get that $(\rho_n)_* : \pi_k(E_n) \to \pi_k(P_n)$ is an isomorphism for all $k$, and so $\rho_n$ is a weak equivalence. Since $E_n$ and $P_n$ are CW-complexes, the map $\rho_n$ is actually a homotopy equivalence. Hence $(\rho_n)_* : [X, E_n]_* \to [X, P_n]_*$ is an isomorphism for all for all based CW-complex $X$. Therefore $T_X^n : \widetilde{h}^n(X) \to \widetilde{k}^n(X)$ is an isomorphism for all based CW-complex $X$ and all $n$. Thus $T$ is an equivalence of reduced cohomology theories. □

Theorem 3.2.14 (Eilenberg-Steenrod).
*Let $\widetilde{h}^*$ and $\widetilde{k}^*$ be ordinary reduced cohomology theories such that there is an isomorphism :*

$$\tau : \widetilde{h}^0(S^0) \xrightarrow{\cong} \widetilde{k}^0(S^0).$$

*Then $\tau$ induces an equivalence of reduced cohomology theories : $T : \widetilde{h}^* \to \widetilde{k}^*$.*

Proof : Let $E$ be the associated $\Omega$-prespectrum of $\widetilde{h}^*$ and $P$ the associated $\Omega$-prespectrum of $\widetilde{k}^*$. Let $G = \widetilde{h}^0(S^0)$ and $G' = \widetilde{k}^0(S^0)$. For all $n$, we get :

$$\begin{aligned} \pi_k(E_n) &= [S^k, E_n]_* \\ &\cong \widetilde{h}^n(S^k) \\ &\cong \widetilde{h}^{n-k}(S^0) \\ &\cong \begin{cases} G, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases} \end{aligned}$$

Therefore $E_n$ is an Eilenberg-MacLane space of type $(G, n)$ for all $n$. Similarly, $P_n$ is an Eilneberg-MacLane space of type $(G', n)$ for all $n$. Using Theorem 2.1.7, for each $n$, there exists a unique based homotopy equivalence $\rho_n : E_n \to P_n$ such that the following diagram commutes :

$$\begin{tikzcd} \pi_n(E_n) \arrow[r, "(\rho_n)_*"] \arrow[d, "\cong"] & \pi_n(P_n) \arrow[d, "\cong"] \\ G \arrow[r, "\tau", "\cong"'] & G'. \end{tikzcd}$$

This defines an equivalence of reduced cohomology theories $T : \widetilde{h}^* \to \widetilde{k}^*$. □

We end this chapter by stating the dual theorem of Brown for homology theories. This was done by Adams, in [Adams, 1962]. This is not done in our work since it involves the concept of Spanier-Whitehead duality that would lead us too far.

Theorem 3.2.15 (Adams).
*Let $\widetilde{h}_*$ be a reduced homology theory defined on based* CW*-complexes. Then there is an $\Omega$-prespectrum $E$ such that there is an equivalence of reduced homology theories : $\widetilde{h}_* \to \widetilde{E}_*$.*

Proof : Omitted, a proof can be found in [Switzer, 1975]. □

# Conclusion

We have succeeded in describing homotopically generalized homology and cohomology theories in terms of objects of stable homotopy theory : the prespectra. For any (reduced) homology theory $\widetilde{E}_*$ and cohomology theory $\widetilde{E}^*$, there exists an $\Omega$-prespectrum, unique up to homotopy, that is often named also $E$, such that :

$$\widetilde{E}_n(X) = \mathrm{colim}_k \pi_{n+k}(X \wedge E_k), \qquad \widetilde{E}^n(X) = [X, E_n]_*,$$

for any $n$ and any CW-complex $X$. We have seen also that ordinary homology and cohomology theories are characterized by the Eilenberg-MacLane $\Omega$-prespectrum $HG$. The fundamental results needed were the Blakers-Massey theorem and the Freudenthal suspension theorem. The required language was the colimit, provided by category theory.

Acknowledgement I would like to thank my professor Kathryn Hess-Bellwald. Her many helpful clarifications and corrections were very instructive. Moreover, without her passion and enthusiasm for algebraic topology, I would probably not have pursued this field of mathematics. I would also like to thank my family and friends for their unconditional support.

# APPENDIX A

# Homotopy of Simplicial and CW Complexes

In this appendix, we gather the main results on CW-complexes needed throughout this paper. We also give an introduction to simplicial complexes, needed for the proof of Blakers-Massey Theorem (in Lemma 1.1.6).

## A.1. Attaching Cells

Definition A.1.1.
From any space $X$, one say that a space $Y$ is obtained by *attaching a* $(n+1)$*-cell* if there exists a map $\varphi : S^n \to X$, called an *attaching map*, such that $Y$ is the following pushout :

$$\begin{tikzcd} S^n \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, hook] \\ D^{n+1} \arrow[r, "\widetilde{\varphi}"] & Y. \end{tikzcd}$$

The resulting map $\widetilde{\varphi}$ is an embedding and is called a *characteristic map* of $Y$, we denote its (closed) image in $Y$ by $e^{n+1}$, and it is called a $(n+1)$*-cell* of $Y$. One usually denotes $Y = X \cup e^{n+1}$. The *interior* $\mathring{e}^{n+1}$ of the cell $e^{n+1}$, is the image of the open $n$-disk, i.e. $\mathring{e}^n = \widetilde{\varphi}(\mathring{D}^{n+1})$.

Recall that a subspace $A$ of a space $X$ is a *strong deformation retract* of $X$ if there exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = x$, $H(x, 1) \in A$ and $H(a, t) = a$, for all $x$ in $X$, $a$ in $A$ and $t$ in $I$. If such happens, it is easy to see that $A$ and $X$ are homotopy equivalent.

Proposition A.1.2.
*If* $Y = X \cup e^{n+1}$*, and* $y \in \mathring{e}^{n+1}$*, then* $X$ *is a strong deformation retract of* $Y - \{y\}$*.*

Proof : Let us prove first that if $x_0$ is in $\mathring{D}^{n+1}$, then $S^n$ is a strong deformation retract of $D^{n+1} - \{x_0\}$. One can easily see that every point $z$ in $D^{n+1} - \{x_0\}$ has a unique representation of the form $z = sx_0 + (1-s)a_z$ for some $a_z$ in $S^n$, and $0 \le s < 1$. Define the map :

$$\begin{array}{rcl} H_{x_0} : \Big(D^{n+1} - \{x_0\}\Big) \times I & \longrightarrow & D^{n+1} - \{x_0\} \\ (z = sx_0 + (1-s)a_z, t) & \longmapsto & (1-t)sx_0 + (1 - s(1-t))a_z \end{array}$$

which is clearly continuous. We get $H_{x_0}(z,0)=z$, $H_{x_0}(z,1)\in S^n$ and $H_{x_0}(z',t)=z'$, for any $z$ in $D^{n+1}-\{x_0\}$, any $z'$ in $S^n$ and any $t$ in $I$.
Name $\varphi : D^{n+1}\to e^{n+1}$ the characteristic map. Denote now $x_0 := \varphi^{-1}(y)$, and define the map :

$$\begin{array}{rcl} H:(Y-\{y\})\times I & \longrightarrow & Y-\{y\} \\ (x,t) & \longmapsto & \begin{cases} x & \text{if } x\in X, \\ \varphi(H_{x_0}(\varphi^{-1}(x),t)) & \text{if } x\notin X. \end{cases} \end{array}$$

It is the required homotopy. □

## A.2. The Simplicial Approximation Theorem

Simplices The following discussion is based on [Rotman, 1998]. A subset $A$ of $\mathbb{R}^n$ is called *affine* if, for every pair of distinct points $x, x'\in A$, the line determined by $x, x'$ is contained in $A$. In particular, the empty set $\emptyset$ and the point-set $\{*\}$ are affine. It is easy to see that any intersection of affine subsets of $\mathbb{R}^n$ is also an affine subset. Thus, one can define the *affine set in* $\mathbb{R}^n$ *spanned* by a subset $X$ of $\mathbb{R}^n$, by the intersection of all affine subsets of $\mathbb{R}^n$ containing $X$. Similarly for convex sets, the *convex set spanned* by a subset $X$ (also called the *convex hull* of $X$) is the intersection of all convex subsets containing $X$. We denote it by $\langle X\rangle$, or Convex hull($X$). An *affine combination* of points $p_0, p_1, \dots, p_m$ in $\mathbb{R}^n$ is a point $x$ with $x=\sum_{i=0}^m t_ip_i$, where $\sum_{i=0}^m t_i = 1$. If moreover $t_i\geq 0$ for all $i$, it is a *convex combination*. One can prove that the affine (respectively convex) set spanned by $\{p_0,p_1,\dots,p_m\}\subset\mathbb{R}^n$ consists of all affine (respectively convex) combinations of these points. An ordered set of points $\{p_0,\dots,p_m\}\subset\mathbb{R}^n$ is *affine independent* if $\{p_1-p_0,\dots,p_m-p_0\}$ is a linearly independent subset of $\mathbb{R}^n$. One can prove the following result.

Proposition A.2.1.
*The following conditions on an ordered set of points $\{p_0,\dots,p_m\}$ in $\mathbb{R}^n$ are equivalent :*

- **(i)** *the set $\{p_0,\dots,p_m\}$ is affine independent;*
- **(ii)** *if $\{s_0,\dots,s_m\}\subset\mathbb{R}$ satisfies $\sum_{i=0}^m s_ip_i=0$ and $\sum_{i=0}^m s_i=0$, then $s_i=0$ for all $i$;*
- **(iii)** *each $x$ in the affine set spanned by $\{p_0,\dots,p_m\}$ has unique expression as an affine combination $x=\sum_{i=0}^m t_ip_i$, where $\sum_{i=0}^m t_i=1$.*

Thus affine independence is a property of the set $\{p_0,\dots,p_m\}$ that is independent of the given ordering. The entries $t_i$ mentioned are called the *barycentric coordinates* of $x$, relative to the set $\{p_0,\dots,p_m\}$, whence we have the following definition.

Definition A.2.2.
Let $\{p_0,\dots,p_m\}$ be an affine independent subset of $\mathbb{R}^n$. The convex set spanned by this set, denoted by $\langle p_0,\dots,p_m\rangle$, is called the *(affine) $m$-simplex* with *vertices* $p_0,\dots,p_m$. We often write $\sigma$ for $\langle p_0,\dots,p_m\rangle$, and we say that $\sigma$ has *dimension* $m$ : $\dim(\sigma)=m$.

Hence, each $x$ in the $m$-simplex $\langle p_0,\dots,p_m\rangle$ has a unique expression of the form :

$$x=\sum_{i=0}^m t_ip_i,\text{ where }\sum t_i=1\,\text{and each } t_i\geq 0.$$

For instance, a 0-simplex is just a point, a 1-simplex is a line segment, a 2-simplex is a triangle with interior, a 3-simplex is a tetrahedron, and so on.

Definition A.2.3.
Let $\sigma = \langle p_0, \ldots, p_m \rangle$ be a $m$-simplex. A $k$-face of $\sigma$ is a $k$-simplex spanned by $k+1$ of the vertices $\{p_0, \ldots, p_m\}$, where $0 \leq k \leq m-1$. The $(m-1)$ *face opposite* $p_i$ is $\langle p_0, \ldots, \widehat{p_i}, \ldots p_m \rangle$, where the circumflex notation means "delete". The *boundary* $\partial\sigma$ of $\sigma$ is the union of its faces. The *interior* $\mathring{\sigma}$ of the simplex $\sigma$ is defined by $\sigma - \partial\sigma$.

Definition A.2.4.
The *barycenter* of a $m$-simplex $\sigma = \langle p_0, \ldots, p_m \rangle$ is defined by : $b_\sigma = \sum_{i=0}^{m} \frac{1}{m+1} p_i$. In particular, if $m = 0$, then $b_\sigma = p_0$.

Finite Simplicial Complexes All the details of the following discussion can be found in [Munkres, 1984]. A *finite simplicial complex* $K$ in $\mathbb{R}^n$ is a finite collection of simplices in $\mathbb{R}^n$, such that every face of a simplex of $K$ is in $K$, and the intersection of any two simplices of $K$ is a face of each them. For instance, every simplex together with its faces forms a simplicial complex. The *dimension* of $K$ is the maximum dimension of any of its simplices. For any finite simplicial complex $K$, let us denote $|K|$ the *underlying space*, or *polytope*, or *realization*, of $K$, which is defined as the union of all the simplices of $K$, endowed with the topology inherited as a subspace of $\mathbb{R}^n$. The realization $|K|$ is then a compact metric space. A space $X$ is a *polyhedron* if there exists a simplicial complex $K$ and a homeomorphism $h : |K| \to X$. The ordered pair $(K, h)$ is called a *triangulation* of $X$. A map $f : |K| \to \mathbb{R}^N$ for some finite simplicial complex $K$ and some natural integer $N$, is said to be *affine* if for any simplex $\sigma = \langle p_0, \ldots, p_m \rangle$ in $K$, and any $x = \sum_{i=0}^{m} t_i p_i$ in $\sigma$, with $\sum_{i=0}^{m} t_i = 1$, we have $f(x) = \sum_{i=0}^{m} t_i f(p_i)$. Such maps are always continuous. If $L$ is a subcollection of $K$ that contains all faces of its elements, then it is also a simplicial complex, and we call it a *subcomplex* of $K$. In that case, the realization $|L|$ of $L$ is a closed subspace of $|K|$. A finite simplicial complex $K'$ is a *subdivision* of another finite simplicial complex $K$ if $|K'| = |K|$, and every simplex in $K'$ is contained in a simplex of $K$. For instance, the *barycentric subdivision* of a finite simplicial complex $K$ in $\mathbb{R}^n$ is a finite simplicial complex $K'$ where its vertices $\{b_\sigma\}$ are the barycenters of the simplices $\{\sigma\}$ of $K$, and where the simplices of $K'$ are :

$$\{\langle b_{\sigma_0}, \ldots, b_{\sigma_m} \rangle \mid \sigma_i \subsetneq \sigma_{i+1} \in K,\ i = 0, \ldots, m-1\}.$$

Notation A.2.5.
For a space $X$, let $Y = X \cup e^{n+1}$. For any $0 < r < 1$, denote $D_r^{n+1} = \{x \in D^{n+1} \mid \|x\| \leq r\}$. Let $\varphi : D^{n+1} \to e^{n+1}$ be the characteristic map. Denote $e_r^{n+1} := \varphi(D_r^{n+1})$.

We now prove the following simplicial approximation theorem; our discussion will be based on [Dodson and Parker, 1997].

Theorem A.2.6 (Simplicial Approximation Theorem).
*Assume that $(X, A)$ is a relative* CW*-complex where $X$ is obtained from $A$ by attaching a $m$-cell $X = A \cup e^m$, and let $K$ be a finite simplicial complex with a subcomplex $L$. Then given a map of pairs $f : (|K|, |L|) \to (X, A)$, there exists a subdivision $(K', L')$ of $(K, L)$ and a map $g : (|K|, |L|) \to (X, A)$ such that :*

**(i)** *$f(x) = g(x)$ for any $x$ in $f^{-1}(A)$;*

**(ii)** *$f \simeq_{f^{-1}(A)} g$;*

**(iii)** *for any simplex $\sigma$ in $K'$, if $g(|\sigma|)$ meets $e_{1/4}^m$, then $g(|\sigma|)$ is contained in the interior of the $m$-cell, and $g$ is a affine map when restricted to $|\sigma|$.*

Proof : Denote by $\varphi : D^m \to e^m$ the characteristic map of $X$. Since $|K|$ is a compact metric space, and $f^{-1}(e^m_{3/4})$ is closed in $|K|$, the restricted composite :

$$\left(\varphi^{-1} \circ f\right)_{|_{f^{-1}(e^m_{3/4})}} : f^{-1}(e^m_{3/4}) \longrightarrow D^m_{3/4},$$

is uniformly continuous. By Lebesgue covering Lemma, we may subdivide $K$ into $K'$ (using barycentric subdivision) such that no simplex of $K'$ has diameter more than $\delta$, where $\delta > 0$ satisfies : for any $x$ and $y$ in $f^{-1}(e^m_{3/4})$, if $\|x-y\| \leq \delta$, then $\|\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))\| \leq 1/4$, where $\|\cdot\|$ denotes the usual euclidean metric.
There are three disjoint classes of simplices of $K'$ : $C_1 = \{\sigma \in K' \mid f(|\sigma|) \subseteq X - e^m_{1/2}\}$, $C_2 = \{\sigma \in K' \mid f(|\sigma|) \subseteq \mathring{e}^m_{1/2}\}$, $C_3 = \{\sigma \in K' \mid f(|\sigma|) \cap \partial e^m_{1/2} \neq \emptyset\}$. We now define $g$ on each of these sets as follows.

