COFIBER SEQUENCES OF THOM SPECTRA OVER $B(\mathbb{Z}/2)^n$

Master Project By
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Introduction

In 1954, in his revolutional paper, "*Quelques Propriétés Globales des Variétés Différentiables*", René Thom introduced new objects, that he simply called *complexes*, that play a unifying role in Algebraic Topology as they display the interconnections between Geometric Topology and Homotopy Theory. These objects are now called *Thom spaces*. The groundbreaking ideas earned René Thom the 1958 Fields Medal at Edinburgh. Thom was able to answer geometric problems such as when is a closed manifold a boundary of a compact manifold with boundary, or when can a homology class in a space be realized by a map of a closed manifold. Subsequently, the study of stable phenomena in Algebraic Topology, i.e., which occur in any sufficiently large dimension, led to a replacement of topological spaces by *spectra* which form an entire new category called the *stable homotopy category* $\mathscr{S}$. The behavior of this new category allows many nice properties not found in the (unstable) homotopy category of spaces, such as the invertibility of the suspension.

In [Mitchell and Priddy, 1983], the authors studied the stable splitting of the classifying spaces of the groups $(\mathbb{Z}/p)^n$ and introduced spectra $M(n)$ and $L(n)$. They proved that $B(\mathbb{Z}/p)^n$ contains (stably) $p^{\binom{n}{2}}$ summands, each equivalent to $M(n)$. They proved also the decomposition $M(n) \simeq L(n) \vee L(n-1)$. The spectra $M(n)$ and $L(n)$ play also an important role in the proof of the mod p Whitehead conjecture which states that :

$$\ker\left(\pi_*(\mathrm{SP}^{p^n}\mathbb{S}) \to \pi_*(\mathrm{SP}^{p^{n+1}}\mathbb{S})\right) = \ker\left(\pi_*(\mathrm{SP}^{p^n}\mathbb{S}) \to \pi_*(\mathrm{SP}\mathbb{S})\right).$$

In [Kuhn and Priddy, 1986], it was proved that the above equality is equivalent to the exactness of the homotopy groups localized at p :

$$\cdots \to \pi_*(L(2)) \to \pi_*(L(1)) \to \pi_*(L(0)) \to \pi_*(H\mathbb{Z}_{(p)}).$$

In [Takayasu, 1999], the author focused on the case $p = 2$ and constructed spectra $L(n, k)$, for $k \in \mathbb{Z}$ which generalize the previous spectra as $L(n, 0) = M(n)$ and $L(n, 1) = L(n)$. They are defined as the stable summands of Thom spectra over $B(\mathbb{Z}/2)^n$. The spectra $L(n, k)$ appeared, for $k \geq 0$, in the description of layers of the Goodwillie tower of the identity functor evaluated at spheres in [Arone and Mahowald, 1999]. This Master Project aims to follow the proof of Takayasu of the existence of a cofibre sequence :

$$\Sigma^k L(n-1, 2k+1) \to L(n, k) \to L(n, k+1),$$

which generalizes the stable splitting $M(n) \simeq L(n) \vee L(n-1)$. We give a more contextual approach and use a different construction of Thom spectrum associated to a virtual vector bundle. Some other arguments will be improved using the category theory with the Grothendieck construction. Let us also notice that this cofiber sequence was also proved more recently by combining Goodwillie calculus with the James fibration, as described in Chapter 2 of [Behrens, 2010].

In Chapter 1, we aim to define the spectrum $L(n,k)$ which we call the *generalized Mitchell-Priddy spectrum*, for all $n \in \mathbb{N}$ and for $k \in \mathbb{Z}$. We recall briefly the Thom space construction and the Thom isomorphism and see how representation of groups induces real vector bundles. We define precisely a model of Thom spectra for virtual vector bundles. Chapter 2 aims to prove the cofibre sequence of Takayasu. It is built from two other cofiber sequences, using spectral sequences of homotopy colimits and the Steenrod algebra module structure of the mod 2 cohomology of $L(n,k)$. In Appendix A, we present in details the Grothendieck construction and give key results needed for the construction of $L(n,k)$ but also for the proofs of Chapter 2. Appendix B is devoted to present a summary of results of the 2-completion of spectra.

We assume that the reader has a prior knowledge on the stable homotopy category $\mathscr{S}$. We refer the reader to [Adams, 1974] for the model used in this paper. A modern reference is given in chapter II of [Rudyak, 1998]. For an axiomatic approach, see [Margolis, 1983]. We make use of homotopy (co)limits in this paper and our reference is [Bousfield and Kan, 1972]. We refer the reader to [James and Liebeck, 2001] for a treatment of the theory of representation of finite groups. Basic properties of (real) vector bundles, principal bundles and classifying spaces will be used throughout this paper, we refer to [Husemoller, 1993]. For a treatment of the mod 2-Steenrod algebra $\mathscr{A}$, we refer to [Mosher and Tangora, 1968] or [Hatcher, 2002].

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Notations

Throughout this paper, we will use the following notations and conventions.

- $\mathbb{Z}/p = \mathbb{F}_p$ denotes the cyclic group of order p, where p is a prime number.
- $\widehat{\mathbb{Z}_p}$ denotes the p-adic numbers.
- A *space* is a topological space. A *map* is a continuous map.
- Homeomorphism and isomorphism are denoted by the symbol $\cong$.
- $\mathscr{S}$ denotes the stable homotopy category as described in [Adams, 1974]. All spectra are considered in this category, and a *map of spectra* is a morphism in this category. Equivalences in this category will be called *homotopy equivalences* and are denoted $\simeq$.
- All our diagrams will be strictly commutative. We will specify when a diagram is commutative up to homotopy.
- $\mathscr{A}$ denotes the mod 2 Steenrod algebra.
- $\mathbb{S}$ denotes the sphere spectrum.
- HG denotes the Eilenberg-Maclane spectrum of the group G.
- The $(n+1)$-disk will be denoted D^{n+1} and its boundary, the n-sphere, is denoted S^n.
- We denote $H^*(X)$ the mod-2 cohomology, meaning that if X is a space, then we denote $H^*(X) := H^*(X;\mathbb{Z}/2)$ and if X is a spectrum, then $H^*(X) := (H\mathbb{Z}/2)^*(X)$. As usual, $\widetilde{H}^*(X)$ denotes the reduced cohomology of a space X.
- We denote $\Sigma^\infty : \mathbf{Top}_* \longrightarrow \mathscr{S}$ the suspension functor (see exemple 2.3 in [Adams, 1974]).
- We denote $(-)_+ : \mathbf{Top} \longrightarrow \mathbf{Top}_*$ the functor :

$$\begin{aligned} (-)_+ : \mathbf{Top} &\longrightarrow \mathbf{Top}_* \\ X &\longmapsto X_+ = X \sqcup \{*\} \\ \left(X \xrightarrow{f} Y\right) &\longmapsto (f_+ : X_+ \to Y_+)\,. \end{aligned}$$

CHAPTER 1

Preliminaries

This chapter is devoted to define the generalized Mitchell-Priddy spectrum $L(n,k)$, for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. In [Takayasu, 1999], the author has chosen the model :

$$B(\mathbb{Z}/2)^n = E(\mathrm{GL}_n(\mathbb{F}_2) \ltimes (\mathbb{Z}/2)^n) \Big/ (\mathbb{Z}/2)^n .$$

The reason was to give a $\mathrm{GL}_n(\mathbb{F}_2)$-action for the ad hoc construction of [Carlsson, 1984] of Thom spectra of virtual representation of $(\mathbb{Z}/2)^n$. But we shall not make use of this construction and our action of $\mathrm{GL}_n(\mathbb{F}_2)$ will be more standard.

1.1. Thom Spectra over $B(\mathbb{Z}/2)^n$

We present the definiton of a Thom spectrum associated to a vector bundle and we wish to consider all the bundles over $(\mathbb{Z}/2)^n$ which originate from the representations of the group $(\mathbb{Z}/2)^n$.

1.1.1 Thom Spaces

Let us start with some motivation. In (co)homology, the suspension induces an isomorphism $\widetilde{H}^m(X) \cong \widetilde{H}^{m+r}(\Sigma^r X)$, for any well-pointed space X. The Thom isomorphism theorem generalizes this for *twisted* suspensions, in a sense to be defined. Let us first notice that :

$$\Sigma^r X_+ = \frac{X \times D^r}{X \times S^{r-1}},$$

for $r \geq 1$. The Thom space is a generalization of this quotient. Let $\mathbb{R}^r \hookrightarrow E \xrightarrow{p} B$ be a real r-dimensional vector bundle. Suppose it is endowed with a metric (e.g. B is paracompact, see Theorem 9.5 in Chapter 3 of [Husemoller, 1993]). We define the *unit sphere bundle* :

$$S(E) = \{v \in E \mid \|v\| = 1\},$$

and the *unit disk bundle* :

$$D(E) = \{v \in E \mid \|v\| \leq 1\},$$

so that we obtain the subbundles $S(E) \subseteq D(E) \subseteq E$. We obtain the relative bundle :

$$(D^r, S^{r-1}) \hookrightarrow (D(E), S(E)) \xrightarrow{p} B.$$

Definition 1.1.1 (Thom Space).
The *Thom space* of the vector bundle p endowed with a metric is the space $\mathrm{Th}(p) := D(E)/S(E)$.

If p is the projection bundle : $E = B \times \mathbb{R}^r$, then :

$$\mathrm{Th}(p) = \frac{B \times D^r}{B \times S^{r-1}} = \Sigma^r B_+.$$

Therefore, for a general p, its Thom space $\mathrm{Th}(p)$ can indeed be regarded as a twisted suspension.

Remark 1.1.2 (Functoriality of Th).
For motivational matters, we presented the definition of Thom spaces for vector bundles with metrics. More generally, for any r-dimensional vector bundle $p : E \to B$, we may as well obtain $\mathrm{Th}(p)$ by applying fiberwise one-point compactification on E, and identifying all the added points to a single basepoint. Clearly these two definitions are homeomorphic. This construction has the advantage to be obviously functorial :

$$\mathrm{Th} : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Top}_*,$$

where $\mathbf{Vect}_{\mathbb{R}}$ is the category of real vector bundles and $\mathbf{Top}_*$ is the category of pointed spaces. Since in this paper all the base spaces of vector bundles will be paracompact, we will allow ourself to switch between the two constructions.

The following theorem is the desired generalization of the isomorphism $\widetilde{H}^m(X) \cong \widetilde{H}^{m+r}(\Sigma^r X)$, for any based space X.

Theorem 1.1.3 (Thom Isomorphism Theorem).
Let $r \geq 1$. Given a r-dimensional vector bundle $p : E \to B$, with B connected and paracompact, there exists a unique non-zero element $u_p \in H^r(D(E), S(E)) \cong \widetilde{H}^r(\mathrm{Th}(p))$, so that each inclusion $(D^r, S^{r-1}) \hookrightarrow (D(E), S(E))$ induces a homomorphism :

$$H^r(D(E), S(E)) \longrightarrow H^r(D^r, S^{r-1}) \cong \mathbb{Z}/2,$$

that maps u_p to the unique non-zero element. Moreover this class u_p defines an isomorphism :

$$\begin{array}{rcl} H^m(B) & \xrightarrow{\cong} & H^{m+r}(D(E), S(E)) \cong \widetilde{H}^{m+r}(\mathrm{Th}(p)) \\ x & \longmapsto & H^m(p)(x) \smile u_p, \end{array}$$

for all $m \geq 0$.

Sketch of the Proof : Let us first notice that :

$$H^m(D^r, S^{r-1}) = \begin{cases} \mathbb{Z}/2, & \text{if } r = m, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the fibration $(D^r, S^{r-1}) \hookrightarrow (D(E), S(E)) \xrightarrow{p} B$ and apply its relative Serre spectral sequence (see Theorem 9.34 in [Davis and Kirk, 2001]). We obtain for the 2-page :

$$\begin{aligned} E_2^{s,t} &= H^s(B; H^t(D^r, S^{r-1})) \\ &\cong H^s(B) \otimes H^t(D^r, S^{r-1}) \\ &\cong \begin{cases} H^s(B), & \text{if } t = r, \\ 0, & \text{if } t \neq r, \end{cases} \end{aligned}$$

using the universal coefficient Theorem. The isomorphism $H^s(B) \to E_2^{s,r}$ is given by $\gamma \mapsto \gamma \otimes \tau$, where $0 \neq \tau \in H^r(D^r, S^{r-1})$. A simple investigation on the spectral sequence implies directly that :

$$E_2^{s,t} = E_\infty^{s,t} = \begin{cases} H^{s+r}(D(E), S(E)), & \text{if } t = r, \\ 0, & \text{otherwise.} \end{cases}$$

So we have obtained the existence of the Thom isomorphism. It remains to prove the formula. If we set $s = 0$, we get in particular $H^r(D(E), S(E)) \cong H^0(B) \cong \mathbb{Z}/2$. Let us define u_p as the unique non-zero element in $H^r(D(E), S(E))$. Then the multiplicativity of the spectral sequence implies the formula. $\square$

Definition 1.1.4 (Thom Class u_p).
Let $p : E \to B$ be as previous theorem. The previous isomorphism is called the *Thom isomorphism*, and u_p is called the *Thom class* of p.

We finish our discussion about Thom spaces by showing how a Thom space can be naturally considered in the stable category.

Proposition 1.1.5.
Let $V \xrightarrow{p} X$ and $W \xrightarrow{q} Y$ be two real vector bundles. Then there is an homeomorphism :

$$\mathrm{Th}(p \times q) \cong \mathrm{Th}(p) \wedge \mathrm{Th}(q).$$

Proof : We have the relative homemorphism :

$$\begin{array}{ccc} (D(p \times q), S(p \times q)) & \longrightarrow & \Big(D(p) \times D(q), S(p) \times D(q) \cup D(p) \times S(q)\Big) \\ (v, w) & \longmapsto & \dfrac{1}{\max(\|v\|, \|w\|)\sqrt{\|v\|^2 + \|w\|^2}}(v, w), \end{array}$$

which induces the desired homeomorphism on the quotient spaces. $\square$

Corollary 1.1.6.
If $p : E \to B$ is a real vector bundle and ε^r is the trivial real r-dimensional vector bundle over B, then there is an isomorphism $\mathrm{Th}(p \oplus \varepsilon^r) \cong \Sigma^r \mathrm{Th}(p)$.