- For any $\sigma$ in $C_1$, define $g(\sigma) := f(\sigma)$.
- For $\sigma = \langle p_0, \dots, p_k\rangle$ in $C_2$, and $x = \sum_{i=0}^k t_i p_i$ in $|\sigma|$, with $\sum_{i=0}^k t_i = 1$, define :

  $$g(x) := \sum_{i=0}^k t_i f(p_i),$$

  so that $g$ is constructed affinely from $f$.
- For $\sigma$ in $C_3$, we proceed inductively on $\dim(\sigma)$. If $\dim(\sigma) = 0$, let $g(\sigma) := f(\sigma)$. Suppose that $g$ is defined on $|\sigma|$, for all $\sigma$ in $C_3$ with $\dim(\sigma) < k$ such that $g(\sigma) \subseteq \langle (f(\sigma)\rangle$. Take $\sigma$ in $C_3$, with $\dim(\sigma) = k$. Write $\sigma = \langle p_0, \dots, p_k\rangle$. Then $g$ is defined on $|\partial\sigma|$. Define $b_\sigma = \sum_{i=0}^k \frac{1}{k+1} p_i$ as the barycenter. Each $x$ in $|\sigma| - \{b_\sigma\}$ has a unique expression of the form $x = tb_\sigma + (1-t)y_x$, for some $t$ in $I$ and $y_x$ in $|\partial\sigma|$. Define $g(b_\sigma) := f(b_\sigma)$ and $g(x) := tf(b_\sigma) + (1-t)g(y_x)$. We get that $g$ is continuous on $|\sigma|$ and $g(\sigma) \subseteq \langle f(\sigma)\rangle$.

The map $g$ has the desired property. It is continuous by the gluing Lemma. Define a homotopy :

$$\begin{array}{rcl} H : |K| \times I & \longrightarrow & X \\ (x,t) & \longmapsto & \begin{cases} f(x) & \text{if } x \in \sigma \in C_1, \\ (1-t)f(x) + tg(x) & \text{if } x \in \sigma \notin C_1. \end{cases} \end{array}$$

On can easily see that $H$ has the desired properties, that is $f \simeq_* g$, via $H$, relative to $f^{-1}(A)$. □

## A.3. CW-Approximation Theorems

Definition A.3.1.
A CW-*approximation* of a pair of spaces $(X, A)$ is a CW-complex $\widetilde{X}$, and a subcomplex $\widetilde{A}$ of $\widetilde{X}$, together with a weak equivalence $(\widetilde{X}, \widetilde{A}) \xrightarrow{\sim} (X, A)$.

We recall the following theorem.

Theorem A.3.2 (Cellular Approximation of Maps).
*Any map $f : (X, A) \to (Y, B)$ between relative* CW-*complexes is homotopic relative to $A$ to a cellular map.*

Proposition A.3.3.
*Let $X$ be a based* CW*-complex and let $(Y,B)$ be a based relative* CW*-complex. If $i:(Y,B)_n \to Y$ is the inclusion, then the induced function $i_* : [X,(Y,B)_n]_* \to [X,Y]_*$ is injective if $\dim X < n$ and surjective if $\dim X \leq n$. In particular, if $X$ and $Y$ are based* CW*-complexes, then the function $i_* : [X,Y_n]_* \to [X,Y]_*$ induced by the inclusion map is injective if $\dim X < n$ and surjective if $\dim X \leq n$. In particular, for any* CW*-complex $Y$, the inclusion $Y_n \hookrightarrow Y$ is an $n$-equivalence.*

Proof : Let $f : X \to Y$ be any map. Then, by the cellular approximation Theorem, it is homotopic to a cellular map $g : (X,*) \to (Y,B)$. If $\dim X \leq n$, then $\operatorname{im} g \subseteq (Y,B)_n$. So $g = i \circ g'$ for some map $g' : X \to (Y,B)_n$, and so $i_*$ is surjective. Let now $f, g : X \to (Y,B)_n$ be cellular maps such that $i \circ f \simeq i \circ g$, via a homotopy $H : X \times I \to Y$. By cellular approximation, since $(X \times I, X \times \partial I \cup \{*\} \times Y)$ is a relative CW-complex, there is a homotopic cellular map $G : X \times I \to Y$ which is a homotopy from $i \circ f$ to $i \circ g$. If $\dim X < n$, then $\operatorname{im} G \subseteq (Y,B)_n$. Thus when corestricted, the map $G : X \times I \to (Y,B)_n$ is a homotopy from $f$ to $g$, and so $i_*$ is injective. $\square$

Lemma A.3.4.
*For any based map $f : S^n \to X$, $[f]_* = 0$ in $\pi_n(X)$ if and only if there exists a based map $\widehat{f} : D^{n+1} \to X$ such that the following diagram commutes :*

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, hook] & X. \\ D^{n+1} \arrow[ur, dashed, "\widehat{f}"'] & \end{tikzcd}$$

Sketch of the Proof : Suppose $\widehat{f}$ exists. Define the map :

$$\begin{aligned} H : S^n \times I &\longrightarrow X \\ (x,t) &\longmapsto \widehat{f}((1-t)x + ts_0), \end{aligned}$$

where $s_0$ is the basepoint of $S^n$. It is a homotopy from $f$ to the identity $\mathrm{id}_{S^n}$.
Conversly, suppose $[f]_* = 0$ through a based homotopy $H : S^n \times I \to X$. Define the map :

$$\begin{aligned} \widehat{f} : D^{n+1} &\longrightarrow X \\ x &\longmapsto \begin{cases} * & \text{if } \|x\| \leq 1/2, \\ H(\frac{x}{\|x\|}, 2 - 2\|x\|) & \text{if } \|x\| \geq 1/2. \end{cases} \end{aligned}$$

Then : $\widehat{f}(x) = H(x,0) = f(x)$ for any $x$ in $S^n$. $\square$

Lemma A.3.5 (Killing homotopy).
*Let $X$ be any* CW*-complex and $n > 0$. There exists a relative* CW*-complex $(X',X)$ with cells in dimension $(n+1)$ only, such that : $\pi_n(X') = 0$, and $\pi_k(X) \cong \pi_k(X')$ for $k < n$.*

Proof : Let the generators of $\pi_n(X)$ be represented by $\{f_j : S^n \to X \mid j \in \mathscr{J}\}$. Define $X'$ as the pushout :

$$\begin{tikzcd} \coprod_{j \in \mathscr{J}} S^n \arrow[r, "\sum_{j \in \mathscr{J}} f_j"] \arrow[d, tail] & X \arrow[d, "i"] \\ \coprod_{j \in \mathscr{J}} D^{n+1} \arrow[r] & X'. \end{tikzcd}$$

The resulting map $i : X \to X'$ is a $n$-equivalence. By the previous lemma, we get that $\pi_n(i)([g]_*) = 0$ for any $[g]_*$ in $\pi_n(X)$. Hence im $\pi_n(i) = 0$. Since the homomorphism is surjective, we obtain $\pi_n(X') = 0$. □

THEOREM A.3.6 (Approximation of spaces by CW-complexes).
*For any space $X$, there is a CW-approximation $\widetilde{X} \xrightarrow{\sim} X$. If $X$ is $n$-connected, $n \geq 1$, then $\widetilde{X}$ can be chosen to have a unique vertex and no $k$-cells for $1 \leq k \leq n$. For a map $f : X \to Y$ and another CW-approximation $\widetilde{Y} \xrightarrow{\sim} Y$, there is a cellular map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$, unique up to homotopy, such that the following diagram is homotopy commutative :*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\sim"] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "\sim"] & Y. \end{tikzcd}$$

SKETCH OF THE PROOF : We give an explicit construction of $\widetilde{X}$. We first build a CW-complex for which the homotopy surjects to $\pi_*(X)$ and then we kill the homotopy in excess, dimension by dimension. We may assume that $X$ is path connected, working one path component at a time. For any $k \geq 1$, let the generators of $\pi_k(X)$ be represented by $\{f_j^k : S^k \to X \mid j \in \mathscr{J}_k\}$. Let $K_1$ be the wedge of spheres :

$$K_1 = \bigvee_{\substack{j \in \mathscr{J}_k \\ k \geq 1}} S^k.$$

We give $K_1$ the CW-structure induced by the usual CW-decomposition of the spheres $S^k$. Notice that if $X$ is $n$-connected, with $n \geq 1$, then $K_1$ has no $k$-cells for $1 \leq k \leq n$. Define the map $\gamma_1 : K_1 \to X$ on each $(k,j)$-th wedge summand as the map $f_j^k : S^k \to X$. It is indeed continuous by the gluing Lemma. The induced homomorphism $(\gamma_1)_*$ is surjective on each dimension : for a fixed $k$, each generator $f_j^k$ is given by $(\gamma_1)_*([\mathrm{id}_{S^k}]_*)$. Inductively, suppose we have constructed $K_m$ and maps $\gamma_m$ for $m \leq n$ such that $(\gamma_m)_*$ is surjective in every dimension, and a bijection for dimensions strictly inferior to $m$. Let $H_n$ be the subgroup of $\pi_n(K_n)$ defined as the kernel of $(\gamma_n)_*$. As in the proof of Lemma A.3.5, we kill the homotopy in excess, making $(\gamma_n)_*$ injective. Let the generators of $H_n$ be represented by $\{g_j : S^n \to K_n \mid j \in \mathscr{H}\}$. For each $j$, by Lemma A.3.4, since $\gamma_n \circ g_j$ is nullhomotopic, there is a map $\widehat{g}_j : D^{n+1} \to X$ which restricts to $\gamma_n \circ g_j$. Define $K_{n+1}$ as the pushout :

$$\begin{tikzcd} \coprod_{j \in \mathscr{H}} S^n \arrow[r, "\sum_{j \in \mathscr{H}} g_j"] \arrow[d, hook] & K_n \arrow[d, "i_n"] \\ \coprod_{j \in \mathscr{H}} D^{n+1} \arrow[r] & K_{n+1}. \end{tikzcd}$$

It is a CW-complex. Define $\gamma_{n+1} : K_{n+1} \to X$ to be $\gamma_n$ on $K_n$ and $\widehat{g}_j$ on the $j$-th coproduct $D^{n+1}$. Then $(\gamma_{n+1})_*$ is surjective on every dimension, since $(\gamma_n)_*$ is. Since $i_n : K_n \to K_{n+1}$ is an $n$-equivalence, $(\gamma_{n+1})_* : \pi_k(K_{n+1}) \to \pi_k(X)$ is a bijection for $k \leq n-1$. By our construction of $K_{n+1}$, $\ker(\gamma_{n+1})_* = 0$ in dimension $n$, which makes $(\gamma_{n+1})_*$ also a bijection for $n$. Notice that $\gamma_{n+1} \circ i_n = \gamma_n$. This allows us to define $\widetilde{X}$ as $\bigcup_{n \geq 1} K_n$, which is a CW-complex and to define the weak equivalence $\widetilde{X} \xrightarrow{\sim} X$ which is induced by the maps $\{\gamma_n\}$. It is indeed a weak equivalence (use theorem C.4.5).

Existence and uniquess of a map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ stem from the Whitehead theorem which states that the weak equivalence $\widetilde{Y} \xrightarrow{\sim} Y$ induces a bijection from $[\widetilde{X}, \widetilde{Y}]_*$ to $[\widetilde{X}, Y]_*$. $\square$

Theorem A.3.7 (Approximation of space by CW-pairs).
*For any pair of spaces $(X, A)$, and any* CW*-approximation $\widetilde{A} \to A$, there is a* CW*-approximation $\widetilde{X} \xrightarrow{\sim} X$ such that $(\widetilde{X}, \widetilde{A}) \xrightarrow{\sim} (X, A)$ is a* CW*-approximation. If $(X, A)$ is $n$-connected, then $(\widetilde{X}, \widetilde{A})$ can be chosen to have no relative $k$-cells for $k \leq n$. If $f : (X, A) \to (Y, B)$ is a map, and $(\widetilde{Y}, \widetilde{B}) \xrightarrow{\sim} (Y, B)$ is another* CW*-approximation, there is a cellular map $\widetilde{f} : (\widetilde{X}, \widetilde{A}) \xrightarrow{\sim} (\widetilde{Y}, \widetilde{B})$, unique up to homotopy, such that the following diagram is homotopy commutative :*

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{A}) \arrow[r, "\sim"] \arrow[d, "\widetilde{f}"'] & (X, A) \arrow[d, "f"] \\
(\widetilde{Y}, \widetilde{B}) \arrow[r, "\sim"] & (Y, B).
\end{tikzcd}$$

Sketch of the Proof : The argument is similar to theorem A.3.6. We may assume that $X$ has a basepoint in $A$, and that $X$ is path connected. For any $k \geq 1$, let the generators of $\pi_k(X)$ be represented by $\{f_j^k : S^k \to X \mid j \in \mathscr{J}_k\}$. Let $K_0$ be defined as :

$$K_0 = \widetilde{A} \vee \left( \bigvee_{\substack{j \in \mathscr{J}_k \\ k \geq 1}} S^k \right).$$

Define $\gamma_0 : K_0 \to X$ with the weak equivalence $\widetilde{A} \xrightarrow{\sim} A$ and the maps $\{f_j^k\}$. Construct $K_1$ from $K_0$ by attaching 1-cells connecting the vertices in the non-basepoint components of $\widetilde{A}$ to the base vertex. The paths in $X$ that connect the images under $\widetilde{A} \xrightarrow{\sim} A$ of the non-basepoint vertices to the basepoint of $X$ give $\gamma_1 : K_1 \to X$. The construction then follows the one of theorem A.3.6. To construct $\widetilde{f} : (\widetilde{X}, \widetilde{A}) \to (\widetilde{Y}, \widetilde{B})$, first build $\widetilde{f} : \widetilde{A} \to \widetilde{B}$, and use the HELP theorem to extend it to $\widetilde{X}$. $\square$

Corollary A.3.8.
*Let $X$ be a based, path-connected space. Then there exists a* CW*-approximation $\widetilde{X} \xrightarrow{\sim} X$, such that $\widetilde{X}$ has a unique vertex and based attaching maps. In particular, the $(n+1)$-skeleton of $\widetilde{X}$ is obtained as the pushout :*

$$\begin{tikzcd}
\bigvee S^n \arrow[r] \arrow[d, tail] & \widetilde{X}_n \arrow[d, tail] \\
\bigvee D^{n+1} \arrow[r] & \widetilde{X}_{n+1}.
\end{tikzcd}$$

Proof : The first statement follows directly from previous theorem, where we set $A = \{*\}$, the basepoint of $X$. For the sake of clarity, rename $\widetilde{X}$ as $X$. Recall that the $(n+1)$-skeleton of $X$ is given by the following pushout :

$$\begin{tikzcd}
\coprod_{j \in \mathscr{J}} S^n \arrow[r, "\sum_{j \in \mathscr{J}} \varphi_j"] \arrow[d, tail] & X_n \arrow[d, tail] \\
\coprod_{j \in \mathscr{J}} D^{n+1} \arrow[r] & X_{n+1}.
\end{tikzcd} \tag{$\star$}$$

Let us recall that for $s_0$ the basepoint of $S^n$, we have :

$$\bigvee_{j \in \mathscr{J}} S^n = (\coprod_{j \in \mathscr{J}} S^n)/((s_0)_j \sim (s_0)_{j'},\ \forall j, j' \in \mathscr{J}).$$

Since the attaching maps $\{\varphi_j\}$ are based, they induce a unique map $\sum_{j \in \mathscr{J}} \widehat{\varphi}_j : \bigvee_{j \in \mathscr{J}} S^n \to X_n$, such that the following diagram commutes :

$$\begin{tikzcd}
\coprod\limits_{j \in \mathscr{J}} S^n \arrow[r, "\sum\limits_{j \in \mathscr{J}} \varphi_j"] \arrow[d, two heads] & X_n. \\
\bigvee\limits_{j \in \mathscr{J}} S^n \arrow[ur, dashed, "\sum\limits_{j \in \mathscr{J}} \widehat{\varphi}_j"'] &
\end{tikzcd}$$

We thus get the following commutative diagram :

$$\begin{tikzcd}
\coprod\limits_{j \in \mathscr{J}} S^n \arrow[r, two heads] \arrow[d, tail] \arrow[rr, bend left, "\sum_{j \in \mathscr{J}} \varphi_j"] & \bigvee\limits_{j \in \mathscr{J}} S^n \arrow[r, "\sum_{j \in \mathscr{J}} \widehat{\varphi}_j"'] \arrow[d, tail] & X_n \arrow[d, tail] \\
\coprod\limits_{j \in \mathscr{J}} D^{n+1} \arrow[r, two heads] \arrow[rr, bend right] & \bigvee\limits_{j \in \mathscr{J}} D^{n+1} \arrow[r] & X_{n+1}.
\end{tikzcd}$$

Since the first square of the last diagram and the diagram $(\star)$ are pushouts, we get that the second square of the last diagram is a pushout, by the universal property of pushouts. $\square$

Definition A.3.9 (CW-triad).
A CW-*triad* $(X; A, B)$ is a CW-complex $X$, together with subcomplexes $A$ and $B$, such that $A \cup B = X$.