Proof : Apply previous proposition where $X = Y = B$, and notice that the Whitney sum $p \oplus \varepsilon^r$ is isomorphic to the bundle $p \times \mathbb{R}^r$ where $\mathbb{R}^r$ is the r-dimensional trivial bundle over a point. As $D^r/S^{r-1} \cong S^r$, we get :

$$\mathrm{Th}(p \oplus \varepsilon^r) \cong \mathrm{Th}(p \times \mathbb{R}^r) \cong \mathrm{Th}(p) \wedge S^r = \Sigma^r \mathrm{Th}(p).$$

This finishes the proof. $\square$

Definition 1.1.7 (Thom Spectrum).
The *Thom spectrum* of a vector bundle p is given by $\Sigma^\infty \mathrm{Th}(p)$.

Since in the stable homotopy category $\mathscr{S}$ there is a desuspension, it is natural to ask if there is also *twisted negative suspensions* in the sense of bundles. This is actually true, and we will show that subsequently when we will describe virtual vector bundles.

1.1.2 The Thom Spectrum of a Representation

Let us begin by recalling this general fact.

Theorem 1.1.8.
Given a topological group G, there exists a principal G-bundle $G \hookrightarrow EG \to BG$, where EG is a contractible space. The construction is functorial so that any continuous homomorphism $\alpha : G \to H$ induces a bundle map

$$\begin{array}{ccc} EG & \xrightarrow{E\alpha} & EH \\ \downarrow & & \downarrow \\ BG & \xrightarrow{B\alpha} & BH, \end{array}$$

compatible with the actions, i.e., for all $x \in EG$ and $g \in G$:

$$E\alpha(gx) = \alpha(g)Ef(x).$$

For any paracompact space B, the function defined by pulling back :

$$\begin{array}{rcl} \Phi : \mathrm{Map}(B, BG) & \longrightarrow & \{\textit{Principal } G\textit{-bundles over } B\} \\ f & \longmapsto & f^*(EG), \end{array}$$

induces a bijection from the homotopy set $[B, BG]$ to the set of isomorphism classes of principal G-bundles over B.

Proof : All the statements can be found in [Husemoller, 1993]. □

Definition 1.1.9.
The space BG is called the *classifying space of* G and $G \hookrightarrow EG \to BG$ is called the *universal principal G-bundle*.

Let G be a finite topological group. Consider the universal principal G-bundle $G \hookrightarrow EG \to BG$. Let us denote ${}_{\mathbb{R}G}\mathbf{Mod}$ the category of finite dimensional left $\mathbb{R}G$-modules. Recall that its objects are finite dimensional real vector spaces endowed with a linear G-action and are uniquely determined by real representations of G. The Borel construction (see [Davis and Kirk, 2001] Definition 4.6) gives an additive functor :

$$\begin{array}{rcl} {}_{\mathbb{R}G}\mathbf{Mod} & \longrightarrow & \mathbf{Vect}_{\mathbb{R}}(BG) \\ V & \longmapsto & \begin{pmatrix} EG \times_G V \\ \downarrow \\ BG \end{pmatrix}, \end{array}$$

where $\mathbf{Vect}_{\mathbb{R}}(BG)$ denotes the category of real vector bundles over BG. Subsequently, we will fudge the distinction between real representations and their image under the above functor.

Definition 1.1.10 (Associated Thom Spectrum of V).
The *Thom spectrum associated to a real representation* V of G is :

$$(BG)^V := \Sigma^\infty \mathrm{Th}(V) = \Sigma^\infty (EG_+ \wedge_G S^V),$$

where S^V is the one-point compactification of the real vector space V, which is homeomorphic to the sphere $S^{\dim_{\mathbb{R}}(V)}$, and $EG_+ \wedge_G S^V$ is the orbit space of the diagonal action of G in $EG_+ \wedge S^V$.

The Regular Representation Let G be a finite group. The group algebra $\mathbb{R}G$ is itself a left $\mathbb{R}G$-module by left multiplication of G, and defines the real regular representation of G. Notice that we have the following elementary result.

Lemma 1.1.11.
The real regular representation $\mathbb{R}G$ of a finite group G can be regarded as a group embedding $\rho : G \hookrightarrow O(|G|)$.

Proof : Recall that as $\mathbb{R}G$ is a finite $|G|$-dimensional real vector space, the identification $\mathbb{R}G \cong \mathbb{R}^{|G|}$ provides a scalar product on $\mathbb{R}G$. Now, since $\mathbb{R}G$ is a left $\mathbb{R}G$-module, it defines a group homomorphism $\rho : G \to \mathrm{GL}_{|G|}(\mathbb{R})$. Now it corestricts to the orthogonal group, as for any elements g, g', h in G, we have :

$$\langle \rho(h)(g), \rho(h)(g') \rangle = \langle hg, hg' \rangle = 1 = \langle g, g' \rangle,$$

i.e., $\rho(h) \in O(|G|)$. As the left action of G on itself is faithful, the regular representation is faithful, i.e., ρ is an embedding. □

Recall that the direct sum decomposition of $\mathbb{R}G$ contains a representative of each isomorphism class of the irreducible real representations of G. If we apply the Definition 1.1.10 on $\mathbb{R}G$, then $(BG)^{\mathbb{R}G}$ is the Thom spectrum of the sum of all the real bundles over BG. The action of G on $\mathbb{R}G$ fixes the sum of all the basis elements in $\mathbb{R}G$. Thus the vector space $\mathbb{R}G$ splits into the one-dimensional trivial representation of G and a $(|G|-1)$-dimensional real representation called the *reduced (real) regular representation of G*. In particular, it defines an embedding $\overline{\rho} : G \hookrightarrow O(|G|-1)$ where the following diagram commutes :

$$\begin{tikzcd} & G \arrow[dl, hook, "\overline{\rho}"'] \arrow[dr, hook, "\rho"] & \\ O(|G|-1) \arrow[rr, hook] & & O(|G|). \end{tikzcd}$$

We are interested into the following special case.

Notation 1.1.12 ($V_n, \overline{\rho}_n, k\overline{\rho}_n$).
Let $n \in \mathbb{N}$ be an integer. We denote by V_n the n-dimensional $\mathbb{F}_2$-vector space $(\mathbb{Z}/2)^n$. We denote by $\overline{\rho}_n$ the real reduced regular representation of V_n. For $k \in \mathbb{N}$, let $k\overline{\rho}_n$ be the direct sum of k copies of $\overline{\rho}_n$. For $n = 0$, the notation V_0 refers to the trivial group. For $k = 0$, the notation $0 \cdot \overline{\rho}_n$ denotes the trivial $\mathbb{R}V_n$-module. The inclusion $V_n \hookrightarrow V_{n+1}$ will always refer to the inclusion onto the first n-th components of V_{n+1}.

In this paper, we study the Thom spectra $BV_n^{k\overline{\rho}_n}$. Let us give an interpretation for the trivial cases. First if $k = 0$, then we get $S^{0 \cdot \overline{\rho}_n} \cong S^0$ for any n, so that $BV_n^{0 \cdot \overline{\rho}_n} = \Sigma^\infty BV_{n+}$. Now, if $n = 0$, then for any k, we get $BV_0^{k\overline{\rho}_0} = \mathbb{S}$, the sphere spectrum.

For a global interpretation, let us first remark that we are considering the real numbers instead of the complex numbers because of the following fact.

Proposition 1.1.13.
For G a finite abelian group, the number of real irreducible characters of G is equal to the number of elements g in G for which $g^2 = 1_G$.

Proof : See exercise 23.2 in [James and Liebeck, 2001]. □

Since every element of V_n satisfies the above property, it suffices to consider the representations over the real numbers. Of course, we could have also used the fact that the irregular representations of a cyclic group of order m are determined by the m-th root of unity, and since the second roots of unity are simply ± 1, all the representations are real. Now recall that each irreducible representation over $\mathbb{C}$ of a finite abelian group is of dimension 1. Therefore, the reduced regular representation $\overline{\rho}_n$ is the sum of all the non-trivial irreducible representations of V_n and we proved that they are all of dimension 1 over $\mathbb{R}$. Thus we interpret $BV_n^{k\overline{\rho}_n}$ as the Thom spectrum associated to k times the sum of all the non-trivial line bundles over BV_n.

1.1.3 A Model for $B(\mathbb{Z}/2)^n$

Let us choose a model for $B(\mathbb{Z}/2)^n$. In order to give a $\mathrm{GL}_n(\mathbb{F}_2)$-action, we use the one involving the Stiefel manifold. We postpone the description of this action onto the end of this chapter.

Let $r \geq 0$ be a fixed integer and let $m \geq 0$ be another integer. The *Stiefel manifold* $\mathscr{V}_r(\mathbb{R}^{r+m})$ is the space of orthonormal r-frames in $\mathbb{R}^{r+m}$:

$$\begin{aligned} \mathscr{V}_r(\mathbb{R}^{r+m}) &= \Big\{ v : \mathbb{R}^r \hookrightarrow \mathbb{R}^{r+m} \mid v \text{ orthogonal linear inclusion} \Big\}, \\ &= \Big\{ A \in \mathrm{Mat}_{r\times(r+m)}(\mathbb{R}) \mid {}^tAA = I_{r+m} \Big\}, \end{aligned}$$

endowed with the subspace topology of $\mathrm{Mat}_{r\times(r+m)}(\mathbb{R}) \cong \mathbb{R}^{r(r+m)}$. Define $\mathbb{R}^{r+m} \hookrightarrow \mathbb{R}^{r+m+1}$ as a subvector space by adding zero in the $(r+m+1)$-coordinate. We obtain :

$$\cdots \hookrightarrow \mathscr{V}_r(\mathbb{R}^{r+m}) \hookrightarrow \mathscr{V}_r(\mathbb{R}^{r+m+1}) \hookrightarrow \cdots,$$

so that we define $\mathscr{V}_r(\mathbb{R}^\infty) = \mathrm{colim}_{m\geq 0}\mathscr{V}_r(\mathbb{R}^{r+m})$. The *Grassmann manifold* $\mathscr{G}_r(\mathbb{R}^{r+m})$ is the set of r-dimensional subspaces of $\mathbb{R}^{r+m}$ endowed with the quotient topology given by the map :

$$\begin{aligned} p : \mathscr{V}_r(\mathbb{R}^{r+m}) &\longrightarrow \mathscr{G}_r(\mathbb{R}^{r+m}) \\ v &\longmapsto \mathrm{Im}(v). \end{aligned}$$

Similarly, we define $\mathscr{G}_r(\mathbb{R}^\infty) = \mathrm{colim}_{m\geq 0}\mathscr{G}_r(\mathbb{R}^{r+m})$. Recall there is a natural continuous right action of $O(r)$ on $\mathscr{V}_r(\mathbb{R}^{r+m})$ given by :

$$\begin{aligned} \mathscr{V}_r(\mathbb{R}^{r+m}) \times O(r) &\longrightarrow \mathscr{V}_r(\mathbb{R}^{r+m}) \\ (v, A) &\longmapsto (v \circ A). \end{aligned}$$

The action is free, and as $\mathrm{Im}(v\circ A) = \mathrm{Im}(v)$ for any v and A as above, the orbits of this action are precisely the orthonormal r-frames spanning a given r-dimensional subspace. In other words : $\mathscr{V}_r(\mathbb{R}^{r+m})/O(r) = \mathscr{G}_r(\mathbb{R}^{r+m})$ and we obtain the following result.

Theorem 1.1.14 (The Universal Principal $O(r)$-Bundle).
The map $p : \mathscr{V}_r(\mathbb{R}^{r+m}) \longrightarrow \mathscr{G}_r(\mathbb{R}^{r+m})$ is a principal $O(r)$-bundle, which is universal in dimensions less or equal to $(m+1)-2$. The induced bundle on colimits $p : \mathscr{V}_r(\mathbb{R}^\infty) \longrightarrow \mathscr{G}_r(\mathbb{R}^\infty)$ is a model of the universal principal $O(r)$-bundle.

Proof : Theorem 6.1 in Chapter 8 of [Husemoller, 1993]. $\square$

The inclusion $\mathbb{R}^r \hookrightarrow \mathbb{R}^{r+1}$ induces an inclusion $O(r) \hookrightarrow O(r+1)$, so that we can define the *infinite orthogonal group* $O := \mathrm{colim}_{r\geq 0}O(r)$. Then $BO = \mathrm{colim}_{r\geq 0}BO(r) = \mathrm{colim}_{r\geq 0}\mathscr{G}_r(\mathbb{R}^\infty)$.

If $G \leq O(r)$ is a closed subgroup, then G acts also on $\mathscr{V}_r(\mathbb{R}^{r+m})$ for all $m \geq 0$ and the previous bundle induces a principal G-bundle.

Theorem 1.1.15.
For any closed subgroup $G \leq O(r)$, the classifying space BG is given by :

$$BG = \mathrm{colim}_{m \geq 0}(\mathscr{V}_r(\mathbb{R}^{r+m})/G),$$

and $G \hookrightarrow EG = \mathscr{V}_r(\mathbb{R}^\infty) \to BG$ is a universal principal G-bundle.

Proof : Theorem 6.16 in Chapter 2 of [Mimura and Toda, 1991]. □

The above construction of BG does not depend of the inclusion of G in $O(r)$. Now for the case $G = V_n$, as the group is finite and endowed with the discrete topology, the reduced regular representation gives then a closed embedding $\overline{\rho}_n : V_n \hookrightarrow O(2^n - 1)$. This gives a model for BV_n, which will be convenient as we will see subsequently for defining a $\mathrm{GL}_n(\mathbb{F}_2)$-action.

Let us describe precisely what happens in the case $n = 1$ in the next part.

1.1.4 Stunted Projective Spaces

Let $n = 1$, so that $V_1 = \mathbb{Z}/2 = O(1)$. The universal principal $\mathbb{Z}/2$-bundle is given as follows. For $m \geq 1$, we have the usual principal $\mathbb{Z}/2$-bundle :

$$\mathbb{Z}/2 \longrightarrow S^{m-1} \longrightarrow \mathbb{R}P^{m-1},$$

obtained by identifying antipodal points on S^{m-1} (see [Husemoller, 1993], Example 2.5 Chapter 4). The standard action of $\mathbb{Z}/2 = O(1)$ on $\mathbb{R}$ gives the reduced regular representation $\overline{\rho}_1$. Then the Borel construction :

$$\mathbb{R} \longrightarrow S^{m-1} \times_{\mathbb{Z}/2} \overline{\rho}_1 \xrightarrow{\lambda_m} \mathbb{R}P^{m-1},$$

recovers the *tautological line bundle* λ_m. We will describe in more details the term tautological subsequently.

Lemma 1.1.16.
For any $k \geq 1$, there is an homeomorphism : $\mathrm{Th}(k\lambda_m) \cong \mathbb{R}P^{m+k-1}/\mathbb{R}P^{k-1}$.

Proof : The unit disk bundle is given by :

$$D(k\lambda_m) = \frac{S^{m-1} \times D^k}{(x, y) \sim (-x, -y)}.$$

If we denote by $[x, y]$ the class of (x, y) in $D(k\lambda_m)$, we obtain :

$$S(k\lambda_m) = \{[x, y] \in D(k\lambda_m) \mid \|y\| = 1\}.$$

Now we define a relative map :

$$\begin{array}{rcl} f : \left(S^{m-1} \times D^k, S^{m-1} \times S^{k-1}\right) & \longrightarrow & \left(S^{m+k-1}, S^{k-1}\right) \\ (x, y) & \longmapsto & \left(y, (1 - \|y\|^2)x\right). \end{array}$$

Since f sends $(-x, -y)$ to $-f(x, y)$, the universal property of the quotient gives the relative map :

$$\widehat{f} : \left(D(k\lambda_m), S(k\lambda_m)\right) \longrightarrow \left(\mathbb{R}P^{m+k-1}, \mathbb{R}P^{k-1}\right).$$

Notice that when f is restricted, it induces an homeomorphism :
$$S^{m-1} \times (D^k - S^{k-1}) \cong S^{m+k-1} - S^{k-1}.$$
Thus g induces an homeomorphism $D(k\lambda_m) - S(k\lambda_m) \cong \mathbb{R}P^{m+k-1} - \mathbb{R}P^{k-1}$. Therefore, we obtain the desired homemorphism : $\mathrm{Th}(k\lambda_m) \cong \mathbb{R}P^{m+k-1}/\mathbb{R}P^{k-1}$. □

Definition 1.1.17 (Stunted Projective Space).
The spaces $\mathbb{R}P^{m+k-1}/\mathbb{R}P^{k-1} =: \mathbb{R}P_k^{m+k-1}$ are called the *stunted projective spaces*. The inclusions $\mathbb{R}P^{m-1} \hookrightarrow \mathbb{R}P^m$ induce maps $\mathbb{R}P_k^{m+k-1} \to \mathbb{R}P_k^{m+k}$, which allow to take the colimit and define the *infinite stunted projective spaces* : $\mathbb{R}P_k^\infty := \mathrm{colim}_m \mathbb{R}P_k^{m+k-1}$.

The previous computation extends in the colimits when considering the universal principal $\mathbb{Z}/2$-bundle :
$$\mathbb{Z}/2 \longrightarrow S^\infty \longrightarrow \mathbb{R}P^\infty = B(\mathbb{Z}/2),$$
together with the Borel construction with $\overline{\rho}_1$, so that we obtain :
$$\mathbb{R} \to S^\infty \times_{\mathbb{Z}/2} \overline{\rho}_1 \xrightarrow{\lambda} \mathbb{R}P^\infty.$$
Proposition 1.1.18.
The Thom spectrum $BV_1^{k\overline{\rho}_1}$ is given by : $(B(\mathbb{Z}/2))^{k\lambda} = \Sigma^\infty \mathbb{R}P_k^\infty$.

1.2. The Thom Spectrum of a Virtual Vector Bundle

By *virtual representation of a group*, we mean a formal difference with respect to direct sum of two ordinary representations. In this section, the goal is to define $BV_n^{k\overline{\rho}_n}$ for negative values of k. We carry on our description in a greater generality. This part differs greatly from the paper [Takayasu, 1999].

1.2.1 The Tautological Bundle

Let us denote the space : $\mathscr{E}_r(\mathbb{R}^{r+m}) = \{(W,x) \in \mathscr{G}_r(\mathbb{R}^{r+m}) \times \mathbb{R}^{r+m} \mid x \in W\}$. This defines the *$r$-dimensional tautological bundle γ_r^m* :
$$\begin{array}{ccccc} \mathbb{R}^r & \hookrightarrow & \mathscr{E}_r(\mathbb{R}^{r+m}) & \xrightarrow{\gamma_r^m} & \mathscr{G}_r(\mathbb{R}^{r+m}) \\ & & (W,x) & \longmapsto & W. \end{array}$$
Endow $\mathbb{R}^r$ with its usual $O(r)$-action. The Borel construction :
$$\mathscr{V}_r(\mathbb{R}^{r+m}) \times_{O(r)} \mathbb{R}^r \longrightarrow \mathscr{G}_r(\mathbb{R}^{r+m}),$$
yields an r-dimensional vector bundle isomorphic to the tautological bundle, via the map :
$$\begin{array}{ccc} \mathscr{V}_r(\mathbb{R}^{r+m}) \times_{O(r)} \mathbb{R}^r & \longrightarrow & \mathscr{E}_r(\mathbb{R}^{r+m}) \\ (v,x) & \longmapsto & (\mathrm{Im}(v), v(x)) \,. \end{array}$$
The construction of γ_r^m can be carried on the colimits over m, so that we obtain the universal tautoligical bundle : $\mathbb{R}^r \hookrightarrow \mathscr{E}_r(\mathbb{R}^\infty) \xrightarrow{\gamma_r} \mathscr{G}_r(\mathbb{R}^\infty)$. When $r = 1$, we recover the canonical line bundle used to introduce the stunted projective spaces (in page 7).

The tautological bundle γ_r can be considered as the universal r-dimensional vector bundle, as we have the following result (see the analogy with Theorem 1.1.8).

Theorem 1.2.1.
For any paracompact space B, the function defined by pulling back :

$$\begin{array}{rcl} \Phi : \mathrm{Map}(B, BO(r)) & \longrightarrow & \{\textit{real vector bundles over } B \textit{ of dimension } r\} \\ f & \longmapsto & f^*(EO(r)), \end{array}$$

induces a bijection from the homotopy set $[B, BO(r)]$ to the set of isomorphism classes of real vector bundles of dimension r.

Proof : See Theorem 7.2 in Chapter 8 of [Husemoller, 1993]. □

Definition 1.2.2 (Classifying Map of Vector Bundles).
If $p : E \to B$ is a real vector bundle of dimension r, then we call *the classifying map of* p the unique corresponding map up to homotopy $f : B \to BO(r)$ such that $f^*(\gamma_r) = p$.

One other useful property of the tautological bundle is the existence of an orthogonal complement. In general, a metric determines a direct sum decomposition between a subbundle and its orthogonal complement. Consider the vector bundle ${}^\perp\gamma_r^m$:

$$\begin{array}{rcl} \mathbb{R}^m \hookrightarrow \{(W, x) \in \mathscr{G}_r(\mathbb{R}^{r+m}) \times \mathbb{R}^{r+m} \mid x \perp W\} & \xrightarrow{{}^\perp\gamma_r^m} & \mathscr{G}_r(\mathbb{R}^{r+m}) \\ (W, x) & \longmapsto & W. \end{array}$$

Proposition 1.2.3.
Let θ^{r+m} be the trivial $(r+m)$-dimensional vector bundle over $\mathscr{G}_r(\mathbb{R}^{r+m})$. Then we have the following isomorphisms of vector bundles :

- **(i)** ${}^\perp\gamma_r^m \cong \gamma_m^r$;
- **(ii)** $\gamma_r^m \oplus {}^\perp\gamma_r^m \cong \theta^{r+m}$.

Proof : The first isomorphism follows from the map :

$$\begin{array}{rcl} \mathscr{G}_r(\mathbb{R}^{r+m}) & \longrightarrow & \mathscr{G}_m(\mathbb{R}^{r+m}) \\ W & \longmapsto & W^\perp. \end{array}$$

The second isomorphism is defined by the map :

$$\begin{array}{rcl} \gamma_r^m \oplus {}^\perp\gamma_r^m & \longrightarrow & \theta^{r+m} \\ \big((W, x), (W, x')\big) & \longmapsto & (W, x + x'), \end{array}$$

using the fact that any vector $y \in \mathbb{R}^{r+m}$ can be written as $x + x'$ where $x \in W$ and $x' \in W^\perp$. □

Definition 1.2.4 (Involution on BO).
Define an involution $\iota : BO \to BO$ as the unique map induced by the composite :

$$\mathscr{G}_r(\mathbb{R}^{r+m}) \longrightarrow \mathscr{G}_m(\mathbb{R}^{r+m}) \longrightarrow \mathscr{G}_m(\mathbb{R}^\infty) \longrightarrow BO,$$

where the left map is as the one in the previous proof, i.e., it is the mapping $W \mapsto W^\perp$.

Be careful that the construction of ${}^\perp\gamma_r^m$ cannot be carried on the colimits over m, which means that the universal tautological bundle γ_r does not have a complementary bundle anymore. However, the best thing we have is the above involution.

1.2.2 Virtual Vector Bundles

In the following argument, let B always be a paracompact space. We are ready to define Thom spectrum for virtual vector bundle. Intuitively, we want to define a twisted negative suspension for spectra, such that a Thom isomorphism is valid. So if we have a vector bundle $p : E \to B$ of dimension r, we would like to consider a corresponding «vector bundle» $-p$ of dimension $-r$. For that, we should be interested on what a «vector space» of dimension $-r$ is. Let us begin with the following observation. A vector space of dimension r can actually be regarded as an element in $BO(r) = \mathrm{colim}_{m\geq 0}(\mathscr{G}_r(\mathbb{R}^{r+m}))$. So, one could consider a virtual vector space as an element in BO. Now, as we saw in Theorem 1.2.1, a vector bundle $p : E \to B$, can be regarded as its classifying map $f : B \to BO(r)$. So a virtual vector bundle should be a map $B \to BO$.

Definition 1.2.5 (Virtual Vector Bundle).
A *virtual vector bundle* over a space B is a map $f : B \to \mathbb{Z} \times BO$ with constant value on the $\mathbb{Z}$ component. The value of this constant is called the *rank* of f.

For each virtual vector bundle f over B with rank r, there is a convergent filtration :

$$B_{-r} \subseteq B_{-r+1} \subseteq \ldots \subseteq B = \mathrm{colim}_{i\geq -r}(B_i),$$

where $B_i = f^{-1}(\{r\} \times BO(r+i))$, for $r+i \geq 0$. Then define E_i as the total space of the pullback $\xi^i(f) = f^*(\gamma_{r+i})$:

$$\begin{tikzcd} E_i \arrow[r] \arrow[d, "\xi^i(f)"'] & \mathscr{E}_{r+i}(\mathbb{R}^\infty) \arrow[d, "\gamma_{r+i}"] \\ B_i \arrow[r, "f"] & BO(r+i). \end{tikzcd}$$

Now notice that the bundle $\xi^{i+1}(f)$ pullbacks to $\xi^i(f) \oplus \varepsilon^1$ along the inclusion $B_i \hookrightarrow B_{i+1}$, where ε^1 is the trivial one-dimensional bundle over B_i. Using the functoriality of Thom spaces, we get a map :

$$\Sigma\mathrm{Th}(\xi^i(f)) \longrightarrow \mathrm{Th}(\xi^{i+1}(f)).$$

By formally (de)suspending $(i+1)$-times both sides after applying the suspension functor Σ^∞ leads to the following definition.

Definition 1.2.6 (Thom Spectrum (Virtual Case)).
For any virtual vector bundle $f : B \to \mathbb{Z} \times BO$ of rank r, its *Thom spectrum* B^f is defined as :

$$B^f = \mathrm{colim}_{r+i\geq 0}\left(\Sigma^{-i}\Sigma^\infty\mathrm{Th}(\xi^i(f)) \longrightarrow \Sigma^{-(i+1)}\Sigma^\infty\mathrm{Th}(\xi^{i+1}(f))\right).$$

Remark 1.2.7.
Notice that if f is homotopic with a map g of same rank, then $B^g \simeq B^f$. This follows directly from Theorem 1.2.1. So we are actually considering elements in $[B, \mathbb{Z} \times BO]$ so that we are in the context of real K-theory, even though we should not make any use of this theory here.

Now, by Theorem 1.2.1, if we consider an actual real vector bundle $p : E \to B$, of dimension $r \geq 0$, then p is obtained as a pullback :

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & \mathscr{E}_r(\mathbb{R}^\infty) \arrow[d, "\gamma_r"] \\ B \arrow[r, "f"] & BO(r), \end{tikzcd}$$

where $f : B \to BO(r)$ is a classifying map of p. Then we get $B_0 = B_1 = \cdots = B$ and thus :

$$B^f = \Sigma^\infty \mathrm{Th}(p),$$

i.e., we recover our previous definition of a Thom spectrum of p. This construction will also remain functorial in the following sense.

Proposition 1.2.8.
Let $f : B \to \mathbb{Z} \times BO$ be a virtual vector bundle of rank r over B. For any map $g : A \to B$, there is a map :

$$g_* : A^{f \circ g} \longrightarrow B^f,$$

such that $(\mathrm{id}_B)_* = \mathrm{id}_{X^f}$ and if $h : A' \to A$ then $(g \circ h)_* = g_* \circ h_*$.

Proof : Let $B_i = f^{-1}(\{r\} \times BO(r+i))$ and :

$$A_i = (f \circ g)^{-1}(\{r\} \times BO(r+i)) = g^{-1}(f^{-1}(\{r\} \times BO(r+i))),$$

for any $i \geq -r$. We get an obvious map $g : A_i \to B_i$. Notice that under this map, the bundle $\xi^i(f)$ pullbacks to $\xi^i(f \circ g)$, using the interchange property of limits. Therefore, using Thom's functoriality, we get a map $\mathrm{Th}(\xi^i(f \circ g)) \to \mathrm{Th}(\xi^i(f))$, for each i, and we get the following commutative diagram :

$$\begin{tikzcd}
\Sigma \mathrm{Th}(\xi^i(f \circ g)) \arrow[r] \arrow[d] & \mathrm{Th}(\xi^{i+1}(f \circ g)) \arrow[d] \\
\Sigma \mathrm{Th}(\xi^i(f)) \arrow[r] & \mathrm{Th}(\xi^{i+1}(f)).
\end{tikzcd}$$

This defines the map of spectra g_* desired. It is straightfoward to see $(\mathrm{id}_B)_* = \mathrm{id}_{X^f}$, and the equality $(g \circ h)_* = g_* \circ h_*$ follows from the commutativity of :

$$\begin{tikzcd}
A'_i \arrow[rr, "h"] \arrow[dr, "g \circ h"'] & & A_i \arrow[dl, "g"] \\
& B_i, &
\end{tikzcd}$$

where $A'_i = (f \circ g \circ h)^{-1}(BO(r+i))$, for each i such that $r + i \geq 0$. $\square$

The Thom spectrum also verifies, as desired, a Thom isomorphism theorem.