Proposition A.3.10.
*Let $(X; A, B)$ be a* CW-*triad. Then the map $A/A \cap B \to X/B$ is an isomorphism of* CW-*complexes.*

Proof : The map $A/A \cap B \to X/B$ is a homemorphism since it is obtained as :

$$\begin{tikzcd}
A \cap B \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads] \\
A/A \cap B \arrow[r, dashed, "\cong"] & X/B,
\end{tikzcd}$$

where the top square is the description of $X$ as a pushout. $\square$

Theorem A.3.11 (Approximation of excisive triads by CW-triads).
*Let $(X; A, B)$ be an excisive triad and let $C = A \cap B$. Then there is a* CW*-triad $(\widetilde{X}; \widetilde{A}, \widetilde{B})$, such that, with $\widetilde{C} = \widetilde{A} \cap \widetilde{B}$, the maps $\widetilde{C} \xrightarrow{\sim} C$, $\widetilde{A} \xrightarrow{\sim} A$, $\widetilde{B} \xrightarrow{\sim} B$, and $\widetilde{X} \xrightarrow{\sim} X$ are all weak equivalences. If $(A, C)$ is $n$-connected, then $(\widetilde{A}, \widetilde{C})$ can be chosen to have no $k$-cells for $k \leq n$, and similarly for $(B, C)$.*

Sketch of the Proof : There is CW-approximation $\widetilde{C} \xrightarrow{\sim} C$ that can be extended to the pairs $(\widetilde{A}, \widetilde{C}) \xrightarrow{\sim} (A, C)$ and $(\widetilde{B}, \widetilde{C}) \xrightarrow{\sim} (B, C)$ such that $\widetilde{A} \cap \widetilde{B} = \widetilde{C}$. Define $\widetilde{X}$ as the following pushout :

$$\begin{tikzcd}
\widetilde{C} \arrow[r, hook] \arrow[d, hook] & \widetilde{A} \arrow[d] \arrow[dr, bend left, "\sim"] & \\
\widetilde{B} \arrow[r] \arrow[dr, bend right, "\sim"'] & \widetilde{X} \arrow[dr, dashed] & A \arrow[d] \\
& B \arrow[r, hook] & X.
\end{tikzcd}$$

Define the desired weak equivalence $\widetilde{X} \to X$ as the (unique) dashed map of the above diagram, given by the universal property of pushouts. It only remains to prove that the map $\widetilde{X} \xrightarrow{\sim} X$ is indeed a weak equivalence, and the reader can find a full detailed proof in chapter 10, paragraph 7, of [May, 1999]. □

# APPENDIX B

# Axiomatic (Co)homology Theories

We present in this appendix the definition of a generalized (co)homology theory in the sense of the axiomatic work of Eilenberg-Steenrod in [Eilenberg and Steenrod, 1952]. We suppose that the reader is familiar with this approach so that we don't present the consequences of these axioms (such as the Mayer-Vietoris sequence). Full details can be found in [May, 1999]. The important result of this appendix is the fact that one can fully determined a generalized (co)homology theory on $\mathbf{Top}_{\mathrm{rel}}$ is completely determined by its associated reduced (co)homology theory defined on $\mathbf{CW}_*$, the category of based CW-complexes.

## B.1. Axioms for Homology

Let us write $\mathbf{Top}_{\mathrm{rel}}$ the category of pairs. Recall that a functor $F : \mathbf{Top}_{\mathrm{rel}} \to \mathscr{C}$ is *homotopy invariant* if : $f \simeq_A g : (X, A) \to (Y, B) \Rightarrow F(f) = F(g) : F(X, A) \to F(Y, B)$.

Definition B.1.1 (The Eilenberg-Steenrod Axioms for a Homology Theory).
A *generalized homology theory* $E_*$ is a family $\{E_n : \mathbf{Top}_{\mathrm{rel}} \to \mathbf{Ab} \mid n \in \mathbb{Z}\}$ of homotopy invariant functors, together with a family $\{\partial : E_n \circ U \Rightarrow E_{n-1} \mid n \in \mathbb{Z}\}$ of natural transformations, where $U : \mathbf{Top}_{\mathrm{rel}} \to \mathbf{Top}_{\mathrm{rel}}$ is a functor which sends any pairs $(X, A)$ to $(A, \emptyset)$, and any map $f : (X, A) \to (Y, B)$ to its restriction $(A, \emptyset) \to (B, \emptyset)$, such that the following axioms are satisfied.

(H1) Exactness For any pair $(X, A)$, let $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ be the inclusions. The following sequence is exact :

$$\cdots \longrightarrow E_n(A, \emptyset) \xrightarrow{E_n(i)} E_n(X, \emptyset) \xrightarrow{E_n(j)} E_n(X, A) \xrightarrow{\partial} E_{n-1}(A, \emptyset) \longrightarrow \cdots .$$

(H2) Excision For any excisive triad $(X; A, B)$, the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces isomorphisms for any $n$ :

$$E_n(A, A \cap B) \xrightarrow{\cong} E_n(X, B).$$

(H3) Additivity For any collection $\{(X_j, A_j)\}_{j \in \mathscr{J}}$ of pairs, the inclusions $i_j : (X_j, A_j) \hookrightarrow \coprod_{j \in \mathscr{J}} (X_j, A_j)$ induce isomorphisms for any $n$ :

$$\sum_{j \in \mathscr{J}} E_n(i_j) : \bigoplus_{j \in \mathscr{J}} E_n(X_j, A_j) \xrightarrow{\cong} E_n\left(\coprod_{j \in \mathscr{J}} (X_j, A_j)\right).$$

(H4) Invariance with Respect to Weak Equivalences If $f : (X, A) \to (Y, B)$ is a weak equivalence, then $E_n(f) : E_n(X, A) \to E_n(Y, B)$ is an isomorphism.

The homology theory $E_*$ is *ordinary* if moreover, there exists an abelian group $G$ such that the following axiom is satisfied.

(H5) Dimension $E_n(\{*\}) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$

We will write then $H_*(-, G) := E_*$.

In [Eilenberg and Steenrod, 1952], and most of the literrature, a generalized homology theory need not satisfy the additivity axiom. This axiom was introduced by Milnor in order to treat infinite dimensional CW-complexes. This was then called an "additive homology theory". However, throughout this paper we always deal with additive homology theories, and so we follow May's convention [May, 1999] and require generalized homology theories to be additive. As we will see, the "invariance with respect to weak equivalence" axiom is introduced in order to pass from homology theory on $\mathbf{Top}_{\mathrm{rel}}$ to an equivalent homology theory on pairs of CW-complexes.

Definition B.1.2.
A *transformation of homology theories* on $\mathbf{Top}_{\mathrm{rel}}$ $T : E_* \to E'_*$ is a family of natural transformations $\{T : E_n \Rightarrow E'_n\}$ that are compatible with $\partial$, i.e., such that the following diagram commutes for all pairs $(X, A)$ and for all $n \in \mathbb{Z}$ :

$$\begin{tikzcd} E_n(X,A) \arrow[r, "\partial"] \arrow[d, "T(X,A)"'] & E_{n-1}(A) \arrow[d, "T(A)"] \\ E'_n(X,A) \arrow[r, "\partial"] & E'_{n-1}(A). \end{tikzcd}$$

The transformation $T$ is called an *equivalence* if each $E_n \Rightarrow E'_n$ is a natural equivalence.

Proposition B.1.3.
*Let $T : E_* \to E'_*$ be a morphism of homology theories. If for all spaces $X$, the homomorphism $T(X) : E_n(X) \to E'_n(X)$ is an isomorphism for each $n$ in $\mathbb{Z}$, then $T$ is an equivalence of homology theories.*

Proof : The proof follows directly from the exactness axiom and the 5-Lemma. $\square$

Before introducing the notion of reduced homology, we begin with a disgression on well-pointed spaces.

Well-Pointed Spaces In many constructions of algebraic topology, such as the cone, the suspension, the mapping cylinder, etc, there are always a reduced and a unreduced case. Usually one uses the reduced case when dealing with based spaces, and the unreduced case for unbased spaces. However when are these constructions equivalent, at least homotopically ? A sufficient answer is given by well-pointed spaces. A *well-pointed* space (or *nondegenerately based* space) is a based space $(X, *)$, such that the inclusion $\{*\} \hookrightarrow X$ is a cofibration. For instance, a CW-complex is well-pointed for any based point in its 0-skeleton (since the inclusion of any $n$-skeleton to the CW-complex is a cofibration).

Lemma B.1.4.
*Suppose we have the following pushout :*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, tail, "j"] & Y. \end{tikzcd}$$

*If $i : A \to X$ is a cofibration and $f : A \to B$ is a homotopy equivalence, then $g : X \to Y$ is a homotopy equivalence.*

Proof : Omitted. □

Hence we often have a homotopy equivalence between reduced and unreduced construction. For instance, if one denotes $SX$ the unreduced suspension of a space $X$, the reduced suspension $\Sigma X$ is given by the pushout :

$$\begin{tikzcd} \{*\} \times I \arrow[r, tail] \arrow[d] & SX \arrow[d] \\ \{*\} \arrow[r, tail] & \Sigma X. \end{tikzcd}$$

The top map is a cofibration since $q \times \mathrm{id}$ is always a cofibration if $q$ is a cofibration. Therefore the induced map $SX \to \Sigma X$ is a homotopy equivalence, since $\{*\} \times I \to \{*\}$ is a homotopy equivalence.

Lemma B.1.5.
*Suppose we have the commutative diagram :*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, tail, "j"] & Y, \end{tikzcd}$$

*in which $i$ and $j$ are cofibrations, and $f$ and $g$ are homotopy equivalences. Then $(g, f) : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs.*

Proof : Omitted, see [May, 1999] page 45. □

We introduce now the notion of reduced homology theory. Let us write ${}_{\mathrm{well}}\mathbf{Top}_*$ the category of well-pointed spaces.

Definition B.1.6 (The Eilenberg-Steenrod Axioms for a Reduced Homology Theory).
A *generalized reduced homology theory* $\widetilde{E}_*$ is a family $\{\widetilde{E}_n : {}_{\mathrm{well}}\mathbf{Top}_* \to \mathbf{Ab} \mid n \in \mathbb{Z}\}$ of homotopy invariant functors that satisfy the following axioms.

($\widetilde{\mathrm{H}}$1) Exactness If $i : A \rightarrowtail X$ is a cofibration, then the following sequence is exact, for any $n$ :

$$\widetilde{E}_n(A) \longrightarrow \widetilde{E}_n(X) \longrightarrow \widetilde{E}_n(X/A).$$

($\widetilde{\mathrm{H}}$2) Suspension For any $n$ and any space $X$, there is a natural isomorphism :

$$\Sigma : \widetilde{E}_n(X) \xrightarrow{\cong} \widetilde{E}_{n+1}(\Sigma X).$$

($\widetilde{\mathrm{H}}3$) Additivity For any collection $\{X_j \mid j \in \mathscr{J}\}$ of based spaces, the inclusions $i_j : X_j \hookrightarrow \bigvee_{j\in\mathscr{J}} X_j$ induce an isomorphism for every $n$ :

$$\sum_{j\in\mathscr{J}} \widetilde{E}_n(i_j) : \bigoplus_{j\in\mathscr{J}} \widetilde{E}_n(X_j) \xrightarrow{\cong} \widetilde{E}_n\left(\bigvee_{j\in\mathscr{J}} X_j\right).$$

($\widetilde{\mathrm{H}}4$) Invariance with Respect to Weak Equivalences If $f : X \to Y$ is a weak equivalence, then $\widetilde{E}_n f : \widetilde{E}_n(X) \longrightarrow \widetilde{E}_n(Y)$ is an isomorphism, for any $n$.

The reduced homology theory is *ordinary* if moreover, there exists an abelian group $G$ such that the following axiom is satisfied.

($\widetilde{\mathrm{H}}5$) Dimension $\widetilde{E}_n(S^0) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$

We will write then $\widetilde{H}_*(-, G) := \widetilde{E}_*$.

Definition B.1.7.
A *transformation of reduced homology theories* $T : \widetilde{E}_* \to \widetilde{E}'_*$ is a family of natural transformations $\{T : \widetilde{E}_n \Rightarrow \widetilde{E}'_n\}$ that are compatible with $\Sigma$, i.e., such that the following diagram commutes for every well-pointed space $X$ and $n \in \mathbb{Z}$ :

$$\begin{tikzcd} \widetilde{E}_n(X) \arrow[r, "\Sigma"] \arrow[d, "T(X)"'] & \widetilde{E}_{n+1}(\Sigma X) \arrow[d, "T(\Sigma X)"] \\ \widetilde{E}'_n(X) \arrow[r, "\Sigma"] & \widetilde{E}'_n(\Sigma X). \end{tikzcd}$$

The transformation $T$ is called an *equivalence* if each $\widetilde{E}_n \Rightarrow \widetilde{E}'_n$ is a natural equivalence.

Theorem B.1.8.
*For any based map $f : A \to X$, we have the following exact sequence :*

$$\cdots \longrightarrow \widetilde{E}_n(A) \longrightarrow \widetilde{E}_n(X) \longrightarrow \widetilde{E}_n(Cf) \longrightarrow \widetilde{E}_{n-1}(A) \longrightarrow \cdots,$$

*where $Cf$ denotes the reduced mapping cone of $f$.*

Proof : Factor $f$ using the reduced mapping cylinder :

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail] & & X \\ & Mf, \arrow[ur, "p_f"'] & \end{tikzcd}$$

where $p_f$ is a homotopy equivalence. Recall that the reduced mapping cone $Cf$ is defined as the pushout :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & X \arrow[d, tail] \\ CA \arrow[r] & Cf. \end{tikzcd}$$

Recall that we get the based homotopy equivalence : $X/A \simeq_* Cf$. The Barratt-Puppe sequence :

$$A \to X \to Cf \to \Sigma A \to \Sigma X \to \Sigma(Cf) \to \cdots,$$

together with the suspension axiom ($\widetilde{\mathrm{H}}2$) and the exactness axiom ($\widetilde{\mathrm{H}}1$) establish the desired exact sequence. $\square$

THEOREM B.1.9.
*A generalized homology theory $E_*$ on $\mathbf{Top}_{\mathrm{rel}}$ determines and is determined by its restriction to a generalized homology theory $E_*$ on pairs of* CW*-complexes. A generalized reduced homology theory $\widetilde{E}_*$ on ${}_{\mathrm{well}}\mathbf{Top}_*$ determines and is determined by its restriction to a generalized homology theory $\widetilde{E}_*$ of based* CW*-complexes.*

PROOF : This follows directly from the CW-approximation theorems of appendix A and the invariance with respect to weak equivalences axiom of Eilenberg-Steenrod. $\square$

THEOREM B.1.10.
*A generalized homology theory $E_*$ on $\mathbf{Top}_{\mathrm{rel}}$ determines and is determined by a generalized reduced homology theory $\widetilde{E}_*$ on ${}_{\mathrm{well}}\mathbf{Top}_*$.*

SKETCH OF THE PROOF : Suppose first that we have a generalized homology theory $E_*$ on pairs. For any based space $X$, we define $\widetilde{E}_*$ by :

$$\widetilde{E}_n(X) := E_n(X, *),$$

for any $n$ in $\mathbb{Z}$. We need to check the Eilenberg-Steenrod axioms.

($\widetilde{\mathrm{H}}1$) Apply exactness of $E_*$ to the pair $(X, *)$. The map $X \to *$ is a retraction, hence the long exact sequence splits in each degree, so that $E_n(X) \cong \widetilde{E}_n(X) \oplus E_n(*)$, for each $n$. When $* \in A \subseteq X$, on obtain the long exact sequence :

$$\cdots \longrightarrow \widetilde{E}_n(A) \longrightarrow \widetilde{E}_n(X) \longrightarrow E_n(X, A) \xrightarrow{\partial} \widetilde{E}_{n-1}(A) \longrightarrow \cdots$$

If $i : A \rightarrowtail X$ is a cofibration, then the quotient map $(X, A) \to (X/A, *)$ induces an isomorphism $E_n(X, A) \overset{\cong}{\to} \widetilde{E}_n(X/A)$ for each $n$. Indeed, let us use excision of $E_*$ to prove it. Let $Ci = X \cup_i CA$ be the mapping cone of $i$. We have an excisive triad :

$$\left(Ci; \frac{X \cup_i A \times [0, 2/3]}{\{*\} \times [0, 2/3]}, \frac{A \times [1/3, 1]}{A \times \{1\} \cup \{*\} \times [1/3, 1]}\right).$$

Let us call $X_1$ and $X_2$ the two components of the triad. Notice that $X_1 \cap X_2 = \frac{A \times [1/3, 2/3]}{\{*\} \times [1/3, 2/3]}$. We have the following commutative diagram :

$$\begin{tikzcd}
(X_1, X_1 \cap X_2) \arrow[r, hook] \arrow[d, "\simeq"] & (Ci, X_2) \arrow[d, "\simeq"] \\
(X, A) \arrow[r] & (X/A, *).
\end{tikzcd}$$

The usual homotopy equivalence $Mi \to X$ induces the left homotopy equivalence when restricted. We conclude using excision.