Theorem 1.2.9 (Thom Isomorphism (Virtual Case)).
Let $f : B \longrightarrow \mathbb{Z} \times BO$ be a virtual vector bundle over B of rank r. Then, for each $m \geq 0$, there is an isomorphism :

$$H^m(B) \xrightarrow{\cong} H^{m+r}(B^f),$$

uniquely defined by a Thom class $u_f \in H^r(B^f)$.

Sketch of the Proof : Each vector bundle $\xi^i(f)$ over B_i, for $r + i \geq 0$, is a $(r+i)$-dimensional vector bundle. In particular, we can apply the usual Thom isomorphism theorem (Theorem 1.1.3), so that $H^m(B) \cong \widetilde{H}^{m+r+i}(\mathrm{Th}(\xi^i(f))$. The isomorphism fits in the commutative diagram :

$$\begin{tikzcd}
\widetilde{H}^{m+r+i+1}(\Sigma \mathrm{Th}(\xi^i(f))) & \widetilde{H}^{m+r+i+1}(\mathrm{Th}(\xi^{i+1}(f))) \arrow[l] \\
H^m(B_i) \arrow[u, "\mathrm{Thom}", "\cong"'] & H^m(B_{i+1}). \arrow[l] \arrow[u, "\cong", "\mathrm{Thom}"']
\end{tikzcd}$$

Now, as all the telescope diagrams considered respect the Mittag-Leffer conditions (see pages 148-149 in [May, 1999]), the cohomology of their colimit is isomorphic to the induced limit on cohomology. Therefore, we obtain an isomorphism :

$$\begin{aligned}
H^m(B) &\cong H^m(\mathrm{colim}_{i\geq -r}(B_i)) \\
&\cong \lim_{i\geq -r}\Big(H^m(B_i) \leftarrow H^m(B_{i+1})\Big) \\
&\cong \lim_{i\geq -r}\Big(H^{m+r}(\Sigma^{-i}\Sigma^\infty \mathrm{Th}(\xi^i(f))) \leftarrow H^{m+r}(\Sigma^{-(i+1)}\Sigma^\infty \mathrm{Th}(\xi^{i+1}(f)))\Big) \\
&\cong H^{m+r}\Big(\mathrm{colim}_{i\geq -r}\Big(\Sigma^{-i}\Sigma^\infty \mathrm{Th}(\xi^i(f))) \to \Sigma^{-(i+1)}\Sigma^\infty \mathrm{Th}(\xi^{i+1}(f)))\Big)\Big) \\
&\cong H^{m+r}(B^f).
\end{aligned}$$

This defines the Thom Isomorphism, and is determined by the cup product of a Thom class u_f under the image of $H^0(B)$. $\square$

Notation 1.2.10.
Let $f, f' : B \to \mathbb{Z} \times BO$ be two virtual vector bundles over B of rank r and s respectively. Then we denote $f \oplus f'$ the virtual vector bundle of rank $r+s$ over B defined as the composite :

$$B \xrightarrow{(f,f')} \Big(\{r\} \times BO\Big) \times \Big(\{s\} \times BO\Big) \xrightarrow{\mu} \{r+s\} \times BO \hookrightarrow \mathbb{Z} \times BO.$$

The associative map $\mu : BO \times BO \longrightarrow BO$ follows from the structure of BO as a H-group. It is given by :

$$BO(n) \times BO(m) \longrightarrow BO(n+m),$$

which is defined by the direct sum (see page 294 in [Aguilar et al., 2002] for more details).

Proposition 1.2.11.
Let $f : B \longrightarrow \mathbb{Z} \times BO$ be a virtual vector bundle of rank r, and let $f' : B \longrightarrow BO(s)$ be the classifying map of real vector bundle $p : E \to B$ of dimension $s \geq 0$. Then there is a map :

$$B^f \longrightarrow B^{(f\oplus f')},$$

which is natural in the sense that, for any map $g : A \to B$, the following diagram commutes :

$$\begin{tikzcd}
A^{f\circ g} \arrow[r] \arrow[d, "g_*"'] & A^{((f\oplus f')\circ g)} \arrow[d, "g_*"] \\
B^f \arrow[r] & B^{(f\oplus f')}.
\end{tikzcd}$$

Proof : Let $B_i = f^{-1}(\{r\}\times BO(r+i))$ for $r+i \geq 0$ and $B'_j = (f\oplus f')^{-1}(\{r+s\}\times BO(r+s+j))$, for $r+s+j \geq 0$. Now the inclusion $B_i \hookrightarrow B_{i+s}$ defines a map $B_i \to B'_i$, for any $r+i \geq 0$, and this fit into the commutative diagram :

$$\begin{tikzcd}
E_i \arrow[r, hook] \arrow[d, "\xi^i(f)"'] & E_i \oplus E \arrow[r] \arrow[d, "\xi^i(f)\oplus p"'] & E'_i \arrow[r] \arrow[d, "\xi^i(f\oplus f')"'] & \mathscr{E}_{r+s+i}(\mathbb{R}^\infty) \arrow[d, "\gamma_{r+s+i}"] \\
B_i \arrow[r, equal] & B_i \arrow[r] & B'_i \arrow[r, "f\oplus f'"] & BO(r+s+i).
\end{tikzcd}$$

This gives a map $\mathrm{Th}(\xi^i(f)) \to \mathrm{Th}(\xi^i(f \oplus f')$ which fits into the commutative diagram :

$$\begin{tikzcd}
\Sigma\mathrm{Th}(\xi^i(f)) \arrow[r] \arrow[d] & \mathrm{Th}(\xi^{i+1}(f)) \arrow[d] \\
\Sigma\mathrm{Th}(\xi^i(f \oplus f')) \arrow[r] & \mathrm{Th}(\xi^{i+1}(f \oplus f')).
\end{tikzcd}$$

Now, given a map $g : A \to B$, the naturality follows from the commutativity of the diagram :

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "g"'] & A'_i \arrow[d, "g"] \\
B_i \arrow[r] & B'_i,
\end{tikzcd}$$

where $A_i = (f \circ g)^{-1}(\{r\} \times BO(r+i))$ and $A'_i = ((f \circ g) \oplus (f' \circ g))^{-1}(\{r+s\} \times BO(r+s+i))$. $\square$

Let us now define precisely what we tried to call $-p$ for a vector bundle p.

Definition 1.2.12 (Associated Virtual Vector Bundle).
Let $p : E \to B$ be a r-dimensional vector bundle, where $r \geq 0$, and $f : B \to BO(r)$ a classifying map. Compose f with the inclusion $BO(r) \hookrightarrow BO$ and the involution $\iota : BO \to BO$, to get a map[1] :

$$B \longrightarrow \{-r\} \times BO \hookrightarrow \mathbb{Z} \times BO$$

called the *associated virtual vector bundle of rank* $-r$, denoted $-f$.

Definition 1.2.13 (Virtual Representation $-V$).
Let G be a finite topological group. Let V be a real representation of G. The *virtual representation* $-V$ of V is the map obtained as above, when V is considered as a vector bundle over BG, via the Borel construction.

The above notation gives a virtual Thom spectrum BG^{-V}. Notice that if 0 denotes the trivial $\mathbb{R}G$-module, then $BG^{-0} = BG^0$. Moreover, if ε^k is the trivial representation of G of dimension $k \geq 0$, we can extend the notation for $k < 0$, i.e., if $k < 0$ then $\varepsilon^k = -\varepsilon^{-k}$. Notice that we get : $BG^{\varepsilon^k \oplus V} = \Sigma^k BG^V$ for any $k \in \mathbb{Z}$ and any representation (virtual or not) V. The previous Proposition 1.2.8 proves the following result. If $H \leq G$ is a subgroup, we denote $V_{|H}$ the induced $\mathbb{R}H$-module.

Proposition 1.2.14.
Let V be a real representation of the group G. Suppose two subgroups $H_1 \leq H_2 \leq G$ are given. Then there exists a map of spectra $BH_1^{-(V_{|H_1})} \to BH_2^{-(V_{|H_2})}$ induced by the inclusion $H_1 \hookrightarrow H_2$. Moreover, if $H_1 \leq H_2 \leq H_3 \leq G$, then the following diagram commutes :

$$\begin{tikzcd}
BH_1^{-V_{|H_1}} \arrow[rr] \arrow[dr] & & BH_2^{-V_{|H_2}} \arrow[dl] \\
& BH_3^{-V_{|H_3}}. &
\end{tikzcd}$$

[1] We are secretly using the inverse map in real K-theory which comes from the H-group structure of BO with the involution ι.

PROOF : Suppose V is of dimension $r \geq 0$. The inclusion $H_1 \hookrightarrow H_2$ gives a map $BH_1 \to BH_2$ by the functoriality of the classifying space. Notice then we get the commutative diagram :

$$\begin{tikzcd} BH_1 \arrow[r] \arrow[rrr, bend left, "V_{|H_1}"] & BH_2 \arrow[r] \arrow[rr, bend right, "V_{|H_2}"'] & BG \arrow[r, "V"] & BO(r), \end{tikzcd}$$

where, by abuse of notation, we denoted by V the classifying map of the vector bundle V over BG. $\square$

1.3. Steinberg Idempotent in $\mathbb{F}_2\left[\mathrm{GL}_n(\mathbb{F}_2)\right]$

This section is purely algebraic and is devoted to find relevant properties of the Steinberg idempotent.

NOTATION 1.3.1.
For an integer $n \geq 1$, we denote by $\mathfrak{S}_n$ the symmetric group (permutations matrices), B_n the upper triangular subgroup of $\mathrm{GL}_n(\mathbb{F}_2)$ and T_n the cyclic subgroup of $\mathfrak{S}_n$ generated by $(1, \ldots, n)$. If $H \leq \mathrm{GL}_n(\mathbb{F}_2)$, we denote by $\overline{H} = \sum_{h \in H} h \in \mathbb{F}_2[\mathrm{GL}_n(\mathbb{F}_2)]$.

DEFINITION 1.3.2 (Steinberg Idempotent).
For $n \geq 1$, the *Steinberg idempotent* is the element $e_n = \overline{B}_n\overline{\mathfrak{S}}_n$, in $\mathbb{F}_2[\mathrm{GL}_n(\mathbb{F}_2)]$. For $n = 0$, let e_0 be the trivial element. The element e_n is indeed an idempotent, see [STEINBERG, 1956].

Consider $1 \leq \ell \leq n$. Let $\mathrm{Pr}_{\mathbb{F}_2}(V_\ell, V_n)$ be the set of all $\mathbb{F}_2$-linear surjections $V_n \to V_\ell$, which can be regarded as a subset of the matrices $\mathrm{Mat}_{\ell\times n}(\mathbb{F}_2)$. Then the set $\mathbb{F}_2[\mathrm{Pr}_{\mathbb{F}_2}(V_\ell, V_n)]$ is a left $\mathrm{GL}_\ell(\mathbb{F}_2)$-module and right $\mathrm{GL}_n(\mathbb{F}_2)$-module.

LEMMA 1.3.3.
Let $H \leq \mathrm{GL}_n(\mathbb{F}_2)$ be a subgroup and x an element of $\mathbb{F}_2[\mathrm{Pr}_{\mathbb{F}_2}(V_\ell, V_n)]$. If the order of the stabilizer $\mathrm{Stab}_H(x) := \{h \in H \mid xh = 0\}$ is even, then $x\overline{H} = 0$.

PROOF : Let $\mathscr{H}$ be a set of representatives in H of the elements of the set $\Big(\mathrm{Stab}_H(x)\Big)\backslash H$ of right cosets. We have :

$$\begin{aligned} x\overline{H} &= \sum_{h\in H} xh \\ &= |\mathrm{Stab}_H(x)| \sum_{h\in\mathscr{H}} xh \\ &= 0, \end{aligned}$$

as $|\mathrm{Stab}_H(x)|$ is even. $\square$

LEMMA 1.3.4.
Let $n \geq \ell \geq 2$. Let $A = (A', v) \in \mathrm{Pr}_{\mathbb{F}_2}(V_\ell, V_n)$, where $A' \in \mathrm{Mat}_{\ell\times(n-1)}(\mathbb{F}_2)$ and $v \in V_\ell$. If $\mathrm{rank}(A') \leq n-2$, then $Ae_n = 0$.

Proof : Write $A = (v_1, \ldots, v_n)$ where $v_i \in V_\ell$ for $1 \leq i \leq n$, such that $A' = (v_1, \ldots, v_{n-1})$ and $v_n = v$. Since $\mathrm{rank}(A') \leq n-2$, the Gauss elimination implies that there exists $s \leq n-1$ and $a_i \in \mathbb{F}_2$ for all $1 \leq i \leq s$, such that $a_s = 1$ and :

$$\sum_{i=1}^{s} a_i v_i = 0.$$

Define the triangular matrix :

$$B = \begin{pmatrix} 1 & & a_1 & & \\ & \ddots & \vdots & 0 & \\ & & a_s & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \in B_n \leq \mathrm{GL}_n(\mathbb{F}_2).$$

We get a matrix $C := AB = (v_1, \ldots, v_{s-1}, 0, v_{s+1}, \ldots v_n) \in \mathrm{Pr}_{\mathbb{F}_2}(V_\ell, V_n)$. We see that $\mathrm{Stab}_{B_n}(C)$ contains a subgroup :

$$\left\{ \begin{pmatrix} 1 & & & 0 & & \\ & \ddots & & & & \\ & & 1 & c_1 \cdots & & c_{n-s} \\ & 0 & & \ddots & & \\ & & & & & 1 \end{pmatrix} \mid c_i \in \mathbb{F}_2 \right\},$$

of order 2^{n-s}. Therefore, by Lemma 1.3.3, we get $A\overline{B}_n = AB\overline{B}_n = C\overline{B}_n = 0$. Thus we obtain : $Ae_n = A\overline{B}_n\overline{\mathfrak{S}_n} = 0$. □

Notation 1.3.5.
Let M be a right $\mathbb{F}_2[\mathrm{GL}_n(\mathbb{F}_2)]$-module and e be an idempotent in $\mathbb{F}_2[\mathrm{GL}_n(\mathbb{F}_2)]$. We denote $M \cdot e$ the sub $\mathbb{F}_2$-module of M which consists of e-invariant elements.