($\widetilde{\mathrm{H}}2$) Since $CX$ is contractible, use the previous sequence to get the natural isomorphism $\widetilde{E}_{n+1}(\Sigma X) \cong \widetilde{E}_{n+1}(CX/X) \cong \widetilde{E}_n(X)$.

($\widetilde{\mathrm{H}}3$) The wedge $\bigvee_{j\in\mathscr{J}} X_j$ is the quotient $\coprod_j X_j$ with its basepoints of each $X_j$. Since the spaces are well-pointed, we use the previous exact sequence to obtain the desired isomorphism using additivity of $E_*$.

($\widetilde{\mathrm{H}}4$) It follows directly from (H4) of $E_*$.

($\widetilde{\mathrm{H}}5$) It follows from $E_n(S^0) \cong \widetilde{E}_n(S^0) \oplus E_n(\{*\})$, for each $n$.

Conversely, suppose we are given a reduced homology theory $\widetilde{E}_*$. Define a functor :

$$\begin{aligned} (-)_+ : \mathbf{Top} &\longrightarrow \mathbf{Top}_* \\ X &\longmapsto X_+ := X \sqcup \{*\} \\ X \xrightarrow{f} Y &\longmapsto f_+ : X_+ \to Y_+. \end{aligned}$$

For any pair $(X, A)$, let $i : A \hookrightarrow X$ be the inclusion, we define :

$$E_n(X, A) := \widetilde{E}_n(C(i_+)).$$

Notice that if $A = \emptyset$, then $E_n(X, \emptyset) = \widetilde{E}_n(X_+)$. So if we have $E_*$ and define $\widetilde{E}_*$ as before, we recover $E_*$ with the above definition. In other words, we have defined an equivalence of homology theories where each $E_n(X, A)$ is maped to $\widetilde{E}_n(C(i_+))$ (we use Proposition B.1.3), and an equivalence of reduced homology theories where each $\widetilde{E}_n(X)$ is maped to $E_n(X, *)$ (because $Ci \cong X$ if $i$ is the inclusion $\{*\} \hookrightarrow X$). Although $X_+$ is not well-pointed, but one can use CW-approximations and work with weakly equivalent CW-complexes which are always well-pointed and use ($\widetilde{\mathrm{H}}4$), i.e., use Theorem B.1.9.
Let us prove now the Eilenberg-Steenrod axioms.

(H1) For any pair $(X, A)$, the sequence associated to the induced inclusion $i_+ : A_+ \to X_+$ of theorem B.1.8 gives the long exact sequence :

$$\cdots \longrightarrow E_n(A, \emptyset) \longrightarrow E_n(X, \emptyset) \longrightarrow E_n(X, A) \xrightarrow{\partial} E_{n-1}(A, \emptyset) \longrightarrow \cdots$$

where the natural transformation $\partial : E_n \circ U \Rightarrow E_{n-1}$ can be given explicitly as follows. From ($\widetilde{\mathrm{H}}2$), we have an isomorphism $\Sigma^{-1} : \widetilde{E}_{n+1}(\Sigma X) \xrightarrow{\cong} \widetilde{E}_n(X)$ for any well-pointed space $X$. For any pair $(X, A)$, using possibly a weak equivalence, we have that $C(i_+)$ is homotopy equivalent to $X_+/A_+ = X/A$ through a map $\psi$. Define a map $p : C(i_+) \to \Sigma(A_+)$ by collapsing $X_+$ to a point[1]. The homomorphism $\partial : E_n(X, A) \to E_{n-1}(A, \emptyset)$ is then given by the composite :

$$E_n(X, A) = \widetilde{E}_n(C(i_+)) \xrightarrow{E_n(p\circ\psi)} \widetilde{E}_n(\Sigma(A_+)) \xrightarrow[\cong]{\Sigma^{-1}} \widetilde{E}_{n-1}(A_+) = E_{n-1}(A, \emptyset).$$

(H2) For any excisive triad $(X; A, B)$ there is a weakly equivalent CW-triad that we shall rename $(X; A, B)$ (see theorem A.3.11). Since in this case $A/(A \cap B) \to X/B$ is an isomorphism of CW-complexes (by Proposition A.3.10), the result follows.

(H3) Let $\{(X_j, A_j)\}_{j\in\mathscr{J}}$ be a collection of based spaces; and $i_j : (X_j, A_j) \hookrightarrow \coprod_{j\in\mathscr{J}}(X_j, A_j)$ the inclusions. Notice that we have the equality of mapping cones $C(\coprod_{j\in\mathscr{J}} i_j) = \bigvee_{j\in\mathscr{J}} C(i_j)$ (use Theorem C.3.4). Using ($\widetilde{\mathrm{H}}3$) of $\widetilde{E}_*$, the result follows.

The weak equivalence and dimension axioms are proved in the same way as before. □

[1] This is done when one proves the Baratt-Puppe sequence.

## B.2. Axioms for Cohomology

Generalized cohomology theory is the dual notion of generalized homology theory. Therefore all the following results are proved in the exact same procedure, and we shall omit the proofs.

Definition B.2.1 (The Eilenberg-Steenrod Axioms for a Cohomology Theory). A *generalized cohomology theory* $E^*$ is a family $\{E^n : \mathbf{Top}_{\mathrm{rel}} \to \mathbf{Ab} \mid n \in \mathbb{Z}\}$ of homotopy invariant contravariant functors, together with a family $\{\delta : E^n \circ U \Rightarrow E^{n+1} \mid n \in \mathbb{Z}\}$ of natural transformations, where $U : \mathbf{Top}_{\mathrm{rel}} \to \mathbf{Top}_{\mathrm{rel}}$ is a covariant functor which sends any pairs $(X, A)$ to $(A, \emptyset)$, and any map $f : (X, A) \to (Y, B)$ to its restriction $(A, \emptyset) \to (B, \emptyset)$, such that the following axioms are satisfied.

(CoH1) Exactness For any pair $(X, A)$, let $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ be the inclusions. The following sequence is exact :

$$\cdots \longrightarrow E^n(X, A) \xrightarrow{E^n(j)} E^n(X, \emptyset) \xrightarrow{E^n(i)} E^n(A, \emptyset) \xrightarrow{\delta} E^{n+1}(X, A) \longrightarrow \cdots .$$

(CoH2) Excision For any excisive triad $(X; A, B)$, the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces isomorphisms for any $n$ :

$$E^n(X, B) \xrightarrow{\cong} E^n(A, A \cap B).$$

(CoH3) Additivity For any collection $\{(X_j, A_j)\}_{j\in\mathscr{J}}$ of pairs, the inclusions $i_j : (X_j, A_j) \hookrightarrow \coprod_{j\in\mathscr{J}}(X_j, A_j)$ induce isomorphisms for any $n$ :

$$(E^n(i_j))_{j\in\mathscr{J}} : E^n\left(\coprod_{j\in\mathscr{J}}(X_j, A_j)\right) \xrightarrow{\cong} \prod_{j\in\mathscr{J}} E^n(X_j, A_j).$$

(CoH4) Invariance with Respect to Weak Equivalences If $f : (X, A) \to (Y, B)$ is a weak equivalence, then $E^n(f) : E^n(Y, B) \to E^n(X, A)$ is an isomorphism.

The cohomology theory $E^*$ is *ordinary* if moreover, there exists an abelian group $G$ such that the following axiom is satisfied.

(CoH5) Dimension $E^n(\{*\}) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$

We will write then $H^*(-, G) := E^*$.

Definition B.2.2 (The Eilenberg-Steenrod Axioms for a Reduced Cohomology Theory). A *generalized reduced cohomology theory* $\widetilde{E}^*$ is a family $\{\widetilde{E}^n : {}_{\mathrm{well}}\mathbf{Top}_* \to \mathbf{Ab} \mid n \in \mathbb{Z}\}$ of homotopy invariant contravariant functors that satisfy the following axioms.

(Co$\widetilde{\text{H}}$1) Exactness If $i : A \rightarrowtail X$ is a cofibration, then the following sequence is exact, for any $n$ :

$$\widetilde{E}^n(X/A) \longrightarrow \widetilde{E}^n(X) \longrightarrow \widetilde{E}^n(A).$$

(Co$\widetilde{\text{H}}$2) Suspension For any $n$ and any space $X$, there is a natural isomorphism :

$$\Sigma : \widetilde{E}^n(X) \xrightarrow{\cong} \widetilde{E}^{n+1}(\Sigma X).$$

(Co$\widetilde{\text{H}}$3) Additivity For any collection $\{X_j \mid j \in \mathscr{J}\}$ of based spaces, the inclusion maps $i_j : X_j \hookrightarrow \bigvee_{j\in\mathscr{J}} X_j$ induce an isomorphism for every $n$ :

$$(\widetilde{E}^n(i_j))_{j\in\mathscr{J}} : \widetilde{E}^n\left(\bigvee_{j\in\mathscr{J}} X_j\right) \xrightarrow{\cong} \prod_{j\in\mathscr{J}} \widetilde{E}^n(X_j).$$

(Co$\widetilde{\text{H}}$4) Invariance with Respect to Weak Equivalences If the map $f : X \to Y$ is a weak equivalence, then $\widetilde{E}^n f : \widetilde{E}^n(Y) \longrightarrow \widetilde{E}^n(X)$ is an isomorphism, for any $n$.

The reduced cohomology theory is *ordinary* if moreover, there exists an abelian group $G$ such that the following axiom is satisfied.

(Co$\widetilde{\text{H}}$5) Dimension $\widetilde{E}^n(S^0) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$

We will write then $\widetilde{H}^*(-, G) := \widetilde{E}^*$.

Definition B.2.3.
A *transformation of cohomology theories* on $\mathbf{Top}_{\text{rel}}$ $T : E^* \to E'^*$ is a family of natural transformations $\{T : E^n \Rightarrow E'^n\}$ that are compatible with $\delta$, i.e., such that the following diagram commutes for all pairs $(X, A)$ and for all $n \in \mathbb{Z}$ :

$$\begin{tikzcd}
E^n(X,A) \arrow[d, "T(X,A)"'] & E^{n-1}(A) \arrow[l, "\delta"'] \arrow[d, "T(A)"] \\
E'^n(X,A) & E'^{n-1}(A). \arrow[l, "\delta"']
\end{tikzcd}$$

A *transformation of reduced cohomology theories* $T : \widetilde{E}^* \to \widetilde{E'}^*$ is a family of natural transformations $\{T : \widetilde{E}^n \Rightarrow \widetilde{E'}^n\}$ that are compatible with $\Sigma$, i.e., such that the following diagram commutes for every well-pointed space $X$ and $n \in \mathbb{Z}$ :

$$\begin{tikzcd}
\widetilde{E}^n(X) \arrow[r, "\Sigma"] \arrow[d, "T(X)"'] & \widetilde{E}^{n+1}(\Sigma X) \arrow[d, "T(\Sigma X)"] \\
\widetilde{E'}^n(X) \arrow[r, "\Sigma"] & \widetilde{E'}^n(\Sigma X).
\end{tikzcd}$$

The transformation $T$ is called an *equivalence* if each natural transformation is a natural equivalence.

Theorem B.2.4.
*For any based map $f : A \to X$, we have the following exact sequence :*

$$\cdots \longleftarrow \widetilde{E}_n(A) \longleftarrow \widetilde{E}_n(X) \longleftarrow \widetilde{E}_n(Cf) \longleftarrow \widetilde{E}_{n-1}(A) \longleftarrow \cdots .$$

Theorem B.2.5.
*A generalized cohomology theory $E^*$ on $\mathbf{Top}_{\text{rel}}$ determines and is determined by its restriction to a generalized cohomology theory $E^*$ on pairs of* CW*-complexes. A generalized reduced cohomology theory $\widetilde{E}^*$ on ${}_{\text{well}}\mathbf{Top}_*$ determines and is determined by its restriction to a generalized cohomology theory $\widetilde{E}^*$ of based* CW*-complexes.*

Theorem B.2.6.
*A generalized cohomology theory $E^*$ on $\mathbf{Top}_{\mathrm{rel}}$ determines and is determined by a generalized reduced cohomology theory $\widetilde{E}^*$ on ${}_{\mathrm{well}}\mathbf{Top}_*$.*

We end this appendix by the following theorem, where a proof can be found in [May, 1999] or [Eilenberg and Steenrod, 1952]. It justifies the notation $H_*(-,G)$ and $H^*(-,G)$. We give an alternative proof for cohomology in page 48, using our work on $\Omega$-prespectra.

Theorem B.2.7 (Eilenberg-Steenrod).
*Let $T : h \to k$ be a map of reduced homology or cohomology theories. If $T(S^0)$ is an isomorphism, then $T$ is an equivalence of reduced homology or cohomology theories.*

# APPENDIX C

# Limits and Colimits

In this appendix, we define the categorical notion of limits and colimits. A wide diversity of objects in algebraic topology are built using limits and colimits. Our summary is based on [Strom, 2011], [May, 1999], [MacLane, 1971] and [Borceux, 1994].

## C.1. Definitions and Examples

Throughout this appendix, let $\mathscr{C}$ be any category and $\mathscr{J}$ be a small category. Denote by $\mathscr{C}^{\mathscr{J}}$ the *functor category* from $\mathscr{J}$ to $\mathscr{C}$, whose objects are functors $F : \mathscr{J} \to \mathscr{C}$, and whose morphisms are natural transformations $F \Rightarrow F'$, between functors $F, F' : \mathscr{J} \to \mathscr{C}$. We call an object of $\mathscr{C}^{\mathscr{J}}$ a *diagram in $\mathscr{C}$ of shape $\mathscr{J}$*, or simply a *$\mathscr{J}$-shaped diagram in $\mathscr{C}$*. This change of terminology reflects the fact that we think of a functor from $\mathscr{J}$ to $\mathscr{C}$ as indexing a family of objects and morphisms in $\mathscr{C}$, as the category $\mathscr{J}$ would often be thought as an index category.

Definition C.1.1 (Cones and Co-cones).
Let $F$ be a $\mathscr{J}$-shaped diagram in $\mathscr{C}$. Let $C$ be an object in $\mathscr{C}$. A *cone from $C$ to $F$* is a family of morphisms $\{\varphi_J : C \to F(J) \mid J \in \mathrm{Ob}\mathscr{J}\}$ in $\mathscr{C}$ such that for every morphism $j : J \to J'$ in $\mathscr{J}$, the following diagram commutes :

$$\begin{tikzcd} & C \arrow[dl, "\varphi_J"'] \arrow[dr, "\varphi_{J'}"] & \\ F(J) \arrow[rr, "F(j)"'] & & F(J'). \end{tikzcd}$$

Dually, a *co-cone from $F$ to $C$* is a family of morphisms $\{\varphi_J : F(J) \to C \mid J \in \mathrm{Ob}\mathscr{J}\}$ in $\mathscr{C}$ such that for every morphism $j : J \to J'$ in $\mathscr{J}$, the following diagram commutes :

$$\begin{tikzcd} F(J) \arrow[rr, "F(j)"] \arrow[dr, "\varphi_J"'] & & F(J') \arrow[dl, "\varphi_{J'}"] \\ & C. & \end{tikzcd}$$

We also say that $(C, \varphi)$ is a *(co-)cone of the $\mathscr{J}$-shaped diagram $F$*.

Definition C.1.2 (Limits).
Let $F : \mathscr{J} \to \mathscr{C}$ be a $\mathscr{J}$-shaped diagram in $\mathscr{C}$. The *limit* of the $\mathscr{J}$-shaped diagram $F$ is a cone

$(L,\varphi)$ of $F$ such that for any other cone $(N,\psi)$ of $F$, there exists a unique morphism $N \xrightarrow{u} L$ in $\mathscr{C}$, such that $\varphi_J \circ u = \psi_J$, for any object $J$ in $\mathscr{J}$. In other words, for any morphism $j : J \to J'$ in $\mathscr{J}$, we have the commutative diagram :

$$\begin{tikzcd} & N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_J"'] \arrow[ddr, bend left, "\psi_{J'}"] & \\ & L \arrow[dl, "\varphi_J"'] \arrow[dr, "\varphi_{J'}"] & \\ F(J) \arrow[rr, "F(j)"'] & & F(J'). \end{tikzcd}$$

The object $L$ is unique up to isomorphism in $\mathscr{C}$, and will be denoted $\lim F := L$.

Definition C.1.3 (Colimits).
The *colimit* of a $\mathscr{J}$-shaped diagram $F$ in $\mathscr{C}$ is a co-cone $(L,\varphi)$ of $F$, such that for any other co-cone $(N,\psi)$ of $F$, there exists a unique morphism $L \xrightarrow{u} N$ in $\mathscr{C}$ such that $u \circ \varphi_J = \psi_J$, for any object $J$ in $\mathscr{J}$. In other words, for any morphism $j : J \to J'$ in $\mathscr{J}$, we have the commutative diagram :

$$\begin{tikzcd} F(J) \arrow[rr, "F(j)"] \arrow[dr, "\varphi_J"] \arrow[ddr, bend right, "\psi_J"'] & & F(J') \arrow[dl, "\varphi_{J'}"'] \arrow[ddl, bend left, "\psi_{J'}"] \\ & L \arrow[d, dashed, "u"] & \\ & N. & \end{tikzcd}$$

The object $L$ is unique up to isomorphism in $\mathscr{C}$, and will be denoted $\mathrm{colim} F := L$.

Example C.1.4 (Products and Coproducts).
Let $\mathscr{J}$ be any (small) discrete category, that is, a category whose only morphisms are the identity morphisms. Then a $\mathscr{J}$-shaped diagram $F$ in $\mathscr{C}$ is just a family of objects in $\mathscr{C}$, indexed by objects of $\mathscr{J}$. Let us write $F = \{C_j\}_{j\in \mathrm{Ob}\mathscr{J}}$ for such a family. If $\mathscr{C} = \mathbf{Set}$, the category of sets, then the limit of $F$ is just the cartesian product :

$$\lim F = \prod_{j\in \mathrm{Ob}\mathscr{J}} C_j,$$

together with the projections $\{\prod_{j\in \mathrm{Ob}\mathscr{J}} C_j \to C_j\}_{j\in \mathrm{Ob}\mathscr{J}}$. If $\mathscr{C}$ is the category of topological spaces, based topological spaces, groups or abelian groups, the previous statement still holds. For instance, when we endow the cartesian product with the product topology, it *is* the product in **Top**. More generally, for any category $\mathscr{C}$, the limit in this case is the usual definition of the product in category theory. We call the *natural projections* the collection $\{\mathrm{pr}_j : \prod_{j\in \mathrm{Ob}\mathscr{J}} C_j \to C_j\}_{j\in \mathrm{Ob}\mathscr{J}}$ associated to the definition of the product.
Similarly, the colimit of such a family is the coproduct in category theory. Therefore, if $\mathscr{C}$ equals **Set** or **Top**, the category of topological spaces, then the colimit is the disjoint union :

$$\mathrm{colim} F = \coprod_{j\in \mathrm{Ob}\mathscr{J}} C_j,$$

together with the inclusions $\{\iota_j : C_j \hookrightarrow \coprod_{j\in \mathrm{Ob}\mathscr{J}} C_j\}_{j\in \mathrm{Ob}\mathscr{J}}$. If $\mathscr{C}$ equals the category of based topological spaces $\mathbf{Top}_*$, then the colimit is the wedge : $\mathrm{colim} F = \bigvee_{j\in \mathrm{Ob}\mathscr{J}} C_j$. If $\mathscr{C}$ equals the category of groups **Gr**, then the colimit is the free product : $\mathrm{colim}F = \ast_{j\in\mathrm{Ob}\mathscr{J}} C_j$. If $\mathscr{C}$ equals the category of abelian groups **Ab**, then the colimit is the free abelian group : $\mathrm{colim}F = \bigoplus_{j\in\mathrm{Ob}\mathscr{J}} C_j$. We call the *canonical inclusions* the collection $\{\iota_j : C_j \hookrightarrow \coprod_{j\in\mathrm{Ob}\mathscr{J}} C_j\}_{j\in\mathrm{Ob}\mathscr{J}}$ associated to the definition of the coproduct, in any category.

Example C.1.5 (Pullbacks and Pushouts).
Let $\mathscr{J}$ be the category whose objects are given by the set $\{-1, 0, 1\}$, and whose morphisms are $(-1) \to 0$, $1 \to 0$, and the identities. One shorthand schematically the description of such a category by : $\mathscr{J} := \{-1 \to 0 \leftarrow 1\}$. Then a $\mathscr{J}$-shaped diagram $F$ in a category $\mathscr{C}$ is given by :

$$B \to A \leftarrow C,$$

where $A$, $B$ and $C$ are objects in $\mathscr{C}$. Its limit is given, by definition, by an object $P = \mathrm{lim}F$ in $\mathscr{C}$ together with morphisms $P \to B$ and $P \to C$, such that for any other object $P'$, and morphisms $P' \to B$ and $P' \to C$, there exists a unique morphism $P' \to P$, such that the following diagram commutes :

$$\begin{tikzcd} & P' \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & P \arrow[dl] \arrow[d] \arrow[dr] & \\ B \arrow[r] & A & C. \arrow[l] \end{tikzcd}$$

This is precisely the definition of the pullback of the diagram $B \to A \leftarrow C$.
Similarly, if $\mathscr{J} = \{-1 \leftarrow 0 \to 1\}$, then the colimit of a $\mathscr{J}$-shaped diagram in a category $\mathscr{C}$ is a pushout in $\mathscr{C}$.

## C.2. The Existence Theorem

This section is not necessary to understand this paper. However, we do this detour as it is much enlightening and helpful to understand the concept of limits and colimits.

Equalizers and Coequalizers If $\mathscr{J} = \{0 \rightrightarrows 1\}$, then a $\mathscr{J}$-shaped diagram $F$ in a category $\mathscr{C}$ is given by a diagram $A \rightrightarrows B$, where $A$ and $B$ are objects in $\mathscr{C}$. The limit of $F$ is called an *equalizer*, and the colimit of $F$ is called a *coequalizer*. If we name $f, g : A \to B$ the morphisms in $\mathscr{C}$ in $F$, then we denote $\mathrm{Equ}(f, g) := \mathrm{lim}F$, and $\mathrm{Coequ}(f, g) := \mathrm{colim}F$. We write $e : \mathrm{Equ}(f, g) \to A$ and $q : B \to \mathrm{Coequ}(f, g)$ the induced morphisms in $\mathscr{C}$, forming the cone $(\mathrm{Equ}(f, g), e)$ and the co-cone $(\mathrm{Coequ}(f, g), q)$. If pullbacks exist in $\mathscr{C}$, then equalizers exist since they are given by the following pullback :

$$\begin{tikzcd} \mathrm{Equ}(f,g) \arrow[r] \arrow[d, "e"'] & B \arrow[d, "\Delta"] \\ A \arrow[r, "{(f,g)}"] & B \times B. \end{tikzcd}$$

Conversely, if products and equalizers exist, then pullbacks exist. Indeed, suppose we have the diagram : $B \xrightarrow{f} A \xleftarrow{g} C$. Then the product $B \times C$ induces two parallel diagonal morphisms to $A$ :

$$\begin{tikzcd}
B \times C \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr] & C \arrow[d, "g"] \\
B \arrow[r, "f"'] & A.
\end{tikzcd}$$

We take the equalizer of these morphisms :

$$\begin{tikzcd}
\mathrm{Equ}(f \circ p_1, g \circ p_2) \arrow[dr, "e"] \arrow[drr, bend left, "p_2 \circ e"] \arrow[ddr, bend right, "p_1 \circ e"'] & & \\
& B \times C \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr] & C \arrow[d, "g"] \\
& B \arrow[r, "f"'] & A.
\end{tikzcd}$$

Then one can easily check that $\mathrm{Equ}(f \circ p_1, g \circ p_2)$ is the pullback of $B \xrightarrow{f} A \xleftarrow{g} C$.
Similarly, if pushouts exist, then coequalizers exist. If coequalizers and coproducts exist, then pushouts exist.

We now state the main theorem of this appendix which states that it suffices to know (co)-products and pushouts/pullbacks in a category to build any (co)-limits.

Definition C.2.1.
A category $\mathscr{C}$ is said to be *complete* if every $\mathscr{J}$-shaped diagram has a limit in $\mathscr{C}$, for any small category $\mathscr{J}$. The category $\mathscr{C}$ is said to be *cocomplete* if every $\mathscr{J}$-shaped diagram has a colimit in $\mathscr{C}$, for any small category $\mathscr{J}$. The category $\mathscr{C}$ is *bicomplete* if it is both complete and cocomplete.

Theorem C.2.2 (A Criterion for Bicompleteness).
*A category $\mathscr{C}$ is complete if and only if $\mathscr{C}$ has pullbacks and products. The category $\mathscr{C}$ is cocomplete if and only if $\mathscr{C}$ has pushouts and coproducts.*

Sketch of the Proof : Let us prove only the completeness, as the proof of the cocompleteness is similar. Let $F : \mathscr{J} \to \mathscr{C}$ be a $\mathscr{J}$-shaped diagram. We construct the limit of $F$ as follows : we take two products and find two arrows between them, of which we take the equalizer. Indeed, first take the product of all objects in $\mathscr{C}$ in the image of $F$ : $\prod_{J \in \mathrm{Ob}\,\mathscr{J}} F(J)$, then take the product of all objects in $\mathscr{C}$ which are in the image of codomains of all morphisms in $\mathscr{J}$ : $\prod_{j \in \mathrm{Mor}\,\mathscr{J}} F(\mathrm{cod}\ j)$. Let us name $\{p_J\}_{J \in \mathrm{Ob}\,\mathscr{J}}$ and $\{p_j\}_{j \in \mathrm{Mor}\,\mathscr{J}}$ the projections of the products. We thus have the following commutative diagram, for any morphism $j$ in $\mathscr{J}$ :

$$\begin{tikzcd}
& F(\mathrm{cod}\ j) \\
\prod\limits_{J \in \mathrm{Ob}\,\mathscr{J}} F(J) \arrow[ur, "p_{\mathrm{cod}\ j}"] \arrow[r, dashed, "f"', "g"] \arrow[d, "p_{\mathrm{dom}\ j}"'] & \prod\limits_{j \in \mathrm{Mor}\,\mathscr{J}} F(\mathrm{cod}\ j) \arrow[u, "p_j"'] \arrow[d, "p_j"] \\
F(\mathrm{dom}\ j) \arrow[r, "F(j)"] & F(\mathrm{cod}\ j).
\end{tikzcd}$$

The universal property of products gives two morphisms $f$ and $g$, dashed in the diagram. We then take the equalizer :

$$\begin{tikzcd}
\mathrm{Equ}(f,g) \arrow[r, "e"] \arrow[d, "\varphi_J"'] & \prod\limits_{J\in \mathrm{Ob}\,\mathscr{J}} F(J) \arrow[r, dashed, shift left, "f"] \arrow[r, dashed, shift right, "g"'] \arrow[dl, "p_J"] & \prod\limits_{j\in \mathrm{Mor}\,\mathscr{J}} F(\mathrm{cod}\ j) \\
F(J). & &
\end{tikzcd}$$

Name $\varphi_J$ the composite $p_J \circ e$, for every object $J$ in $\mathscr{J}$. We get a cone $(\mathrm{Equ}(f,g), \varphi)$, as one can easily check. The limit of $F$ is the cone $(\mathrm{Equ}(f,g), \varphi)$. $\square$

Therefore, the categories of sets, topological spaces, pointed topological spaces, CW-complexes, groups, abelian groups, modules over a ring and vector spaces over a field, are all bicomplete. However, the category of fields is neither complete or cocomplete. For instance, the product fails : the cartesian product $\mathbb{R} \times \mathbb{R}$ is not a field with the laws induced by the projections.

## C.3. Interchange in Limits and Colimits

Let $\mathscr{J}$ and $\mathscr{K}$ be small categories. We want to prove in this part that for any given functor $F : \mathscr{J} \times \mathscr{K} \longrightarrow \mathscr{C}$, there is a interchange property :

$$\lim_{J\in\mathscr{J}}(\lim_{K\in\mathscr{K}} F(J,K)) \cong \lim_{K\in\mathscr{K}}(\lim_{J\in\mathscr{J}} F(J,K)),$$

as long as the involved limits exist. Before showing this statement, we define all the notations used. We start with the following definition.

Definition C.3.1.
The *product of two categories* $\mathscr{C}$ and $\mathscr{D}$ is the category $\mathscr{C} \times \mathscr{D}$ defined in the following way.

- **(i)** The objects of $\mathscr{C} \times \mathscr{D}$ are the pairs $(C, D)$ with $C$ object in $\mathscr{C}$ and $D$ object in $\mathscr{D}$.
- **(ii)** The morphism $(C, D) \to (C', D')$ of $\mathscr{C} \times \mathscr{D}$ are the pairs $(c, d)$, where $c : C \to C'$ is a morphism in $\mathscr{C}$ and $j : D \to D'$ is a morphism in $\mathscr{D}$.
- **(iii)** The composition in $\mathscr{C} \times \mathscr{D}$ is that induced by the compositions of $\mathscr{C}$ and $\mathscr{D}$, in other words : $(c', d') \circ (c, d) = (c' \circ c, d' \circ d)$.

We emphasize the following argument that we shall used in our next discussion.

Lemma C.3.2.
*Let $\mathscr{C}$ be a category, and $\mathscr{J}$ be a small category. If a cone $(L, \varphi)$ is a limit of a $\mathscr{J}$-shaped diagram $F$ in $\mathscr{C}$, then two morphisms $f, g : C \to L$ in $\mathscr{C}$ are equal as long $\varphi_J \circ f = \varphi_J \circ g$, for every object $J$ in $\mathscr{J}$.*

Proof : It follows directly from universal property of the limits and the cones $(C, \{\varphi_J \circ f\}_{J\in\mathscr{J}})$ and $(C, \{\varphi_J \circ g\}_{J\in\mathscr{J}})$ of the functor $F$. $\square$

Next, we define the notation $\lim_{J\in\mathscr{J}}(\lim_{K\in\mathscr{K}} F(J,K))$. The description of the other term in the equation $\lim_{K\in\mathscr{K}}(\lim_{J\in\mathscr{J}} F(J,K))$ will be completely analogous and shall be omitted. Let $\mathscr{J} \times \mathscr{K} \longrightarrow \mathscr{C}$ be a fixed functor. For every object $J$ in $\mathscr{J}$, there is a functor :

$$F(J, -) : \mathscr{K} \longrightarrow \mathscr{C},$$

defined by : $F(J,-)(K) = F(J,K)$ for every object $K$ in $\mathscr{K}$, and $F(J,-)(k) = F(\mathrm{id}_J, k)$, for every morphism $k$ in $\mathscr{K}$. By $\lim_{K\in\mathscr{K}} F(J,K)$, we mean the limit of the $\mathscr{K}$-shaped diagram $F(J,-)$ in $\mathscr{C}$. Now every morphism $j : J \to J'$ in $\mathscr{J}$ induces a morphism $F(j, \mathrm{id}_K)$ in $\mathscr{C}$, for every object $K$ in $\mathscr{K}$. This leads to a natural transformation $F(j,-) : F(J,-) \Rightarrow F(J',-)$ since we have the commutativity of the following diagram for every morphism $k : K \to K'$ in $\mathscr{K}$ :

$$\begin{tikzcd}
F(J,K) \arrow[r, "F(j,\mathrm{id}_K)"] \arrow[d, "F(\mathrm{id}_J,k)"'] \arrow[dr, "F(j,k)"] & F(J',K) \arrow[d, "F(\mathrm{id}_{J'},k)"] \\
F(J,K') \arrow[r, "F(j,\mathrm{id}_{K'})"'] & F(J',K').
\end{tikzcd}$$

For every $K_0$ in $\mathscr{K}$, we write $\varphi_{K_0} : \lim_{K\in\mathscr{K}} F(J,K) \to F(J,K_0)$ and $\varphi'_{K_0} : \lim_{K\in\mathscr{K}} F(J',K) \to F(J',K_0)$ the family of morphisms in $\mathscr{K}$, forming the cones of the limits of the $\mathscr{K}$-shaped diagrams $F(J,-)$ and $F(J',-)$. The composite :

$$\lim_{K\in\mathscr{K}} F(J,K) \xrightarrow{\varphi_{K_0}} F(J,K_0) \xrightarrow{F(j,\mathrm{id}_{K_0})} F(J',K_0),$$

constitutes a cone on the functor $F(J',-)$. Hence, by the universal property of limits, there is a unique morphism denoted :

$$\lim_{K\in\mathscr{K}} F(j,\mathrm{id}_K) : \lim_{K\in\mathscr{K}} F(J,K) \longrightarrow \lim_{K\in\mathscr{K}} F(J',K)$$

such that the following diagram commutes for every object $K_0$ in $\mathscr{K}$ :

$$\begin{tikzcd}
\lim_{K\in\mathscr{K}} F(J,K) \arrow[r, "\lim_{K\in\mathscr{K}} F(j,\mathrm{id}_K)"] \arrow[d, "\varphi_{K_0}"] & \lim_{K\in\mathscr{K}} F(J',K) \arrow[d, "\varphi'_{K_0}"] \\
F(J,K_0) \arrow[r, "F(j,\mathrm{id}_K)"'] & F(J',K_0).
\end{tikzcd} \tag{$\diamond$}$$

If all the $\mathscr{K}$-shaped diagrams $F(J,-)$ have limits for every object $J$ in $\mathscr{J}$, we can define a functor :

$$L : \mathscr{J} \longrightarrow \mathscr{C},$$

by $L(J) = \lim_{K\in\mathscr{K}} F(J,K)$ and $L(j) = \lim_{K\in\mathscr{K}} F(j,\mathrm{id}_K)$, for every object $J$ in $\mathscr{J}$ and morphism $j$ in $\mathscr{J}$.