Proposition 1.3.6.
Suppose $\ell \leq n-2$. For any right $\mathbb{F}_2[\mathrm{GL}_\ell(\mathbb{F}_2)]$-module M, we have :

$$\Big(M \otimes_{\mathbb{F}_2[\mathrm{GL}_\ell(\mathbb{F}_2)]} \mathbb{F}_2[\mathrm{Pr}_{\mathbb{F}_2}(V_\ell, V_n)]\Big) \cdot e_n = 0.$$

Proof : Immediate from Lemma 1.3.4. □

Lemma 1.3.7.
Let $I_{n-1,n} = (I_{n-1}, 0) \in \mathrm{Pr}(V_{n-1}, V_n)$ where $I_{n-1} \in \mathrm{Mat}_{n-1 \times n-1}$ is the unit matrix. Then in $\mathbb{F}_2[\mathrm{Pr}(V_{n-1}, V_n)]$, the following equality holds :

$$(I_{n-1,n}\overline{T}_n)e_n = I_{n-1,n}e_n = e_{n-1}(I_{n-1,n}\overline{T}_n).$$

Proof : Omitted. See [Nishida, 1986], Lemma 1.1. □

Proposition 1.3.8.
For any right $\mathbb{F}_2[\mathrm{GL}_{n-1}(\mathbb{F}_2)]$-module M, we have an isomorphism :

$$M \cdot e_{n-1} \cong M \otimes_{\mathbb{F}_2[\mathrm{GL}_{n-1}(\mathbb{F}_2)]} \mathbb{F}_2[\mathrm{Pr}_{\mathbb{F}_2}(V_{n-1}, V_n)] \cdot e_n.$$

PROOF : Consider the case $M = \mathbb{F}_2[\mathrm{GL}_{n-1}(\mathbb{F}_2)]$. The general case will follow by the fact that for any module M : $M \otimes_{\mathbb{F}_2[\mathrm{GL}_{n-1}(\mathbb{F}_2)]} \mathbb{F}_2[\mathrm{GL}_{n-1}(\mathbb{F}_2)] \cong M$. Define a homomorphism :

$$\begin{array}{rcl} f : \mathbb{F}_2[\mathrm{GL}_{n-1}(\mathbb{F}_2)] \cdot e_{n-1} & \longrightarrow & \mathbb{F}_2[\mathrm{Pr}(V_{n-1}, V_n)] \cdot e_n \\ A & \longmapsto & AI_{n-1,n}\overline{T}_n. \end{array}$$

From Lemma 1.3.7, we see that f is a monomorphism. To show that f is an epimorphism, consider any element $A = (A', v)$ in $\mathrm{GL}_{n-1}(\mathbb{F}_2)$. If $\mathrm{rank}(A') \leq n-2$, then Lemma 1.3.4 implies that $Ae_n = 0$. If $\mathrm{rank}(A') = n-1$, then by Gauss elimination, there exists $C \in \mathrm{GL}_{n-1}(\mathbb{F}_2)$ and $B \in B_n$ such that $CAB = I_{n-1,n}$. Then :

$$\begin{array}{rcl} Ae_n & = & C^{-1}I_{n-1,n}B^{-1}e_n \\ & = & C^{-1}I_{n-1,n}e_n, \text{ as } B^{-1} \in B_n, \\ & = & C^{-1}e_{n-1}I_{n-1,n}\overline{T}_n, \text{ by Lemma 1.3.7}, \\ & = & f(C^{-1}). \end{array}$$

Therefore f is an isomorphism of $\mathbb{F}_2$-vector spaces. □

1.4. THE GENERALIZED MITCHELL-PRIDDY SPECTRUM $L(n, k)$

In order to define the spectrum $L(n, k)$ we need to give a $\mathrm{GL}_2(\mathbb{F}_2)$-action on $BV_n^{k\overline{\rho}_n}$. We shall introduce this action in a greater generality. We define intermediate spectra $L((n, \ell), k)$ that will be useful subsequently, as they will allow us to relate the spectra $L(n, k)$ and form cofibre sequences.

We now consider all our spectra after 2-completion and work with the category $\mathscr{S}_2$. We refer the reader to Appendix B. We shall omit the functor L_2 from the notation, i.e., all spectra are implicitly completed at the prime two. Notice that the Thom isomorphism proves that all our spectra are connective and locally of finite type.

1.4.1 THE FUNCTOR $X^k_{n,\ell}$

NOTATION 1.4.1.
Let $\mathbf{C}(n, \ell)$ denote the poset of $\mathbb{F}_2$-subspaces W of V_n such that $\dim_{\mathbb{F}_2}(W) \leq \ell$, ordered by inclusion. We denote $\mathbf{C}(n) := \mathbf{C}(n, n)$.

For any element W in $\mathbf{C}(n, \ell)$, one can construct the Thom spectrum $BW^{k\overline{\rho}_n|_W}$ for any $k \in \mathbb{Z}$. Notice we write unambiguously $k\overline{\rho}_n|_W$ as : $(k\overline{\rho}_n)|_W \cong k(\overline{\rho}_n)|_W$ by the interchange property of the limits. For $W_1 \leq W_2 \leq V_n$, we define maps :

$$i_{W_2,W_1} : BW_1^{k\overline{\rho}_n|_{W_1}} \longrightarrow BW_2^{k\overline{\rho}_n|_{W_2}},$$

as follows. We seperate the cases $k \geq 0$ and $k < 0$ this time only to be explicit, but the virtual case cover both cases, so that we do not have to consider that. For the positive case $k \geq 0$, the inclusion $W_1 \hookrightarrow W_2$ defines a map $BW_1 \to BW_2$ and a map $EW_1 \to EW_2$ using the functoriality of the classifying space. Since we also obtain a vector bundle map :

$$\begin{tikzcd} EW_1 \times_{W_1} k\overline{\rho}_n|_{W_1} \arrow[r] \arrow[d] & EW_1 \times_{W_2} k\overline{\rho}_n|_{W_2} \arrow[d] \\ BW_1 \arrow[r] & BW_2, \end{tikzcd}$$

via the natural inclusion $k\overline{\rho}_{n|W_1} \hookrightarrow k\overline{\rho}_{n|W_2}$, the functoriality of Thom spaces gives the map of spectra i_{W_2,W_1}. For the case $k < 0$, Proposition 1.2.14 defines i_{W_2,W_1}. It is elementary to see that for $W_1 \leq W_2 \leq W_3 \leq V_n$, we get $i_{W_3,W_2} \circ i_{W_2,W_1} = i_{W_3,W_1}$ (see Proposition 1.2.14). Therefore we obtain a functor :

$$\begin{array}{rcl} X^k_{n,\ell} : \mathbf{C}(n,\ell) & \longrightarrow & \mathscr{S}_2 \\ W & \longmapsto & BW^{k\overline{\rho}_{n|W}} \\ \Big(W_1 \hookrightarrow W_2\Big) & \longmapsto & \Big(i_{W_2,W_1} : BW_1^{k\overline{\rho}_{n|W_1}} \longrightarrow BW_2^{k\overline{\rho}_{n|W_2}}\Big). \end{array}$$

The $\mathrm{GL}_n(\mathbb{F}_2)$-Action We present in a very explicit way the left action of $\mathrm{GL}_n(\mathbb{F}_2)$ on the spectra $BV_n^{k\overline{\rho}_n}$, for $k \in \mathbb{Z}$. We carry on our description of maps $BV_n^{k\overline{\rho}_n} \to BV_n^{k\overline{\rho}_n}$ for each $A \in \mathrm{GL}_n(\mathbb{F}_2)$, in a greater generality by defining maps :

$$BW^{k\overline{\rho}_n|_W} \longrightarrow B(A(W))^{k\overline{\rho}_n|_{A(W)}},$$

for any $W \leq V_n$. The functoriality of the classifying space provides a map $BW \to B(A(W))$ which can be described explicitly as follow. For clarity, let us denote $\overline{\mathbb{R}V_n}$ the reduced regular representation as a real vector space. If we write $r := 2^n - 1$, then we have the embedding $\overline{\rho}_n : V_n \hookrightarrow O(r)$ and the identification $\mathbb{R}^r \cong \overline{\mathbb{R}V_n}$. Then one can write formally, for any $w \in W \leq V_n$:

$$\begin{array}{rcl} \overline{\rho}_n(w) : \overline{\mathbb{R}V_n} & \longrightarrow & \overline{\mathbb{R}V_n} \\ x & \longmapsto & x + w. \end{array}$$

Moreover, each element $A \in \mathrm{GL}_n(\mathbb{F}_2)$, as an automorphism of V_n, can be regarded as an orthogonal matrix in $O(r)$ via the identification :

$$\begin{array}{rcl} A : \overline{\mathbb{R}V_n} & \longrightarrow & \overline{\mathbb{R}V_n} \\ x & \longmapsto & Ax. \end{array}$$

Then, for any $w \in W \leq V_n$, the following diagram commutes by linearity of A :

$$\begin{tikzcd} \overline{\mathbb{R}V_n} \arrow[r, "A"] \arrow[d, "\overline{\rho}_n(A^{-1}(w))"'] & \overline{\mathbb{R}V_n} \arrow[d, "\overline{\rho}_n(w)"] \\ \overline{\mathbb{R}V_n} \arrow[r, "A"] & \overline{\mathbb{R}V_n}. \end{tikzcd} \qquad (\diamond)$$

The map $BW \to B(A(W))$ is defined from :

$$\begin{array}{rcl} \mathscr{V}_r(\mathbb{R}^\infty)/W & \longrightarrow & \mathscr{V}_r(\mathbb{R}^\infty)/(A(W)) \\ [v] & \longmapsto & \Big[v \circ A^{-1}\Big]. \end{array}$$

It is well-defined as $[v \circ \overline{\rho}_n(w) \circ A^{-1}] = [v \circ A^{-1} \circ \overline{\rho}_n(A(w))]$ by the commutativity of $(\diamond)$. Using the naturality of Thom spectrum (Proposition 1.2.8) together with the commutativity of the diagram :

$$\begin{tikzcd} BW \arrow[r] \arrow[rr, bend right, "|k|\overline{\rho}_n|_W"'] & B(A(W)) \arrow[r, "|k|\overline{\rho}_n|_{A(W)}"] & BO(r) \end{tikzcd}$$

gives the desired map. Actually, Proposition 1.2.8 defines a natural transformation :

$$A_* : X^k_{n,\ell} \Longrightarrow X^k_{n,\ell} \circ A,$$

where each element $A \in \mathrm{GL}_n(\mathbb{F}_2)$ defines a functor :

$$\begin{array}{rcl} A : \mathbf{C}(n,\ell) & \longrightarrow & \mathbf{C}(n,\ell) \\ W & \longmapsto & A(W). \end{array}$$

Using again Proposition 1.2.8, we get that $(I_n)_*$ is the identity transformation $\mathrm{id}_{X^k_{n,\ell}}$, and that, for all A, C in $\mathrm{GL}_n(\mathbb{F}_2)$, the equality $(AC)_* = (A_*)_C \circ C_*$ holds. Using Proposition A.9 expresses the following result.

PROPOSITION 1.4.2.
For all $k \in \mathbb{Z}$, the spectrum $BV_n^{k\overline{\rho}_n}$ is endowed with a left $\mathrm{GL}_n(\mathbb{F}_2)$-action. More generally, the functor $X^k_{n,\ell} : \mathbf{C}(n,\ell) \to \mathscr{S}_2$ is a $\mathrm{GL}_n(\mathbb{F}_2)$-diagram, in the sense of Definition A.8*, for all $0 \leq \ell \leq n$ and $k \in \mathbb{Z}$.*

1.4.2 Stable Splittings

Recall that in the stable homotopy category $\mathscr{S}$, the set of all maps between spectra X and Y up to homotopy, denoted $[X,Y]$, is an abelian group, and $[X,X]$ is a ring.

NOTATION 1.4.3 ($e \cdot X$).
Let $f : R \to [X,X]$ be a ring homomorphism. Let e be an idempotent of R. We denote by $e \cdot X$ the mapping telescope :

$$\operatorname{hocolim}\left(X \xrightarrow{f(e)} X \xrightarrow{f(e)} X \xrightarrow{f(e)} \cdots\right).$$

There are natural maps $e \cdot X \to X$ and $X \to e \cdot X$.

LEMMA 1.4.4.
Let X and e be as above. Then for any spectrum E, the isomorphism : $E^*(e \cdot X) \cong E^*(X) \cdot e$ holds.

PROOF : Since $e^2 = e$, the Mittag-Leffer conditions provide the isomorphism. □

The maps $X \to e \cdot X$ and $e \cdot X \to X$ provides a splitting of X :

$$X \simeq \Big(e \cdot X\Big) \vee \Big((1-e) \cdot X\Big).$$

For any 2-completed spectrum X with a G-action, there exists a ring homomorphism (by Proposition B.9) :

$$\widehat{\mathbb{Z}_2}[G] \longrightarrow [X,X].$$

By the Hensel's Lemma, any idempotent in $\mathbb{F}_2[G]$ can be lifted to an idempotent in $\widehat{\mathbb{Z}_2}[G]$ via the epimorphism $\widehat{\mathbb{Z}_2} \to \mathbb{Z}/2$. Therefore, if we take $G = \mathrm{GL}_n(\mathbb{F}_2)$, one can consider the Steinberg idempotent e_n (Definition 1.3.2) in $\widehat{\mathbb{Z}_2}[\mathrm{GL}_n(\mathbb{F}_2)]$.

DEFINITION 1.4.5 (Generalized Mitchell-Priddy Spectrum $L(n,k)$).
For $k \in \mathbb{Z}$, and $n \geq 0$, the *generalized Mitchell-Priddy spectrum*, denoted[2] $L(n,k)$, is defined as :

$$L(n,k) = e_n \cdot BV_n^{k\overline{\rho}_n}.$$

[2]In [TAKAYASU, 1999], it is denoted $M(n)_k$.