It is indeed a functor. Let $J \xrightarrow{j} J' \xrightarrow{j'} J''$ be morphisms in $\mathscr{J}$. Using commutativity of the diagram ($\diamond$), we have for all $K$ in $\mathscr{K}$ :

$$\begin{aligned}
\varphi''_K \circ L(j') \circ L(j) &= F(j', \mathrm{id}_K) \circ \varphi'_K \circ L(j) \\
&= F(j', \mathrm{id}_K) \circ F(j, \mathrm{id}_K) \circ \varphi_K \\
&= F(j' \circ j, \mathrm{id}_K) \circ \varphi_K
\end{aligned}$$

Hence $L(j') \circ L(j) = L(j' \circ j)$ by Lemma C.3.2. Similarly, we have $L(\mathrm{id}_J) = \mathrm{id}_{L(J)}$. So $L$ is a functor. We denote by : $\lim_{J\in\mathscr{J}}(\lim_{K\in\mathscr{K}} F(J,K))$ the limit of the $\mathscr{J}$-shaped diagram $L$, if it exists.

The interchange property :

$$\lim_{J\in\mathscr{J}}(\lim_{K\in\mathscr{K}} F(J,K)) \cong \lim_{K\in\mathscr{K}}(\lim_{J\in\mathscr{J}} F(J,K)),$$

means that the *canonical morphisms* connecting these two limits are in fact isomorphisms. Let us define what we mean by canonical morphism. Starting with the limit of $L$ defined above, we have the composites for every $J$ and $K$ :

$$\mathrm{lim}L \xrightarrow{\varphi_J} \mathrm{lim}_{K\in\mathscr{K}}F(J,K) \xrightarrow{\varphi_K} F(J,K).$$

Let $K$ be a fixed object in $\mathscr{K}$ and $j : J \to J'$ be a fixed morphism in $\mathscr{J}$. The composites $\varphi_K \circ \varphi_J$ form a cone of the functor $F(-,K)$ since we have :

$$\begin{aligned} F(j,\mathrm{id}_K)\circ\varphi_K\circ\varphi_J &= \varphi'_K\circ\mathrm{lim}_{K\in\mathscr{K}}F(j,\mathrm{id}_K)\circ\varphi_J \\ &= \varphi'_K\circ\varphi_{J'}. \end{aligned}$$

Therefore, there exists a unique morphism $\lambda_K : \mathrm{lim}L \to \mathrm{lim}_{J\in\mathscr{J}}F(J,K)$ such that the following diagram commutes, for every $J$ and $K$ :

$$\begin{tikzcd} \mathrm{lim}L \arrow[r, "\lambda_K"] \arrow[d, "\varphi_J"] & \mathrm{lim}_{J\in\mathscr{J}}F(J,K) \arrow[d, "\overline{\varphi}_J"] \\ \mathrm{lim}_{K\in\mathscr{K}}F(J,K) \arrow[r, "\varphi_K"] & F(J,K). \end{tikzcd}$$

where $\overline{\varphi}_J$ is the morphism which stems from the cone $\mathrm{lim}_{J\in\mathscr{J}}F(J,K)$. Let $k : K \to K'$ be a morphism in $\mathscr{K}$. Define $\overline{\varphi}'_J$ the morphism constituting the cone $\mathrm{lim}_{J\in\mathscr{J}}F(J,K')$. We have :

$$\begin{aligned} \overline{\varphi}'_J\circ\mathrm{lim}_{J\in\mathscr{J}}F(\mathrm{id}_J,k)\circ\lambda_K &= F(\mathrm{id}_J,k)\circ\overline{\varphi}_J\circ\lambda_K \\ &= F(\mathrm{id}_J,k)\circ\varphi_K\circ\varphi_J \\ &= \varphi_{K'}\circ\varphi_J \\ &= \overline{\varphi}'_K\circ\lambda_{K'}. \end{aligned}$$

Therefore we get : $\mathrm{lim}_{J\in\mathscr{J}}F(\mathrm{id}_J,k)\circ\lambda_K = \lambda_{K'}$, using Lemma C.3.2. Thus the morphisms $\lambda_K$ form a cone, and so there exists a unique morphism $\lambda : \mathrm{lim}L \to \mathrm{lim}_{K\in\mathscr{K}}(\mathrm{lim}_{J\in\mathscr{J}}F(J,K))$, which is one of the desired canonical morphism. Analogously we can define the other canonical morphism :

$$\mu : \mathrm{lim}_{K\in\mathscr{K}}(\mathrm{lim}_{J\in\mathscr{J}}F(J,K)) \longrightarrow \mathrm{lim}_{J\in\mathscr{J}}(\mathrm{lim}_{K\in\mathscr{K}}F(J,K)).$$

Theorem C.3.3 (Interchange Property of Limits).
*Let $\mathscr{C}$ be a complete category. Let $\mathscr{J}$ and $\mathscr{K}$ be small categories. Given a functor $F : \mathscr{J}\times\mathscr{K} \to \mathscr{C}$, and using previous notations, the following interchange property holds :*

$$\mathrm{lim}_{J\in\mathscr{J}}(\mathrm{lim}_{K\in\mathscr{K}}F(J,K)) \cong \mathrm{lim}_{K\in\mathscr{K}}(\mathrm{lim}_{J\in\mathscr{J}}F(J,K)).$$

Proof : We want to prove that the canonical morphisms $\lambda$ and $\mu$ defined in our previous discussion are mutual inverse isomorphisms. Let us prove that $\mu\circ\lambda = \mathrm{id}$. By Lemma C.3.2, it suffices to prove :

$$\varphi_K\circ\varphi_J\circ\mu\circ\lambda = \varphi_K\circ\varphi_J,$$

for every objects $J$ and $K$. But this is straightfoward from the definition of the canonical morphisms. Similarly, we have $\lambda\circ\mu = \mathrm{id}$. □

Of course, all of our work can be dualized for colimits. We have therefore the dual theorem.

Theorem C.3.4 (Interchange Property of Colimits).
*Let $\mathscr{C}$ be a cocomplete category. Let $\mathscr{J}$ and $\mathscr{K}$ be small categories. Given a functor $F : \mathscr{J}\times\mathscr{K} \to \mathscr{C}$, the following interchange property holds :*

$$\mathrm{colim}_{J\in\mathscr{J}}(\mathrm{colim}_{K\in\mathscr{K}}F(J,K)) \cong \mathrm{colim}_{K\in\mathscr{K}}(\mathrm{colim}_{J\in\mathscr{J}}F(J,K)).$$

## C.4. Towers and Telescopes

In algebraic topology, we are often faced with special cases of diagrams called towers and telescopes. For instance CW-complexes and stable homotopy groups are colimits of telescopes.

Towers If $\mathscr{J} = \{0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\}$, then a $\mathscr{J}$-shaped diagram $F$ in a category $\mathscr{C}$ is just a sequence of morphisms in $\mathscr{C}$ of the form :

$$C_0 \xleftarrow{f_0} C_1 \xleftarrow{f_1} C_2 \longleftarrow \cdots \longleftarrow C_n \xleftarrow{f_n} C_{n+1} \longleftarrow \cdots .$$

We call such a diagram a *tower* and we denote its limit $\mathrm{lim} F$ by $\mathrm{lim}_n C_n$. Let us emphasize the universal property in this case. Write $L$ the limit. There is a collection of morphisms $\{\varphi_n : L \to C_n\}_n$ in $\mathscr{C}$, satisfying $f_{n-1} \circ \varphi_n = \varphi_{n-1}$ for any $n \geq 1$, such that for any object $Y$ together with morphisms $\{\psi_n : Y \to C_n\}_n$ in $\mathscr{C}$ with $f_{n-1} \circ \psi_n = \psi_{n-1}$, there exists a unique morphism $u : Y \to L$ in $\mathscr{C}$, such that $\varphi_n \circ u = \psi_n$, for any $n$, i.e., the following diagram commutes :

$$\begin{tikzcd}
 & & \vdots \arrow[d] \\
 & & C_{n+1} \arrow[d, "f_n"] \\
 & & C_n \arrow[d] \\
Y \arrow[r, dashed, "u"] \arrow[uuurr, bend left] \arrow[uurr, bend left] \arrow[urr, bend left] \arrow[drr, bend right] \arrow[ddrr, bend right] & L \arrow[uuur] \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \vdots \arrow[d, "f_1"] \\
 & & C_1 \arrow[d, "f_0"] \\
 & & C_0.
\end{tikzcd}$$

Define the *shift map* :

$$\mathrm{sh} : \prod_{n\geq 0} C_n \longrightarrow \prod_{n \geq 0} C_n,$$

by the formula $\mathrm{sh} = (f_0 \circ \mathrm{pr}_1, f_1 \circ \mathrm{pr}_2, \ldots, f_n \circ \mathrm{pr}_{n+1}, \ldots)$, where $\mathrm{pr}_j : \prod_{n\geq 0} C_n \to C_j$ are the natural projections given by the definition of the product. Then it easy to see that the limit $L = \mathrm{lim}_n C_n$ is given by the following pullback (actually an equalizer) :

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & \prod_{n\geq 0} C_n \arrow[d, "{(\mathrm{sh},\mathrm{id})}"] \\
\prod_{n\geq 0} C_n \arrow[r, "\Delta"'] & \prod_{n\geq 0} C_n \times \prod_{n\geq 0} C_n.
\end{tikzcd}$$

Indeed, name $\varphi_j : L \to C_j$ the morphism defined as the composite :

$$L \longrightarrow \prod_{n\geq 0} C_n \xrightarrow{\mathrm{pr}_j} C_j,$$

where the unlabeled morphism is the one given by the pullback (they are both the same as it is actually an equalizer). It follows that $f_{n-1} \circ \varphi_n = \varphi_{n-1}$, for any $n \geq 1$ (thanks to the shift map). Now suppose we have an object $Y$ together with morphisms $\{\psi_n : Y \to C_n\}_n$ in $\mathscr{C}$ with $f_{n-1} \circ \psi_n = \psi_{n-1}$. By the universal property of products, there is a unique morphism $\Psi : Y \to \prod_{n\geq 0} C_n$ such that $\mathrm{pr}_n \circ \Psi = \psi_n$, for any $n \geq 0$. Hence, the universal property of pullbacks (or equalizers) gives a unique morphism $u : Y \to L$ such that the following diagram commutes :

$$\begin{tikzcd}
Y \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\Psi"] \arrow[ddr, bend right, "\Psi"'] & & \\
& L \arrow[r] \arrow[d] & \prod\limits_{n\geq 0} C_n \arrow[d, "(\mathrm{sh},\mathrm{id})"] \\
& \prod\limits_{n\geq 0} C_n \arrow[r, "\Delta"'] & \prod\limits_{n\geq 0} C_n \times \prod\limits_{n\geq 0} C_n.
\end{tikzcd}$$

It follows that $\varphi_n \circ u = \psi_n$, for every $n \geq 0$, and therefore $L$ is indeed the limit of the tower. Thus, in $\mathscr{C} = \mathbf{Set}$, the limit is given by $L = \{(x_n)_{n\geq 0} \in \prod_{n\geq 0} C_n \mid f_n(x_n) = x_n, \forall n \geq 0\}$. We can carry out this construction for the categories **Gr**, **Ab**, **Top**, etc.

Telescopes If $\mathscr{J} = \{0 \to 1 \to 2 \to 3 \to \cdots\}$, then a $\mathscr{J}$-shaped diagram $F$ in a category $\mathscr{C}$ is just a sequence of morphisms in $\mathscr{C}$ :

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \longrightarrow \cdots \longrightarrow C_n \xrightarrow{f_n} C_{n+1} \longrightarrow \cdots .$$

We call such a diagram a *telescope*, and we denote its colimit $\mathrm{colim} F$ by $\mathrm{colim}_n C_n$. We again emphasize the universal property in this case. Let $L$ denote the colimit. There is a collection of morphisms $\{\varphi_n : C_n \to L\}_n$ satisfying $\varphi_{n+1} \circ f_n = \varphi_n$, for any $n \geq 0$, such that for any other object $Y$ in $\mathscr{C}$, together with morphisms $\{\psi_n : C_n \to Y\}_n$, such that $\psi_{n+1} \circ f_n = \psi_n$, there exists a unique morphism $u : L \to Y$ such that $u \circ \varphi_n = \psi_n$, for any $n \geq 0$, i.e., the following diagram commutes :

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] \arrow[drr] \arrow[ddrr, bend right] & C_1 \arrow[r, "f_1"] \arrow[dr] \arrow[ddr, bend right] & \cdots \arrow[r] & C_n \arrow[r, "f_n"] \arrow[dl] \arrow[ddl, bend left] & C_{n+1} \arrow[r] \arrow[dll] \arrow[ddll, bend left] & \cdots \arrow[dlll] \arrow[ddlll, bend left] \\
& & L \arrow[d, dashed, "u"] & & & \\
& & Y. & & &
\end{tikzcd}$$

Define the *shift map* :

$$\mathrm{sh} : \coprod_{n\geq 0} C_n \longrightarrow \coprod_{n\geq 0} C_n,$$

by the formula $\mathrm{sh} = (\iota_1 \circ f_0, \iota_2 \circ f_1, \ldots, \iota_{n+1} \circ f_n, \ldots)$, where $\iota_j : C_j \to \coprod_{n\geq 0} C_n$ are the canonical inclusions given by the definition of the coproduct. Then it easy to see that the colimit $L = \mathrm{colim}_n C_n$ is given, by an argument dual to the previous case, by the following pushout (again, actually a coequalizer) :

$$\begin{array}{ccc} \coprod_{n\geq 0} C_n \sqcup \coprod_{n\geq 0} C_n & \xrightarrow{(\mathrm{sh},\mathrm{id})} & \coprod_{n\geq 0} C_n \\ \nabla\big\downarrow & & \big\downarrow \\ \coprod_{n\geq 0} C_n & \longrightarrow & L. \end{array}$$

Therefore, for $\mathscr{C} = \mathbf{Set}$, the colimit is given by $L = \coprod_{n\geq 0} C_n/\sim$, where $\sim$ is an equivalence relation given by $x_n \sim f_n(x_n)$, for every $n \geq 0$ and $(x_n)_{n\geq 0}$ in $\coprod_{n\geq 0} C_n$. We can carry out this construction for the categories **Gr**, **Ab**, **Top**, etc.
For instance, if $\mathscr{C} = \mathbf{Top}$ and the maps $f_n$ are inclusions, then the colimit of the telescope is just the union $\bigcup_{n\geq 0} C_n$ endowed with its coarsest topology such that the canonical inclusion $\iota_j : C_j \to \bigcup_{n\geq 0} C_n$ are all continuous, i.e., its weak topology : $A$ is closed in $\bigcup_{n\geq 0} C_n$ if and only if $A \cap C_n$ is closed in $C_n$, for each $n \geq 0$. This is exactly the construction used in the definition of a CW-complex. Another example can be given for $\mathscr{C} = \mathbf{Ab}$, with the following proposition, which is a just a reformulation of what we have just proved.

Proposition C.4.1.
*Suppose we have a telescope $A_0 \overset{f_0}{\to} A_1 \overset{f_1}{\to} A_2 \overset{f_2}{\to} \cdots$ of abelian groups. Then there is a short exact sequence :*

$$0 \longrightarrow \bigoplus_{j\geq 0} A_j \xrightarrow{\ \alpha\ } \bigoplus_{j\geq 0} A_j \xrightarrow{\ \beta\ } \mathrm{colim}_j A_j \longrightarrow 0,$$

*where $\alpha(a_j) = a_j - f_j(a_j)$ for each $a_j$ in $A_j$, and the restriction of $\beta$ to $A_j$ is the canonical homomorphism $\varphi_j : A_j \to \mathrm{colim}_j A_j$, forming the co-cone $(\mathrm{colim}_j A_j, \varphi_j)$.*[1]

We now present various useful results on telescopes. Let us first introduce the dual of Lemma C.3.2.

Lemma C.4.2.
*Let $\mathscr{C}$ be a category, and $\mathscr{J}$ be a small category. If a co-cone $(L, \varphi)$ is a colimit of a $\mathscr{J}$-shaped diagram $F$ in $\mathscr{C}$, then two morphisms $f, g : L \to C$ in $\mathscr{C}$ are equal as long $f \circ \varphi_J = g \circ \varphi_J$, for every object $J$ in $\mathscr{J}$.*

Proof : Follows directly from the universal property of colimits. □

This argument leads to the following two results.

Theorem C.4.3 (Colimits of Telescopes Preserve Exactness).
*Suppose $\mathscr{C} = \mathbf{Ab}$. Let $A_0 \overset{f_0}{\to} A_1 \overset{f_1}{\to} A_2 \overset{f_2}{\to} \cdots$, $B_0 \overset{g_0}{\to} B_1 \overset{g_1}{\to} B_2 \overset{g_2}{\to} \cdots$, and $C_0 \overset{h_0}{\to} C_1 \overset{h_1}{\to} C_2 \overset{h_2}{\to} \cdots$*

[1]The homomorphism $\beta \circ \alpha$ is actually the coequalizer of the shift map with the identity.

*be telescopes such that there is a commutative diagram :*

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "u_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "u_1"] & A_2 \arrow[r] \arrow[d, "u_2"] & \cdots \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "u_n"] & A_{n+1} \arrow[r] \arrow[d, "u_{n+1}"] & \cdots \\ B_0 \arrow[r, "g_0"] \arrow[d, "v_0"] & B_1 \arrow[r, "g_1"] \arrow[d, "v_1"] & B_2 \arrow[r] \arrow[d, "v_2"] & \cdots \arrow[r] & B_n \arrow[r, "g_n"] \arrow[d, "v_n"] & B_{n+1} \arrow[r] \arrow[d, "v_{n+1}"] & \cdots \\ C_0 \arrow[r, "h_0"] & C_1 \arrow[r, "h_1"] & C_2 \arrow[r] & \cdots \arrow[r] & C_n \arrow[r, "h_n"] & C_{n+1} \arrow[r] & \cdots , \end{tikzcd} \tag{C.1}$$

*where $A_j \overset{u_j}{\to} B_j \overset{v_j}{\to} C_j$ are exact, for each $j \geq 0$. Then there is an exact sequence :*

$$\mathrm{colim}_n A_n \xrightarrow{u} \mathrm{colim}_n B_n \xrightarrow{v} \mathrm{colim}_n C_n.$$

Proof : We first begin by constructing the canonical homomorphism $u$ and $v$. There exist morphisms $\varphi_j : A_j \to \mathrm{colim}_n A_n$ forming the co-cone with the colimit $\mathrm{colim}_n A_n$ of the telescope $A_0 \overset{f_0}{\to} A_1 \overset{f_1}{\to} A_2 \overset{f_2}{\to} \cdots$ . There exist morphisms $\psi_j : B_j \to \mathrm{colim}_n B_n$ forming the co-cone with the colimit $\mathrm{colim}_n B_n$ of the telescope $B_0 \overset{g_0}{\to} B_1 \overset{g_1}{\to} B_2 \overset{g_2}{\to} \cdots$. Hence, we have a co-cone $(\mathrm{colim}_n B_n, \{\psi_n \circ u_n\}_n)$ of the telescope $A_0 \overset{f_0}{\to} A_1 \overset{f_1}{\to} A_2 \overset{f_2}{\to} \cdots$. By the universal property of colimits, there exists a unique homomorphism $u : \mathrm{colim}_n A_n \to \mathrm{colim}_n B_n$, such that : $u \circ \varphi_n = \psi_n \circ u_n$, for every $n \geq 0$. Similarly, the universal property of colimits gives a unique homomorphism $v : \mathrm{colim}_n B_n \to \mathrm{colim}_n C_n$ such that : $v \circ \psi_n = \theta_n \circ v_n$, for each $n \geq 0$, where $\theta_j : C_j \to \mathrm{colim}_n C_n$ are the morphisms forming the co-cone with the colimit $\mathrm{colim}_n C_n$ of the telescope $C_0 \overset{h_0}{\to} C_1 \overset{h_1}{\to} C_2 \overset{h_2}{\to} \cdots$. Let us prove now the exactness.

im $u \subseteq \ker v$ We want to see that $v \circ u$ is the trivial homomorphism. From Lemma C.4.2, we only need to see that $v \circ u \circ \varphi_n$ is the trivial homomorphism, for each $n$. So let $a$ be in $A_n$. We have :

$$\begin{aligned} (v \circ u \circ \varphi_n)(a) &= v(\psi_n(u_n(a))) \\ &= \theta_n((v_n \circ u_n)(a)) \\ &= 0. \end{aligned}$$

Therefore im $u \subseteq \ker v$.

$\ker v \subseteq$ im $u$ From Lemma C.4.2, it suffices to suppose that we have $b$ in $B_n$ such that $(v \circ \psi_n)(b) = 0$, i.e., $(\theta_n \circ v_n)(b) = 0$. We get $(\theta_{n+1} \circ h_n \circ v_n)(b) = 0$. So $(h_n \circ v_n)(b) = 0$ by Proposition C.4.1. From commutativity of (C.1), we have : $(v_{n+1} \circ g_n)(b) = 0$. By exactness, there is $a$ in $A_{n+1}$ such that $u_{n+1}(a) = g_n(b)$. Therefore we get :

$$\begin{aligned} (u \circ \varphi_{n+1})(a) &= (\psi_{n+1} \circ u_{n+1})(a) \\ &= (\psi_{n+1} \circ g_n)(b) \\ &= \psi_n(b). \end{aligned}$$

Thus im $u \subseteq \ker v$.

Therefore im $u = \ker v$. □

Proposition C.4.4.
*Let $\mathscr{C}$ be any category. Let $A_0 \overset{i_0}{\to} A_1 \overset{i_1}{\to} A_2 \overset{i_2}{\to} \cdots$ and $B_0 \overset{j_0}{\to} B_1 \overset{j_1}{\to} B_2 \overset{j_2}{\to} \cdots$ be telescopes in $\mathscr{C}$,*

*whose colimits exist. Let* $(A = \mathrm{colim}_n A_n, \iota)$ *and* $(B = \mathrm{colim}_n B_n, \lambda)$ *denote the co-cones forming the colimit of the telescopes. Let* $r, s : \mathbb{N} \to \mathbb{N}$ *be increasing functions. Let* $f_n : A_n \to B_{r(n)}$ *and* $g_n : B_n \to A_{s(n)}$ *be morphism in* $\mathscr{C}$ *for every* $n$*. As shown in the proof of Theorem C.4.3, they induce canonical morphisms* $f : A \to B$ *and* $g : B \to A$*. Then we have the following results.*

**(i)** *The following diagrams commute for all* $n \geq 0$ *:*

$$\begin{tikzcd} A_n \arrow[r, "\iota_n"] \arrow[rrr, bend right, "\iota_n"'] & A \arrow[r, "f"] & B \arrow[r, "g"] & A, \end{tikzcd} \qquad \begin{tikzcd} B_n \arrow[r, "\lambda_n"] \arrow[rrr, bend right, "\lambda_n"'] & B \arrow[r, "g"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

*if and only if* $f : A \to B$ *is an isomorphism in* $\mathscr{C}$*, with inverse* $g : B \to A$*.*

**(ii)** *Suppose that* $r$*, or* $s$*, is a stricly increasing function. If for all* $n$*, the following diagrams commute :*

$$\begin{tikzcd} A_n \arrow[r, "f_n"] \arrow[rr, bend right, "i_{s(r(n))-1} \circ \cdots \circ i_n"'] & B_{r(n)} \arrow[r, "g_{r(n)}"] & A_{s(r(n))}, \end{tikzcd} \qquad \begin{tikzcd} B_n \arrow[r, "g_n"] \arrow[rr, bend right, "j_{r(s(n))-1} \circ \cdots \circ j_n"'] & A_{s(n)} \arrow[r, "f_{s(n)}"] & B_{r(s(n))}, \end{tikzcd}$$

*then* $f : A \to B$ *is an isomorphism in* $\mathscr{C}$*, with inverse* $g : B \to A$*.*

Proof : To prove that $f$ and $g$ are mutual inverses, we have to check that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. The first statement follows then directly from Lemma C.4.2. The second statement follows from the first since we obtain the following commutative diagram :

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & A \\ A_n \arrow[u, "\iota_n"] \arrow[r, "f_n"] \arrow[rr, bend right, "i_{s(r(n))-1} \circ \cdots \circ i_n"'] & B_{r(n)} \arrow[u, "\lambda_{r(n)}"] \arrow[r, "g_{r(n)}"] & A_{s(r(n))}, \arrow[u, "\iota_{s(r(n))}"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] & A \arrow[r, "f"] & B \\ B_n \arrow[u, "\lambda_n"] \arrow[r, "g_n"] \arrow[rr, bend right, "j_{r(s(n))-1} \circ \cdots \circ j_n"'] & A_{s(n)} \arrow[u, "\iota_{s(n)}"] \arrow[r, "f_{s(n)}"] & B_{r(s(n))}, \arrow[u, "\lambda_{r(s(n))}"] \end{tikzcd}$$

for all $n \geq 0$. □

We now give the behavior of homotopy and homology groups with respect to particular colimits of telescopes.

Theorem C.4.5 (The Homotopy of Telescopes).

*Suppose there is a sequence of inclusions* $X_0 \overset{i_0}{\hookrightarrow} X_1 \overset{i_1}{\hookrightarrow} \cdots \hookrightarrow X_j \overset{i_j}{\hookrightarrow} X_{j+1} \hookrightarrow \cdots$ *of based topological* $T_1$*-spaces*[2]*, where the basepoint is chosen in* $X_0$*. Then for each* $n \geq 0$*, there exists a natural isomorphism :*

$$\mathrm{colim}_j \pi_n(X_j) \xrightarrow{\cong} \pi_n(\mathrm{colim}_j X_j).$$

Proof : We start by building the natural homomorphism $\mathrm{colim}_j \pi_n(X_j) \to \pi_n(\mathrm{colim}_j X_j)$. For $n \geq 0$ fixed, the colimit of the telescope :

$$\begin{tikzcd} X_0 \arrow[r, hook, "i_0"] \arrow[drrr, "\iota_0"] & X_1 \arrow[r, hook, "i_1"] \arrow[drr, "\iota_1"] & X_2 \arrow[r, hook] \arrow[dr, "\iota_2"] & \cdots \arrow[r, hook] & X_j \arrow[r, hook, "i_j"] \arrow[dl, "\iota_j"'] & X_{j+1} \arrow[r, hook] \arrow[dll, "\iota_{j+1}"'] & \cdots \arrow[dlll] \\ & & & \mathrm{colim}_j X_j, & & & \end{tikzcd}$$

[2]We only need that each point in the spaces are closed, but one can replace the $T_1$-property by the Hausdorff property if one is not familiar with $T_1$-spaces

induces the desired (unique) homomorphism $\Phi$ (or set map for the case $n = 0$), using the universal property of colimits, which is obviously natural :

$$\begin{tikzcd}
\pi_n(X_0) \arrow[r, "i_{0*}"] \arrow[drrr] \arrow[ddrrr, bend right, "\iota_{0*}"'] & \pi_n(X_1) \arrow[r, "i_{1*}"] \arrow[drr] \arrow[ddrr, bend right, "\iota_{1*}"'] & \pi_n(X_2) \arrow[r] \arrow[dr] \arrow[ddr, bend right, "\iota_{2*}"'] & \cdots \arrow[r] & \pi_n(X_j) \arrow[r, "i_{j*}"] \arrow[dl] \arrow[ddl, bend left, "\iota_{j*}"] & \pi_n(X_{j+1}) \arrow[r] \arrow[dll] \arrow[ddll, bend left, "\iota_{j+1*}"] & \cdots \arrow[dlll] \arrow[ddlll, bend left] \\
 & & & \mathrm{colim}_j \pi_n(X_j) \arrow[d, dashed, "\Phi"] & & & \\
 & & & \pi_n(\mathrm{colim}_j X_j). & & &
\end{tikzcd}$$

Here naturality means that if there is another telescope diagram $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} Y_2 \to \cdots$ in $\mathbf{Top}_*$ such that there are based maps $f_j : X_j \to Y_j$, for any $j \geq 0$, provided that $f_{j+1} \circ i_j = g_{j+1} \circ f_j$, then there exist a map $f : \mathrm{colim}_j X_j \to \mathrm{colim}_j Y_j$ induced by $\{f_j\}$ and a homomorphism $f_* : \mathrm{colim}_j \pi_n(X_j) \to \mathrm{colim}_j \pi_n(Y_j)$ induced by $\{f_{j*}\}$ such that the following diagram commutes[3] :

$$\begin{tikzcd}
\mathrm{colim}_j \pi_n(X_j) \arrow[r, "f_*"] \arrow[d, "\Phi"'] & \mathrm{colim}_j \pi_n(Y_j) \arrow[d, "\Phi"] \\
\pi_n(\mathrm{colim}_j X_j) \arrow[r, "\pi_n(f)"'] & \pi_n(\mathrm{colim}_j Y_j).
\end{tikzcd}$$

Let $X := \mathrm{colim}_j X_j = \bigcup_{j \geq 0} X_j$ be the colimit. We have the following claim.

Claim *For any compact space $K$, and for any map $f : K \to X$, there exists $m \in \mathbb{N}$ such that* $\mathrm{im}(f) \subseteq X_m$.

Proof of the Claim : Suppose that such a $m$ does not exist. Since $\mathrm{im}(f)$ is a compact subspace of $X$, let us use the characterisation of the compactness by the finite intersection property (see theorem 26.9 in [Munkres, 2000]) to get a contradiction. Since $\mathrm{im}(f) \nsubseteq X_m$ for any $m$, then there exists an element $x_m$ in $\mathrm{im}(f) - (\mathrm{im}(f) \cap X_m)$, for every $m$. Since $X_k \subseteq X_m$ if $k \leq m$, we get $x_m \notin X_k$ whenever $k \leq m$. Therefore we obtain $|\{X_m \cap \{x_k \mid k \in \mathbb{N}\}\}| < m$ for any $m$. But, $\{x_k\}$ is closed in $X_k$, for all $k$ in $\mathbb{N}$. Hence $X_m \cap \{x_k \mid k \in \mathbb{N}\}$ is a closed subset of $X_m$, for each $m$. Thus $\{x_k \mid k \in \mathbb{N}\}$ is a closed subset of $X$, whence of $\mathrm{im}(f)$ as well. Define $C_m := \{x_k \mid k \geq m\}$. We get a collection of closed subsets of $\mathrm{im}(f)$ : $\{C_m\}_{m \in \mathbb{N}}$. Notice we have $C_{m_1} \cap \ldots \cap C_{m_r} = C_{\max(m_k, 1 \leq k \leq r)} \neq \emptyset$, for any $m_1, \ldots, m_r$. But $\bigcap_{m \in \mathbb{N}} C_m = \emptyset$. We get a contradiction with the finite intersection property of the compact subspace $\mathrm{im}(f)$. $\square$

We now prove that $\Phi$ is an isomorphism, using the claim. Let $[f]_*$ be an element of $\pi_n(X)$. Since $S^n$ is compact, there exists $m$ such that $\mathrm{im}(f) \subseteq X_m$. Hence $[f]_*$ is in $\pi_n(X_m)$. Surjectivity follows when using the homomorphism $\pi_n(X_m) \to \mathrm{colim}_j \pi_n(X_j)$. Suppose now $\Phi([f]_*) = 0$ for some $[f]_*$ in $\mathrm{colim}_j \pi_n(X_j)$. Then there exists a homotopy $H : S^n \times I \to X$ from a representative

[3] Here the argument for an existence of a natural morphism $\Phi$ remains valid for any telescope diagram $X_j \to X_{j+1}$ (not necessarily inclusions). In fact, it holds for any colimit of any diagram (not necessarily in $\mathbf{Top}_*$), and any functor defined on the category (not necessarly $\pi_n$).

of $\Phi([f]_*)$ to the identity. Using the compactness of $S^n \times I$, we obtain a homotopy from a well-chosen representative of $[f]_*$ to the identity. Therefore $[f]_* = 0$. So $\Phi$ is injective. $\square$

Corollary C.4.6.
*Let $X$ and $Y$ be CW-complexes. Let us denote $(X \times Y)^w$ the product endowed with its weak topology. Then we obtain the following isomorphism for all $n \geq 0$ :*

$$\pi_n((X \times Y)^w) \cong \pi_n(X) \oplus \pi_n(Y).$$

Proof : For the sake of clarity, let us write $X \times Y$ instead of $(X \times Y)^w$. It is a CW-complex with $j$-skeleton :

$$(X \times Y)_j = \bigcup_{i=0}^{j} X_j \times Y_{j-i},$$

where $X_j$ and $Y_j$ denote the $j$-skeletons of the CW-complexes $X$ and $Y$ respectively. We first show that : $\mathrm{colim}_j (X \times Y)_j \cong \mathrm{colim}_j (X_j \times Y_j)$. Consider the two telescopes induced by the inclusions of skeletons :

$$X_0 \times Y_0 \hookrightarrow X_1 \times Y_1 \hookrightarrow \cdots \hookrightarrow X_j \times Y_j \hookrightarrow X_{j+1} \times Y_{j+1} \hookrightarrow \cdots,$$

$$(X \times Y)_0 \hookrightarrow (X \times Y)_1 \hookrightarrow \cdots \hookrightarrow (X \times Y)_j \hookrightarrow (X \times Y)_{j+1} \hookrightarrow \cdots.$$

Define for all $j \geq 0$ the inclusions :

$$X_j \times Y_j \xhookrightarrow{f_j} \bigcup_{i=0}^{2j} X_j \times Y_{2j-i} = (X \times Y)_{2j},$$

and the inclusions :

$$(X \times Y)_j \xhookrightarrow{g_j} X_j \times Y_j.$$

We obtain the following commutative diagrams for all $j \geq 0$ :

$$\begin{tikzcd}
X_j \times Y_j \arrow[rr, hook] \arrow[dr, hook, "f_j"'] & & X_{2j} \times Y_{2j}, \\
& (X \times Y)_{2j} \arrow[ur, hook, "g_{2j}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \times Y)_j \arrow[rr, hook] \arrow[dr, hook, "g_j"'] & & (X \times Y)_{2j}. \\
& X_j \times Y_j, \arrow[ur, hook, "f_j"'] &
\end{tikzcd}$$

Therefore by Proposition C.4.4, we obtain the desired homeomorphism :

$$\mathrm{colim}_j (X \times Y)_j \cong \mathrm{colim}_j (X_j \times Y_j).$$

Thus we get :

$$\begin{aligned}
\pi_n(X \times Y) &= \pi_n(\mathrm{colim}_j (X \times Y)_j) \\
&\cong \pi_n(\mathrm{colim}_j (X_j \times Y_j)), \text{ by previous homeomorphism,} \\
&\cong \mathrm{colim}_j \pi_n(X_j \times Y_j), \text{ by Theorem C.4.5,} \\
&\cong \mathrm{colim}_j (\pi_n(X_j) \oplus \pi_n(Y_j)) \\
&\cong \mathrm{colim}_j \pi_n(X_j) \oplus \mathrm{colim}_j \pi_n(Y_j), \text{ by Theorem C.3.4,} \\
&\cong \pi_n(X) \oplus \pi_n(Y), \text{ by Theorem C.4.5,}
\end{aligned}$$

for all $n \geq 0$. $\square$

Theorem C.4.7 (The Homology of Telescopes).