Using Proposition A.12 with Proposition 1.4.2 allows to consider intermediate spectra :

$$L((n,\ell),k) = e_n \cdot \text{hocolim } X^k_{n,\ell}.$$

The intermediate spectra $L((n,\ell),k)$ are not complicated once $\ell \leq n-2$. Indeed, as we will see in Corollary 2.1.5, the spectrum $L((n,\ell),k)$ is contractible for $\ell \leq n-2$. For the case $\ell = n$, we recover the generalized Mitchell-Priddy spectrum $L(n,k)$, as we see in the following lemma.

Lemma 1.4.6.
For any $\mathrm{GL}_n(\mathbb{F}_2)$*-diagram* $F : \mathbf{C}(n) \to \mathscr{S}_2$*, we have the equivalence :* $e_n \cdot \text{hocolim } F \simeq e_n \cdot F(V_n)$*. In particular, we have :* $L((n,n),k) \simeq L(n,k)$*.*

Proof : Notice that $\mathbf{C}(n)$ has terminal object $V_n = (\mathbb{Z}/2)^n$. Therefore the natural map $F(V_n) \longrightarrow \text{hocolim } F$ is a weak equivalence (see [Bousfield and Kan, 1972] Chapter XII, 3.1). Thus we have the homotopy equivalence $F(V_n) \simeq \text{hocolim } F$. It is $\mathrm{GL}_n(\mathbb{F}_2)$-equivariant. □

The case $\ell = n-1$ is the only non-trivial case and will be the object of study of the next chapter. We answer on how does the spectrum $L((n,n-1),k)$ relates to the generalized Mitchell-Priddy spectrum. We first define maps that connects the spectra if we change the indices n or k.

1.4.3 The Map $j_{n,k}$

For all $k \in \mathbb{Z}$, the inclusion $k\overline{\rho}_n \hookrightarrow (k+1)\overline{\rho}_n$ defines a map of spectra $j_{n,k} : BV_n^{k\overline{\rho}_n} \to BV_n^{(k+1)\overline{\rho}_n}$, and more generally a natural transformation :

$$j_{n,k} : X^k_{n,\ell} \Longrightarrow X^{k+1}_{n,\ell}.$$

The explicit definitions goes as follows. Proposition 1.2.11 provides a map :

$$BW^{k\overline{\rho}_n|_W} \longrightarrow BW^{k\overline{\rho}_n|_W \oplus \overline{\rho}_n|_W} = BW^{(k+1)\overline{\rho}_n|_W},$$

for any $k \in \mathbb{Z}$ and $W \in \mathbf{C}(n,\ell)$, such that the following diagram commutes :

$$\begin{tikzcd}
BW^{k\overline{\rho}_n|_W} \arrow[r, "j_{n,k}"] \arrow[d, "A_*"'] & BW^{(k+1)\overline{\rho}_n|_W} \arrow[d, "A_*"] \\
(BA(W))^{k\overline{\rho}_n|_{A(W)}} \arrow[r, "j_{n,k}"] & (BA(W))^{(k+1)\overline{\rho}_n|_{A(W)}},
\end{tikzcd}$$

for any A in $\mathrm{GL}_n(\mathbb{F}_2)$. Therefore Proposition A.9 gives the following result.

Proposition 1.4.7.
For all $k \in \mathbb{Z}$*, the natural transformation* $j_{n,k} : X^k_{n,\ell} \Longrightarrow X^{k+1}_{n,\ell}$ *is a* $\mathrm{GL}_n(\mathbb{F}_2)$*-map, in the sense of* Definition A.11*.*

The previous proposition induces the maps :

$$j_{n,k} : L(n,k) \longrightarrow L(n,k+1),$$

and more generally the maps :

$$j_{n,k} : L((n,\ell),k) \longrightarrow L((n,\ell),k+1).$$

1.4.4 The Map $i_{n,k}$

In the following argument, let $n \geq 1$. Denote the inclusion $\iota : V_{n-1} \hookrightarrow V_n$ of the first $(n-1)$-terms. It defines a functor for any $0 \leq \ell \leq n-1$:

$$\begin{array}{rcl} \iota : \mathbf{C}(n-1,\ell) & \longrightarrow & \mathbf{C}(n,\ell) \\ W & \longmapsto & \iota(W). \end{array}$$

Therefore we obtain a functor :

$$\begin{array}{rcl} \iota^* : \mathscr{S}_2^{\mathbf{C}(n,\ell)} & \longrightarrow & \mathscr{S}_2^{\mathbf{C}(n-1,\ell)} \\ X & \longmapsto & X \circ \iota. \end{array}$$

By the functoriality of homotopy colimits, we get a natural map :

$$\text{hocolim } \iota^* : \text{hocolim}(\iota^*(X)) \longrightarrow \text{hocolim}(X).$$

The inclusion $\iota : V_{n-1} \hookrightarrow V_n$ induces an inclusion $\text{Aut}(V_{n-1}) \hookrightarrow \text{Aut}(V_n)$, explicitly :

$$\begin{array}{rcl} \text{GL}_{n-1}(\mathbb{F}_2) & \hookrightarrow & \text{GL}_n(\mathbb{F}_2) \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

Then if X is a $\text{GL}_n(\mathbb{F}_2)$-diagram, it is also a $\text{GL}_{n-1}(\mathbb{F}_2)$-diagram. Proposition A.12 endows hocolim X and hocolim $\iota^*(X)$ with a $\text{GL}_{n-1}(\mathbb{F}_2)$ action, such that the natural map (hocolim ι^*) is $\text{GL}_{n-1}(\mathbb{F}_2)$-equivariant.

Notation 1.4.8.
For any functor $X : \mathbf{C}(n,\ell) \to \mathscr{S}$, let us denote $(X)|_{\mathbf{C}(n-1,\ell)} := \iota^*(X)$.

Lemma 1.4.9.
For $n \geq 1$, we have the isomorphism of $\mathbb{R}V_{n-1}$-modules : $(\overline{\rho}_n)|_{V_{n-1}} \cong 2\overline{\rho}_{n-1} \oplus \varepsilon$, where ε denotes the one-dimensional trivial representation of V_{n-1}.

Proof : In general, for groups $H \leq G$, recall that $G = \coprod_{g \in H\backslash G} Hg$. Then a simple investigation of the action of H gives the isomorphism :

$$(\mathbb{R}G)_{|_H} \cong \bigoplus_{H\backslash G} \mathbb{R}H.$$

Since $V_n/V_{n-1} \cong \mathbb{F}_2$ as vector spaces, we obtain : $(\rho_n)_{|_{V_{n-1}}} \cong \rho_{n-1} \oplus \rho_{n-1} = 2\rho_{n-1}$. Since the trivial representation of V_n clearly restricts to the trivial reprensentation of V_{n-1}, the result follows. □

Theorem 1.4.10.
For $n \geq 1$ and $\ell \leq n-1$, the functors $\Sigma^k X^{2k}_{n-1,\ell}$ and $(X^k_{n,\ell})|_{\mathbf{C}(n-1,\ell)}$ are naturally isomorphic via a $\text{GL}_{n-1}(\mathbb{F}_2)$-map, in the sense of Definition A.11.

Proof : The previous lemma generalizes directly to :

$$k(\overline{\rho}_n)|_W \cong 2k(\overline{\rho}_{n-1})|_W \oplus \varepsilon^k,$$

for $k \geq 0$ and $W \in \mathbf{C}(n-1,\ell)$. Then Proposition 1.2.8 provides that the induced maps :

$$BW^{k\overline{\rho}_n|_W} \longrightarrow BW^{2k\overline{\rho}_{n-1}|_W \oplus \varepsilon^k} = \Sigma^k BW^{2k\overline{\rho}_{n-1}|_W}$$

are homotopy equivalences and define a natural transformation which is a $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-map. $\square$

We now define the map $i_{n,k}$. First define :

$$i_{n,k} : \Sigma^k L((n-1,\ell),2k) \longrightarrow L((n,\ell),k),$$

as the composite of :

$$e_{n-1} \cdot \Sigma^k \mathrm{hocolim}\left(X^{2k}_{n-1,\ell}\right) \xrightarrow{\simeq} e_{n-1} \cdot \mathrm{hocolim}\left((X^k_{n,\ell})|_{\mathbf{C}(n-1)}\right) \longrightarrow e_n \cdot \mathrm{hocolim}(X^k_{n,\ell})$$

where the right map is induced by $\mathrm{hocolim}(\iota^*)$. We also denote $i_{n,k}$ the map :

$$i_{n,k} : \Sigma^k L(n-1,2k) \longrightarrow L(n,k),$$

which is defined using the previous case with the composite :

$$\Sigma^k L((n-1,n-1),2k) \xrightarrow{i_{n,k}} L((n,n-1),k) \longrightarrow L((n,n),k) \simeq L(n,k)$$

where the right map is induced by the inclusion $\mathbf{C}(n,n-1) \hookrightarrow \mathbf{C}(n,n) = \mathbf{C}(n)$. When $k=0$, it is the map induced by the $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-equivariant map :

$$B\iota : \Sigma^\infty BV_{n-1+} \longrightarrow \Sigma^\infty BV_{n+}.$$

CHAPTER 2

The Cofibre Sequence

This chapter is devoted to present the cofiber sequence of Takayasu (Theorem 2.3.3) :

$$L(n-1, 2k+1) \longrightarrow L(n,k) \xrightarrow{j_{n,k}} L(n,k+1).$$

For this matter, we shall make use of two cofiber sequences which involves the intermediate spectra $L((n,\ell),k)$. The first one (Corollary 2.1.8) :

$$\Sigma^k L((n-1,n-2),2k) \longrightarrow \Sigma^k L(n-1,2k) \xrightarrow{i_{n,k}} L((n,n-1),k)$$

uses one of the maps $i_{n,k}$, and the second uses the map $j_{n,k}$ (Proposition 2.3.2) :

$$L((n,n-1),k) \longrightarrow L(n,k) \xrightarrow{j_{n,k}} L(n,k+1)$$

2.1. A First Cofibre Sequence

The proof of the first cofibre sequence uses the spectral sequences of homotopy colimits.

Theorem 2.1.1 (Cohomology Spectral Sequence for Homotopy Colimits).
For any poset $\mathbf{C}$ and functor $X : \mathbf{C} \to \mathscr{S}$, there is a cohomological spectral sequence :

$$E_r^{p,q}(X) \Rightarrow H^{p+q}(\mathrm{hocolim}\ X),$$

where the first page is given by : $E_1^{p,q}(X) = \bigoplus_{(W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_p) \in \mathbf{C}} H^q(X(W_0))$, and the differentials $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ are given by the alterning sum $\sum_{i=0}^{p} (-1)^i \partial^i$, where the differentials ∂^i :

$$\partial^i : \bigoplus_{(W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_p) \in \mathbf{C}} H^q(X(W_0)) \longrightarrow \bigoplus_{(W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{p+1}) \in \mathbf{C}} H^q(X(W_0))$$

are described as follows. Notice that any sequence $(W_0 \subsetneq \ldots \subsetneq W_{p+1})$ give a new sequence $(W_0, \subsetneq \ldots \subsetneq W_p)$ by omitting one term. If $i > 0$, then the terms W_0 are the same and ∂^i is induced by the identity $\mathrm{id} : H^q(X(W_0)) \to H^q(X(W_0))$. If $i = 0$, then the inclusion $W_0 \hookrightarrow W_1$ provides a morphism $H^q(X(W_1)) \to H^q(X(W_0))$ which induces ∂^0.

Proof : See [Bousfield and Kan, 1972] Chapter XII, 5.8. □

Let G be a group. Let $\mathbf{C}$ be a poset together with a G-action Let $X : \mathbf{C} \to \mathscr{S}_2$ be a G-diagram in $\mathscr{S}_2$, in the sense of Definition A.8. From Proposition A.12, the spectrum hocolim X is endowed with a (left) G-action. If $e \in \widehat{\mathbb{Z}_2}[G]$ is an idempotent, then the spectral sequence of Theorem 2.1.1 defines the spectral sequence :

$$\overline{E}_r^{p,q} := E_r^{p,q}(X) \cdot e \Rightarrow H^{p+q}(e \cdot \text{hocolim } X),$$

since the following functor is exact :

$$\begin{array}{rcl} \mathbf{Mod}_{\mathbb{F}_2[G]} & \longrightarrow & {}_{\mathbb{F}_2}\mathbf{Mod} \\ M & \longmapsto & M \cdot e. \end{array}$$

Definition 2.1.2 (The Cochain Complex $K_\bullet^q(Z)$).
Let $n \geq 1$. Let $Z : \mathbf{C} \to \mathscr{S}$ be a $\mathrm{GL}_n(\mathbb{F}_2)$-diagram in the sense of Definition A.8, such that $\mathbf{C}(n) \leq \mathbf{C}$ is a full subcategory and the action of $\mathrm{GL}_n(\mathbb{F}_2)$ on $\mathbf{C}$ agrees with its standard action on $\mathbf{C}(n)$. For any $q \in \mathbb{Z}$, denote the cochain complex $K_\bullet^q(Z)$:

$$K_\bullet^q(Z) : 0 \longrightarrow H^q\Big(e_n \cdot Z(V_n)\Big) \xrightarrow{\iota_n^*} H^q\Big(e_{n-1} \cdot Z(V_{n-1})\Big) \xrightarrow{\iota_{n-1}^*} \cdots.$$

where $\iota_{j+1} : e_j \cdot Z(V_j) \to e_{j+1} \cdot Z(V_{j+1})$ is the composite of the maps :

$$e_j \cdot Z(V_j) \longrightarrow Z(V_j) \longrightarrow Z(V_{j+1}) \longrightarrow e_{j+1} \cdot Z(V_{j+1}).$$

The middle arrow is induced by the inclusion of $\iota_{j+1} : V_j \hookrightarrow V_{j+1}$ by adding a zero on the $(j+1)$-term, for all $j \geq 0$. Let us denote the cochain complex $K_\bullet^q(Z)[-1]$ given by :

$$K_\bullet^q(Z)[-1] : 0 \longrightarrow H^q\Big(e_{n-1} \cdot Z(V_{n-1})\Big) \xrightarrow{\iota_{n-1}^*} H^q\Big(e_{n-2} \cdot Z(V_{n-2})\Big) \xrightarrow{\iota_{n-2}^*} \ldots,$$

We simply shifted the previous cochain complex. We will see subsequently that $K_\bullet^q(Z)$ is indeed a chain complex.

Notation 2.1.3.
For *any* functor $X_n : \mathbf{C}(n) \longrightarrow \mathscr{S}$, denote by $X_{n,\ell} : \mathbf{C}(n,\ell) \to \mathscr{S}$, for $0 \leq \ell \leq n$, the composite :

$$\mathbf{C}(n,\ell) \longhookrightarrow \mathbf{C}(n,n) = \mathbf{C}(n) \xrightarrow{X_n} \mathscr{S}.$$

Lemma 2.1.4.
For any $\mathrm{GL}_n(\mathbb{F}_2)$*-diagram* $X_{n,\ell} : \mathbf{C}(n,\ell) \to \mathscr{S}$*, we have :*

- **(i)** $\overline{E}_1^{p,q} = E_1^{p,q}(X_{n,\ell}) \cdot e_n = 0$*, if* $\ell \leq n-2$*;*
- **(ii)** $\overline{E}_2^{p,q} = E_2^{p,q}(X_{n,n-1}) \cdot e_n \cong H^p(K_\bullet^q(X_{n,n-1})[-1])$*, the* p*-th cohomology of the cochain complex.*

Proof : The E_1 of the spectral sequence is given by :

$$E_1^{p,q}(X_{n,\ell}) \cong \bigoplus_{(W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_p) \in \mathbf{C}(n,\ell)} H^q(X_{n,\ell}(W_0))$$

which is isomorphic to :

$$\bigoplus_{a_0 < \cdots < a_p \leq \ell} H^q(X_{n,\ell}(V_{a_0})) \otimes_{\mathbb{F}_2[\mathrm{GL}_{a_0}]} \mathbb{F}_2[\mathrm{Pr}(V_{a_0}, V_{a_1})] \otimes_{\mathbb{F}_2[\mathrm{GL}_{a_1}]} \cdots \otimes_{\mathbb{F}_2[\mathrm{GL}_{a_p}]} \mathbb{F}_2[\mathrm{Pr}(V_{a_p}, V_n)].$$

Now according to Proposition 1.3.6, we see directly that for $\ell \leq n-2$, the above direct sum is trivial once we applied e_n on the right. This shows **(i)**. For $\ell = n-1$, we apply Proposition 1.3.8. For instance, for $p=0$, we have :

$$\overline{E}_1^{0,q} \cong \bigoplus_{a_0 \leq n-1} H^q(X_{n,n-1}(V_{a_0})) \otimes_{\mathbb{F}_2[\mathrm{GL}_{a_0}]} \mathbb{F}_2[\mathrm{Pr}(V_{a_0}, V_{n-1})] \cdot e_n,$$

which is trivial for all the terms $a_0 \leq n-2$ by Proposition 1.3.6, and for the remaining $a_0 = n-1$, it is by Proposition 1.3.8 :

$$\overline{E}_1^{0,q} \cong H^q(X_{n,n-1}(V_{n-1})) \cdot e_{n-1} \cong H^q(e_{n-1} \cdot X_{n,n-1}(V_{n-1})).$$

For $p=1$, we get that the summand :

$$\overline{E}_1^{1,q} \cong \bigoplus_{a_0 < a_1 \leq n-1} H^q(X_{n,n-1}(V_{a_0})) \otimes_{\mathbb{F}_2[\mathrm{GL}_{a_0}]} \mathbb{F}_2[\mathrm{Pr}(V_{a_0}, V_{a_1})] \otimes_{\mathbb{F}_2[\mathrm{GL}_{a_1}]} \mathbb{F}_2[\mathrm{Pr}(V_{a_1}, V_{n-1})] \cdot e_n,$$

is trivial for the terms $a_1 \leq n-2$ by Proposition 1.3.6, and so we get by Proposition 1.3.8 :

$$\overline{E}_1^{1,q} \cong \bigoplus_{a_0 < n-1} H^q(X_{n,n-1}(V_{a_0})) \otimes_{\mathbb{F}_2[\mathrm{GL}_{a_0}]} \mathbb{F}_2[\mathrm{Pr}(V_{a_0}, V_{n-1})] \cdot e_{n-1}.$$

The above direct sum is always trivial except for $a_0 = n-2$, and so :

$$\overline{E}_1^{1,q} \cong H^q(e_{n-2} \cdot X_{n,n-1}(V_{n-2})).$$

The computations carries on for $p \geq 2$. To finish the proof, notice that we get the commutative diagram :

$$\begin{tikzcd}
\left(\bigoplus_{(W_0 \subsetneq \cdots \subsetneq W_p)} H^q(X_{n,n-1}(W_0))\right) \cdot e_n \arrow[r, "d_1^{p,q}"] \arrow[d, "\cong"'] & \left(\bigoplus_{(W_0 \subsetneq \cdots \subsetneq W_{p+1})} H^q(X_{n,n-1}(W_0))\right) \cdot e_n \arrow[d, "\cong"] \\
H^q(X_{n,n-1}(V_{n-p-1})) \cdot e_{n-p-1} \arrow[r] \arrow[d, "\cong"'] & H^q(X_{n,n-1}(V_{n-p-2})) \cdot e_{n-p-2} \arrow[d, "\cong"] \\
H^q(e_{n-p-1} \cdot X_{n,n-1}(V_{n-p-1})) \arrow[r, "\iota_{n-p-1}^*"] & H^q(e_{n-p-2} \cdot X_{n,n-1}(V_{n-p-2})).
\end{tikzcd}$$

Therefore by definition of the $\overline{E}_2$-page, we obtain the result **(ii)**. □

As promised, the following corollary shows that $L((n,\ell),k)$ is contractible whenever $\ell \leq n-2$, for all $k \in \mathbb{Z}$.

Corollary 2.1.5 (The Case $\ell \leq 2$).
Let $X_{n,\ell} : \mathbf{C}(n,\ell) \to \mathscr{S}_2$ be any $\mathrm{GL}_n(\mathbb{F}_2)$-diagram. For $\ell \leq n-2$, there is an homotopy equivalence $e_n \cdot \mathrm{hocolim}\, X_{n,\ell} \simeq *$.

Proof : Use previous lemma and Corollary B.8. □

Now recall the Notation A.14. The Grothendieck construction $\mathbf{Cone}(\mathbf{C}(n,n-1), \mathbf{C}(n))$ can be regarded as a poset with a $\mathrm{GL}_n(\mathbb{F}_2)$-action which agrees with the standard action of $\mathrm{GL}_n(\mathbb{F}_2)$ on its full subcategory $\mathbf{C}(n)$.

Lemma 2.1.6.
For any $\mathrm{GL}_n(\mathbb{F}_2)$-diagram $Y : \mathbf{Cone}(\mathbf{C}(n, n-1), \mathbf{C}(n)) \longrightarrow \mathscr{S}$, its $\overline{E}_2$-page is given by :

$$\overline{E}_2^{p,q} = E_2^{p,q}(Y) \cdot e_n \cong H^p(K_\bullet^q(Y)).$$

Proof : Let us first compute the E_1-page :

$$E_1^{p,q}(Y) = \bigoplus_{(W_0 \subsetneq \ldots \subsetneq W_p) \in \mathbf{Cone}(\mathbf{C}(n,n-1),\mathbf{C}(n))} H^q(Y(W_0)).$$

Recalling the definition of the Grothendieck construction of $\mathbf{Cone}(\mathbf{C}(n, n-1), \mathbf{C}(n))$, we can separate the direct sums as follows :

$$E_1^{p,q}(Y) = \left(\bigoplus_{(W_0 \subsetneq \ldots \subsetneq W_p) \in \mathbf{C}(n)} H^q(Y(W_0)) \right) \oplus \left(\bigoplus_{(\underbrace{W_0 \subsetneq \ldots \subsetneq W_{p-1}}_{\in \mathbf{C}(n,n-1)} \subsetneq *)} H^q(Y(W_0)) \right).$$

Now as Lemma 2.1.4, once we apply e_n on the right, the left direct sum becomes for all $p \geq 0$:

$$H^q(e_{n-p} \cdot Y(V_{n-p-1})) \oplus H^q(e_{n-p-1} \cdot Y(V_{n-p-1})),$$

and the right direct sum becomes :

$$H^q(e_{n-p} \cdot Y(V_{n-p})),$$

for $p \geq 1$, and is trivial for $p = 0$. Investigating on the differential d_1, we obtain a cochain complex $\overline{E}_1^{\bullet,q} = (\overline{E}_1^{\bullet,q}, d_1^{\bullet,q})$ depicted as follows :

$$\overline{E}_1^{0,q} \xrightarrow{\qquad d_1^{0,q} \qquad} \overline{E}_1^{1,q} \xrightarrow{\qquad d_1^{1,q} \qquad} \overline{E}_1^{2,q} \xrightarrow{\quad d_1^{2,q} \quad} \cdots$$

$$\begin{tikzcd}
H^q(e_n \cdot Y(V_n)) \arrow[r, "\iota_n^*"] & H^q(e_{n-1} \cdot Y(V_{n-1})) \arrow[r, "\iota_{n-1}^*"] & H^q(e_{n-2} \cdot Y(V_{n-2})) \arrow[r, "\iota_{n-2}^*"] & \cdots \\
\oplus & \oplus & \oplus & \\
H^q(e_{n-1} \cdot Y(V_{n-1})) \arrow[r, "-\iota_{n-1}^*"] \arrow[uur, equal] \arrow[ddr, equal] & H^q(e_{n-2} \cdot Y(V_{n-2})) \arrow[r, "-\iota_{n-2}^*"] \arrow[uur, equal] \arrow[ddr, equal] & H^q(e_{n-3} \cdot Y(V_{n-3})) \arrow[r, "-\iota_{n-3}^*"] & \cdots \\
 & \oplus & \oplus & \\
 & H^q(e_{n-1} \cdot Y(V_{n-1})) \arrow[r, "\iota_{n-1}^*"] & H^q(e_{n-2} \cdot Y(V_{n-2})) \arrow[r, "\iota_{n-2}^*"] & \cdots .
\end{tikzcd}$$

We see that the middle and bottom row form the cochain complex $\mathbf{cone}(\mathrm{id}_{K_\bullet^q(Y)[-1]})$ which is the mapping cone (see definition on page 650 of [Rotman, 2009]) of the identity cochain map $\mathrm{id}_{K_\bullet^q(Y)[-1]} : K_\bullet^q(Y)[-1] \to K_\bullet^q(Y)[-1]$. The top row is the cochain complex $K_\bullet^q(Y)$. So the cochain complex $\overline{E}_1^{\bullet,q}$ splits as $K_\bullet^q(Y) \oplus \mathbf{cone}(\mathrm{id}_{K_\bullet^q(Y)[-1]})$ and we therefore obtain the short exact sequence of cochain complexes :

$$0 \longrightarrow K_\bullet^q(Y) \longrightarrow \overline{E}_1^{\bullet,t} \longrightarrow \mathbf{cone}(\mathrm{id}_{K_\bullet^q(Y)[-1]}) \longrightarrow 0.$$

As $\mathrm{id}_{K_\bullet^q(Y)[-1]}$ is a quasi-isomorphism, its mapping cone $\mathbf{cone}(\mathrm{id}_{K_\bullet^q(Y)[-1]})$ is an acyclic cochain complex (see Corollary 10.41 in [Rotman, 2009]). Therefore the cohomology of the cochain complexes $\overline{E}_1^{\bullet,q}$ and $K_\bullet^q(Y)$ are isomorphic. Thus $\overline{E}_2^{p,q} = H^p(\overline{E}_1^{\bullet,q}) \cong H^p(K_\bullet^q(Y))$. □

Theorem 2.1.7 (The Case $\ell = n-1$).
Let $X_{n,\ell} : \mathbf{C}(n,\ell) \to \mathscr{S}_2$ be any $\mathrm{GL}_n(\mathbb{F}_2)$-diagram. When $\ell = n-1$, there exists a cofibre sequence :

$$e_{n-1}\cdot \text{hocolim}\left((X_{n,n-2})|_{\mathbf{C}(n-1,n-2)}\right) \to e_{n-1}\cdot \text{hocolim}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right) \to e_n \cdot \text{hocolim } X_{n,n-1}.$$

Proof : The inclusion $\mathbf{C}(n-1,n-2) \hookrightarrow \mathbf{C}(n-1,n-1) = \mathbf{C}(n-1)$, induces a map :

$$e_{n-1}\cdot \text{hocolim}\left((X_{n,n-2})|_{\mathbf{C}(n-1,n-2)}\right) \to e_{n-1}\cdot \text{hocolim}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right).$$

Now Theorem A.17 applied to the mapping cone diagram $\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)$ of shape $\mathbf{C}(n-1)$ over $\mathbf{C}(n-1,n-2)$ implies that $e_{n-1}\cdot \text{hocolim}\left(\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)\right)$ is the mapping cone of the above map. Let us show that there exists a commutative diagram :

$$\begin{tikzcd}
e_{n-1}\cdot \text{hocolim}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right) \arrow[r] \arrow[dr] & e_n \cdot \text{hocolim } X_{n,n-1} \arrow[d, dashed, "\gamma"', "\simeq"] \\
 & e_{n-1}\cdot \text{hocolim}\left(\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)\right).
\end{tikzcd}$$

The diagonal map is induced by the inclusion $\mathbf{C}(n-1) \hookrightarrow \mathbf{Cone}(\mathbf{C}(n-1,n-2),\mathbf{C}(n-1))$. The horizontal map is induced by the inclusion $\mathbf{C}(n-1) \hookrightarrow \mathbf{C}(n,n-1)$. For the vertical homotopy equivalence, let us first consider the functor :

$$\begin{array}{rcl} F : \mathbf{C}(n,n-1) & \longrightarrow & \mathbf{Cone}(\mathbf{C}(n-1,n-2),\mathbf{C}(n-1)) \\ W & \longmapsto & \begin{cases} W, & \text{if } W \subseteq V_{n-1} \\ *, & \text{otherwise.} \end{cases} \end{array}$$

Notice it preserves the order structure. We define a natural transformation :

$$\gamma : X_{n,n-1} \Longrightarrow \mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right) \circ F,$$

as follows : if $W \subseteq V_{n-1}$, then $\gamma_W : X_{n,n-1}(W) \to X_{n,n-1}(W)$ is the identity map, if $W \nsubseteq V_{n-1}$, then $\gamma_W : X_{n,n-1}(W) \to *$ is the unique trivial map. This defines a natural map which we will denote again γ :

$$\gamma : \text{hocolim } X_{n,n-1} \longrightarrow \text{hocolim}\left(\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)\right).$$