*Suppose there is a sequence of inclusions* $X_0 \overset{i_0}{\hookrightarrow} X_1 \overset{i_1}{\hookrightarrow} \cdots \hookrightarrow X_j \overset{i_j}{\hookrightarrow} X_{j+1} \hookrightarrow \cdots$ *of topological* $T_1$*-spaces. Let* $E_*$ *be a generalized homology theory. Then for each* $n$ *in* $\mathbb{Z}$*, there exists a natural isomorphism :*

$$\mathrm{colim}_j E_n(X_j) \xrightarrow{\cong} E_n(\mathrm{colim}_j X_j).$$

Proof : Set $X := \mathrm{colim}_j X_j$. Working with path components, we may assume that $X$ and $X_j$ are path connected, by the additivity axiom. We can then work with based spaces with base point chosen in $X_0$. We construct what is called generally a *homotopy colimit* : special case of a much more general construction. For $j \geq 0$, let $M_{j+1}$ be the mapping cylinder, defined as the pushout :

$$\begin{tikzcd} X_j \arrow[r] \arrow[d, hook, "i_j"'] & X_j \times [j, j+1] \arrow[d] \\ X_{j+1} \arrow[r] & M_{j+1}, \end{tikzcd}$$

where the top row unlabeled map sends each $x$ of $X_j$ to $(x, j+1)$. As usual, we have homotopy equivalences : $M_{j+1} \xrightarrow{\simeq} X_{j+1}$, such that the following diagram commutes :

$$\begin{tikzcd} M_n \arrow[r, tail] & M_{n+1} \\ X_n \arrow[u, "\simeq"] \arrow[r, hook, "i_n"] & X_{n+1}. \arrow[u, "\simeq"'] \end{tikzcd}$$

We define a new telescope of spaces $Y_0 \to Y_1 \to Y_2 \to \cdots$ inductively. Set $Y_0 := X_0 \times \{0\}$. Let $Y_1 = M_1$. Of course $X_1 \times \{1\} \subseteq Y_1$. Suppose we have constructed $Y_j \supseteq X_j \times \{j\}$. Let $Y_{j+1}$ be the double mapping cylinder defined as the pushout :

$$\begin{tikzcd} X_j \times \{j\} \arrow[r, hook] \arrow[d] & Y_j \arrow[d, hook, "y_j"] \\ M_{j+1} \arrow[r] & Y_{j+1}. \end{tikzcd}$$

Again, we have the usual homotopy equivalences $Y_{j+1} \xrightarrow{\simeq} M_{j+1}$, and $Y_0 \cong X_0$. We get that :

$$Y_{n+1} = M_1 \coprod_{X_1 \times \{1\}} M_2 \coprod_{X_2 \times \{2\}} \cdots \coprod_{X_{n-1} \times \{n-1\}} M_n.$$

Define the space $T := \mathrm{hocolim}_j X_j := \mathrm{colim}_j Y_j$ as the *homotopy colimit* of $X_0 \to X_1 \to \cdots$. Therefore, we have :

$$T = \bigcup_{j \geq 0} Y_j = \left( \bigcup_{j \geq 0} X_j \times [j, j+1] \right) / \sim,$$

where $X_j \times [j, j+1] \ni (x_j, j+1) \sim (i_j(x_j), j+1) \in X_{j+1} \times [j+1, j+2]$ for each $j$ and $x_j$ in $X_j$. We have composites of homotopy equivalences that give homotopy equivalences $r_j : Y_j \xrightarrow{\simeq} X_j$, for all $j \geq 0$, such that we have commutativity of the following diagram :

$$\begin{tikzcd} Y_j \arrow[r, hook, "y_j"] \arrow[d, "\simeq"', "r_j"] & Y_{j+1} \arrow[d, "\simeq"', "r_{j+1}"] \\ X_j \arrow[r, hook, "i_j"] & X_{j+1}. \end{tikzcd}$$

The homotopy equivalences $r_j$ composed with the maps $X_j \to \mathrm{colim}_j X_j$ induce a (unique) map :

$$r : T = \mathrm{colim}_j Y_j \longrightarrow \mathrm{colim}_j X_j = X.$$

It induces isomorphisms on homotopy groups, since we have for all $n$ :

$$\begin{array}{rcl} \pi_n(T) & = & \pi_n(\mathrm{colim}_j Y_j) \\ & \cong & \mathrm{colim}_j \pi_n(Y_j), \text{ by Theorem C.4.5,} \\ & \cong & \mathrm{colim}_j \pi_n(X_j), \text{ by the homotopy equivalences } r_j, \\ & \cong & \pi_n(\mathrm{colim}_j X_j), \text{ by Theorem C.4.5,} \\ & = & \pi_n(X). \end{array}$$

Hence $r$ is a weak equivalence. By the invariance with respect to weak equivalences axiom, $r$ induces an isomorphism on homology, i.e., $E_n(T) \stackrel{\cong}{\to} E_n(\mathrm{colim}_j X_j)$ for all $n$. Therefore, it suffices to prove that the natural homomorphism :

$$\mathrm{colim}_j E_n(X_j) \cong \mathrm{colim}_j E_n(Y_j) \to E_n(\mathrm{hocolim}_j X_j) = E_n(T),$$

is an isomorphism, for each $n$ in $\mathbb{Z}$. Define $A$ and $B$ to be open subspaces of $T$ given by choosing $0 < \zeta < 1$ and setting :

$$A := \left( X_0 \times I \sqcup \coprod_{j \geq 1} (X_{2j-1} \times ]2j-\zeta, 2j] \cup X_{2j} \times [2j, 2j+1]) \right) / \sim,$$

and :

$$B := \left( \coprod_{j \geq 0} X_{2j} \times ]2j+1-\zeta, 2j+1] \cup X_{2j+1} \times [2j+1, 2j+2] \right) / \sim .$$

The spaces $A$ and $B$ are indeed open in $T$, and $A \cup B = T$. Let $C$ be the intersection $A \cap B$, we get :

$$C = \left( \coprod_{j \geq 0} X_j \times ]j+1-\zeta, j+1] \right) / \sim .$$

We obtain an excisive triad $(T; A, B)$. The homotopy equivalence $]2j-\zeta, 2j] \to \{2j\}$ leads to the homotopy equivalences : $A \simeq_* \coprod_{j \geq 0} X_{2j}$, $B \simeq_* \coprod_{j \geq 0} X_{2j+1}$ and $C \simeq_* \coprod_{j \geq 0} X_j$, such that the inclusion $C \hookrightarrow A$ induces a map taht becomes under the homotopy equivalence above : $\coprod_{j \geq 0} X_j \to \coprod_{j \geq 0} X_{2j}$ which is $\mathrm{id} : X_{2j} \to X_{2j}$ when restricted to $X_{2j}$, and $i_{2j+1} : X_{2j+1} \hookrightarrow X_{2j+2}$ when restricted to $X_{2j+1}$, for each $j \geq 0$; and the inclusion $C \hookrightarrow B$ induces a map : $\coprod_{j \geq 0} X_j \to \coprod_{j \geq 0} X_{2j+1}$ which is $\mathrm{id} : X_{2j+1} \to X_{2j+1}$ when restricted to $X_{2j+1}$, and $i_{2j} : X_{2j} \hookrightarrow X_{2j+1}$ when restricted to $X_{2j}$, for each $j \geq 0$.
Using that $E_*$ is a homotopy invariant functor, and the additivity axiom, we obtain for each $n$ on $\mathbb{Z}$ the isomorphisms :

$$E_n(A) \cong \bigoplus_{j \geq 0} E_n(X_{2j}),\ E_n(B) \cong \bigoplus_{j \geq 0} E_n(X_{2j+1}),\ \text{and } E_n(C) \cong \bigoplus_{j \geq 0} E_n(X_j).$$

We now obtain the following commutative diagram where the rows are exact, for each $n \in \mathbb{Z}$;

we shall explain it in details, row by row :

$$\begin{tikzcd}
\cdots \arrow[r] & E_n(C) \arrow[r, "\psi_n"] \arrow[d, "\cong"] & E_n(A) \oplus E_n(B) \arrow[r, "\varphi_n"] \arrow[d, "\cong"] & E_n(T) \arrow[r] \arrow[d, "E_n(r)"', "\cong"] & \cdots \\
\cdots \arrow[r] & \bigoplus_{j\geq 0} E_n(X_j) \arrow[r, "f_n"] & \bigoplus_{j\geq 0} E_n(X_j) \arrow[r, "g_n"] & E_n(X) \arrow[r] & \cdots \\
0 \arrow[r] & \bigoplus_{j\geq 0} E_n(X_j) \arrow[r, "\alpha"] \arrow[u, "\cong"', "\mathrm{alt}"] & \bigoplus_{j\geq 0} E_n(X_j) \arrow[r, "\beta"] \arrow[u, "\cong"', "\mathrm{alt}"] & \mathrm{colim}_j E_n(X_j) \arrow[r] \arrow[u, dashed] & 0.
\end{tikzcd} \tag{$\star$}$$

First row of ($\star$) It is the Mayer-Vietoris sequence, where $\psi_n(e) = (E_n(\ell_A)(e), E_n(\ell_B)(e))$, for each $e$ in $E_n(C)$, with $\ell_A : C \hookrightarrow A$ and $\ell_B : C \hookrightarrow B$ the inclusions; and where $\varphi_n(a, b) = E_n(s_A)(a) - E_n(s_B)(b)$, for each $a$ in $E_n(A)$ and $b$ in $E_n(B)$, with the inclusions $s_A : A \hookrightarrow T$ and $s_B : B \hookrightarrow T$, as usual. This row is therefore exact.

Second row of ($\star$) It is the previous Mayer-Vietoris sequence but seen via our homotopy equivalences and the weak equivalence $r$. Recall that elements in $\bigoplus_{j\geq 0} E_n(X_j)$ are $\sum_{j\geq 0} e_j$ where $|\{j \geq 0 \mid E_n(X_j) \ni e_j \neq 0\}| < \infty$. We define the homomorphisms :

$$\begin{aligned} f_n : \bigoplus_{j\geq 0} E_n(X_j) &\longrightarrow \bigoplus_{j\geq 0} E_n(X_j) \\ \sum_{j\geq 0} e_j &\longmapsto \sum_{j\geq 0} e_j + E_n(i_j)(e_j), \end{aligned}$$

and :

$$\begin{aligned} g_n : \bigoplus_{j\geq 0} E_n(X_j) &\longrightarrow E_n(X) \\ \sum_{j\geq 0} e_j &\longmapsto \sum_{j\geq 0} (-1)^j E_n(k_j)(e_j), \end{aligned}$$

where $k_j : X_j \hookrightarrow X = \mathrm{colim}_j X_j$ is the inclusion, for each $j \geq 0$. From our previous talk about how the inclusions $\ell_A : C \hookrightarrow A$ and $\ell_B : C \hookrightarrow B$ restrict via the homotopy equivalences, it follows that the upper lefthand square of ($\star$) commutes. The upper righthand square commutes since we have the following commutative diagrams :

$$\begin{tikzcd}
A \arrow[r, hook, "s_A"] \arrow[d, "\simeq"'] & T \arrow[d, "r"] \\
\coprod_{j\geq 0} X_{2j} \arrow[r, hook, "\sum_{j\geq 0} k_{2j}"'] & X,
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, hook, "s_B"] \arrow[d, "\simeq"'] & T \arrow[d, "r"] \\
\coprod_{j\geq 0} X_{2j+1} \arrow[r, hook, "\sum_{j\geq 0} k_{2j+1}"'] & X.
\end{tikzcd}$$

Therefore, the second row is exact.

Third row of ($\star$) It is the short exact sequence which stems from proposition C.4.1 applied to the telescope $E_n(X_0) \xrightarrow{E_n(i_0)} E_n(X_1) \xrightarrow{E_n(i_1)} \cdots$. We define the automorphism :

$$\begin{aligned} \mathrm{alt} : \bigoplus_{j\geq 0} E_n(X_j) &\longrightarrow \bigoplus_{j\geq 0} E_n(X_j) \\ \sum_{j\geq 0} e_j &\longmapsto \sum_{j\geq 0} (-1)^j e_j. \end{aligned}$$

It is easy to check that the bottom left square commutes. From the universal property of colimits, there exists a homomorphism $\mathrm{colim}_j E_n(X_j) \to E_n(X)$ (dashed in $(\star)$), which makes the bottom right square commutes.

We get that the dashed map in $(\star)$ is an isomorphism, using an argument similar to the 5-Lemma. $\square$

# References


[Adams, 1962] Adams, J. F. (1962). *A Variant of E.H. Brown's Representabitilty Theorem*. Annals of Mathematics, 75:603–632.

[Adams, 1974] Adams, J. F. (1974). *Stable Homotopy and Generalized Homology*. University of Chicago Press.

[Adams, 1978] Adams, J. F. (1978). *Infinite Loop Spaces*. Princeton University Press and University of Tokyo Press.

[Aguilar et al., 2002] Aguilar, M., S. Gitler and C. Prieto (2002). *Algebraic Topology from a Homotopical Viewpoint*. Springer-Verlag.

[Arkowitz, 2011] Arkowitz, M. (2011). *Introduction to Homotopy Theory*. Springer-Verlag.

[Borceux, 1994] Borceux, F. (1994). *Handbook of Categorical Algebra*, vol. 1. Cambridge University Press.

[Dodson and Parker, 1997] Dodson, C.T.J. and P. Parker (1997). *A User's Guide to Algebraic Topology*. Kluwer Academic Publisher.

[Eilenberg and Steenrod, 1952] Eilenberg, S. and N. E. Steenrod (1952). *Foundations of Algebraic Topology*. Princeton University Press.

[Fritsch and Piccinini, 1990] Fritsch, R. and R. A. Piccinini (1990). *Cellular Structures in Topology*. Cambridge University Press.

[Hatcher, 2002] Hatcher, A. (2002). *Algebraic Topology*. Cambridge University Press.

[Kochman, 1996] Kochman, S.O. (1996). *Bordism, Stable Homotopy and Adams Spectral Sequences*. American Mathematical Society.

[Kono and Tamaki, 2002] Kono, A. and D. Tamaki (2002). *Generalized Cohomology*. Iwanami Shoten.

[MacLane, 1971] MacLane, S. (1971). *Categories for the Working Mathematician*. Springer-Verlag.

[May, 1999] May, J. P. (1999). *A Concise Course in Algebraic Topology*. University of Chicago Press.

[Munkres, 1984] Munkres, J.R. (1984). *Elements of Algebraic Topology*. Benjamin/Cummings Publishing Company.

[Munkres, 2000] Munkres, J.R. (2000). *Topology*. Prentice Hall.

[Piccinini, 1992] Piccinini, R.A. (1992). *Lectures on Homotopy Theory*. North-Holland Mathematical Studies 171.

[Ravenel, 1992] Ravenel, D.C. (1992). *Nilpotence and Periodicity in Stable Homotopy Theory*. Princeton University Press.

[Rotman, 1998] Rotman, J.J. (1998). *Introduction to Algebraic Topology*. Springer-Verlag.

[Strom, 2011] Strom, J. (2011). *Modern Classical Homotopy Theory*. American Mathematical Society.

[Switzer, 1975] Switzer, R. M. (1975). *Algebraic Topology - Homology and Homotopy*. Springer-Verlag.

[TomDieck, 2008] TomDieck, T. (2008). *Algebraic Topology*. European Mathematical Society.

[Whitehead, 1978] Whitehead, G.W. (1978). *Elements of Homotopy Theory*. Springer-Verlag.