The vertical map is then the composite of : $(e_n \cdot \text{hocolim } X_{n,n-1} \to \text{hocolim } X_{n,n-1})$, γ and the map

$$\text{hocolim}\left(\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)\right) \longrightarrow e_{n-1}\cdot \text{hocolim}\left(\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)\right).$$

We will denote it again γ subsequently. We may apply Lemma 2.1.6 to the $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-diagram $\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)$ (by Proposition A.16). Notice that in particular, we have the identities of cochain complexes by definition :

$$K^q_\bullet\left(\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)\right) = K^q_\bullet\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right).$$

The natural transformation γ is a $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-map, in the sense of Definition A.11, and it induces a map of cochain complexes :

$$K^q_\bullet\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right) \longrightarrow K^q_\bullet(X_{n,n-1})[-1],$$

depicted as follow :

$$\begin{tikzcd} 0 \arrow[r] & H^q(e_{n-1}\cdot X_{n,n-1}(V_{n-1})) \arrow[r] \arrow[d, "\gamma_{V_{n-1}}"] & H^q(e_{n-2}\cdot X_{n,n-1}(V_{n-2})) \arrow[r] \arrow[d, "\gamma_{V_{n-2}}"] & \cdots \\ 0 \arrow[r] & H^q(e_{n-1}\cdot X_{n,n-1}(V_{n-1})) \arrow[r] & H^q(e_{n-2}\cdot X_{n,n-1}(V_{n-2})) \arrow[r] & \cdots . \end{tikzcd}$$

It is simply the identity, and from the convergence of the spectral sequences $\overline{E}$, we obtain an isomorphism :

$$\gamma^* : H^*\left(e_{n-1}\cdot \text{hocolim } X_{n,n-1}\right) \longrightarrow H^*\left(e_n \cdot \text{hocolim}\left(\mathscr{M}\left((X_{n,n-1})|_{\mathbf{C}(n-1)}\right)\right)\right).$$

Thus we obtained the desired homotopy equivalence γ, which clearly commutes in the diagram by construction. $\square$

Corollary 2.1.8 (The First Cofibre Sequence).
There is a cofibre sequence :

$$\Sigma^k L((n-1,n-2),2k) \longrightarrow \Sigma^k L(n-1,2k) \xrightarrow{i_{n,k}} L((n,n-1),k).$$

Proof : Applying previous Theorem 2.1.7 together with Theorem 1.4.10, we obtain the commutative diagram :

$$\begin{tikzcd} e_{n-1}\cdot \text{hocolim}\left((X^k_{n,n-2})|_{\mathbf{C}(n-1,n-2)}\right) \arrow[r] & e_{n-1}\cdot \text{hocolim}\left((X^k_{n,n-1})|_{\mathbf{C}(n-1)}\right) \arrow[r] & e_n \cdot \text{hocolim } X^k_{n,n-1} \\ e_{n-1}\cdot \text{hocolim}\left(\Sigma^k X^{2k}_{n-1,n-2}\right) \arrow[r] \arrow[u, "\simeq"] & e_{n-1}\cdot \text{hocolim}\left(\Sigma^k X^{2k}_{n-1}\right) \arrow[r, "i_{n,k}"] \arrow[u, "\simeq"] & e_n \cdot \text{hocolim } X^k_{n,n-1}. \arrow[u, equal] \end{tikzcd}$$

Using the interchange property of colimits on Σ^k and e_n gives the desired cofibre sequence. $\square$

2.2. A Long Exact Sequence in Cohomology

2.2.1 The Euler Class

Recall our discussion (see Theorem 1.1.3) about the Thom isomorphism Theorem and the Thom classes. Let $p : E \to B$ be an r-dimensional vector bundle. Then the projection $p : D(E) \to B$ is a homotopy equivalence, with inverse the zero-section $B \to D(E)$.

Definition 2.2.1 (Euler Class).
The *Euler class* ω_p of a real r-dimensional vector bundle $p : E \to B$ is the image in $H^r(B)$ of the Thom class $u_p \in \widetilde{H}^r(\text{Th}(p))$ under the homomorphism $\widetilde{H}^r(\text{Th}(p)) \to H^r(B)$ induced by the composite map :

$$(B,\emptyset) \xrightarrow{\simeq} (D(E),\emptyset) \hookrightarrow (D(E),S(E)) \twoheadrightarrow D(E)/S(E) = \text{Th}(p).$$

We therefore obtain the commutativity of the diagram :

$$\begin{tikzcd} \widetilde{H}^{*+r}(\text{Th}(p)) \arrow[r] & H^{*+r}(B) \\ H^*(B). \arrow[u, "\text{Thom}", "\cong"'] \arrow[ur, "\smile\omega_p"'] & \end{tikzcd} \tag{2.1}$$

Proposition 2.2.2 (Whitney Sum Formula).
Let p and q be two real vector bundles, of dimension r and s respectively, over a space B. Then we get : $\omega_{p\oplus q} = \omega_p \smile \omega_q$.

Proof : Since $p\oplus q$ is the pullback of $p\times q$ along the diagonal map $\Delta : B \to B\times B$, Proposition 1.1.5 gives the followings commutative diagram :

$$\begin{tikzcd}
\widetilde{H}^{r+s}\Big(\mathrm{Th}(p\oplus q)\Big) \arrow[d] & \widetilde{H}^{r+s}\Big(\mathrm{Th}(p\times q)\Big) \arrow[l] \arrow[d] & \widetilde{H}^{r}(\mathrm{Th}(p))\otimes \widetilde{H}^{s}(\mathrm{Th}(q)) \arrow[l, "\cong"', "\text{Künneth}"] \arrow[d] \\
H^{r+s}(B) & H^{r+s}(B\times B) \arrow[l, "\Delta^*"'] & H^r(B)\otimes H^s(B). \arrow[l, "\text{Künneth}"'] \arrow[ll, bend left, "\text{cup product}"]
\end{tikzcd}$$

An inspection on the commutativity gives the desired result. □

Notice that, in general, it makes no sense to define an Euler class in $H^*(BG)$ for the virtual case (BG^{-V}), as the cohomology is trivial in negative dimensions.

2.2.2 $\mathscr{A}$-Module Structure of $L(n,k)$

For $1 \le i \le n$, let us denote $x_i \in H^1(BV_n)$ the element which corresponds to the unique non-zero element in $H^1(BV_1)$ via the natural inclusion onto the i-th component :

$$\mathbb{Z}/2 = H^1(BV_1) \hookrightarrow \prod_{j=1}^{n} H^1(BV_1) \cong H^1(BV_n) = (\mathbb{Z}/2)^n.$$

Then there is an isomorphism of graded rings (see Theorem 3.19 in [Hatcher, 2002]) :

$$H^*(BV_n) \cong \mathbb{F}_2[x_1,\dots,x_n].$$

This isomorphism is $\mathrm{GL}_n(\mathbb{F}_2)$-equivariant, where the right action of $\mathbb{F}_2[x_1,\dots,x_n]$ is given by :

$$\begin{array}{ccc}
\mathbb{F}_2[x_1,\dots,x_n]\times \mathrm{GL}_n(\mathbb{F}_2) & \longrightarrow & \mathbb{F}_2[x_1,\dots,x_n] \\
(f(x_1,\dots,x_n),\gamma) & \longmapsto & f(\gamma^{-1}(x_1),\dots,\gamma^{-1}(x_n)).
\end{array}$$

Let us denote $\omega_n = \omega_{\overline{\rho}_n} \in H^{2^n-1}(BV_n)$ the Euler class of $\overline{\rho}_n$. As two non-isomorphic line bundles over the same space give different Euler classes, and since the reduced regular representation $\overline{\rho}_n$ is the sum of all non-trivial line bundles over BV_n, we get from the Whitney sum formula :

$$\omega_n = \prod_{(a_1,\dots,a_n)\in V_n - 0} \Big(a_1x_1+\cdots+a_nx_n\Big).$$

Then ω_n is clearly $\mathrm{GL}_n(\mathbb{F}_2)$-invariant. Let us denote by $\omega_n^{-1}H^*(BV_n)$ the localization of the ring $H^*(BV_n)$ away from ω_n. As ω_n is a non-zero divisor, the natural ring homomorphism :

$$\begin{array}{ccc}
H^*(BV_n) & \hookrightarrow & \omega_n^{-1}H^*(BV_n) \\
1 & \longmapsto & 1
\end{array}$$

is a monomorphism. A theorem of Wilkerson (see Theorem 2.1 in [Wilkerson, 1977]) extends the $\mathscr{A}$-module structure of $H^*(BV_n)$ into its localization $\omega_n^{-1}H^*(BV_n)$, i.e., the above map can be considered as a $\mathscr{A}$-module monomorphism. Explicitly, if we denote Sq the total Steenrod square, as it is a ring homomorphism, the $\mathscr{A}$-structure is determined uniquely by the equality $\mathrm{Sq}(\omega_n^{-1}) = (\mathrm{Sq}(\omega_n))^{-1}$. The Thom isomorphism provides an $\mathscr{A}$-module structure for $H^*(BV_n^{k\overline{\rho}_n})$ as a $H^*(BV_n)$-submodule of $\omega_n^{-1}H^*(BV_n)$, via the multiplication with ω_n^k :

$$\begin{array}{ccccc} H^{*+k(2^n-1)}(BV_n^{k\overline{\rho}_n}) & \xrightarrow[\text{Thom}]{\cong} & H^*(BV_n) & \longrightarrow & \omega_n^{-1}H^*(BV_n) \\ u_{k\overline{\rho}_n} & \longmapsto & 1 & \longmapsto & \omega_n^k. \end{array}$$

Notice the similarity with the diagram (2.1) in the definition of the Euler class. In other words, ω_n^k is the "Euler class" of $k\overline{\rho}_n$ in the localization $\omega_n^{-1}H^*(BV_n)$, even for $k < 0$. Denoting $\omega_n^{-1}H^*(BV_n) \cong \mathbb{F}_2[x_1, \dots, x_n][\omega_n^{-1}]$, we get :

$$H^*(BV_n^{k\overline{\rho}_n}) \cong \mathbb{F}_2[x_1, \dots, x_n] \cdot \{\omega_n^k\}.$$

And as ω_n is $GL_n(\mathbb{F}_2)$-invariant, we obtain as $\mathscr{A}$-modules :

$$H^*(L(n,k)) \cong \Big(\mathbb{F}_2[x_1, \dots, x_n] \cdot \{\omega_n^k\}\Big) \cdot e_n.$$

Notice that the map $j_{n,k} : BV_n^{k\overline{\rho}_n} \to BV_n^{(k+1)\overline{\rho}_n}$ induces the commutativity of the diagram :

$$\begin{tikzcd} H^{*+k(2^n-1)}(BV_n^{k\overline{\rho}_n}) \arrow[d, "\text{Thom}"', "\cong"] & H^{*+(k+1)(2^n-1)}(BV_n^{(k+1)\overline{\rho}_n}) \arrow[l, "j_{n,k}^*"'] \arrow[d, "\cong"', "\text{Thom}"] \\ H^*(BV_n) \arrow[d, "\cdot\omega_n^k"'] & H^*(BV_n) \arrow[d, "\cdot\omega_n^{k+1}"] \\ \mathbb{F}_2[x_1, \dots, x_n] \cdot \{\omega_n^k\} & \mathbb{F}_2[x_1, \dots, x_n] \cdot \{\omega_n^{k+1}\}. \arrow[l, hook'] \end{tikzcd}$$

From this observation, we deduce that the map $j_{n,k}^* : H^*(L(n,k+1)) \longrightarrow H^*(L(n,k))$ is a monomorphism of $\mathscr{A}$-modules.

2.2.3 Basis of $H^*(L(n,k))$ as an $\mathscr{A}$-Module

We want to understand the $\mathscr{A}$-module structure of $H^*(L(n,k))$. We already have a result for $k = 0$.

Theorem 2.2.3 (Mitchell-Priddy Theorem).
A basis as an $\mathscr{A}$-module of $H^*(L(n,0))$ is given by :

$$\Big\{\mathrm{Sq}^I\left(x_1^{-1} \cdots x_n^{-1}\right) \mid I = (i_1, \dots, i_n) : \text{admissible and } i_n \geq 1\Big\}.$$

Proof : See [Mitchell and Priddy, 1983]. □ Let $J = (j_1, \dots, j_n)$ be a n-tuple in $\mathbb{Z}$. As usual, we say that J is *admissible* if $j_i \geq 2j_{i+1}$ for $1 \leq i \leq n$. We say that J is *positive* if $j_i > 0$ for $1 \leq i \leq n$.

Notation 2.2.4.
Let $n \geq 2$. For an admissible n-tuple $J = (j_1, \dots, j_n)$ in $\mathbb{Z}$, we denote an $(n-1)$-tuple $\widetilde{J} = (\widetilde{j}_1, \dots, \widetilde{j}_{n-1})$ by :

$$\widetilde{j}_i = j_i - 2^{n-i}(j_n - 1),$$

for $1 \leq i \leq n-1$. Notice that $\widetilde{J}$ is admissible and positive.

Definition 2.2.5.
For an admissible n-tuple J in $\mathbb{Z}$, define the element θ_J in $\mathbb{F}_2[x_1,\ldots,x_n][\omega_n^{-1}]$ inductively as follows.

- For a 1-tuple (j) :
$$\theta_{(j)} = \omega_1^{j-1} = x_1^{j-1} \in \mathbb{F}_2[x_1][\omega_1^{-1}].$$
- For a n-tuple $J = (j_1,\ldots,j_n)$, with $n \geq 2$:
$$\theta_J = \omega_n^{j_n-1}(\theta_{\widetilde{J}} I_{n-1,n}\overline{T}_n) \in \mathbb{F}_2[x_1,\ldots,x_n][\omega_n^{-1}].$$

We first prove that the elements θ_J are in $H^*(L(n, j_n-1))$. Let us start by showing that θ_J are in $\mathbb{F}_2[x_1,\ldots,x_n]\cdot\{\omega_n^{j_n-1}\}$.

Lemma 2.2.6.
For an admissible n-tuple J in $\mathbb{Z}$, the element θ_J is a homogeneous element in the graded module $\mathbb{F}_2[x_1,\ldots,x_n][\omega_n^{-1}]$, and the degree of θ_J is given by :
$$|\theta_J| = \sum_{i=1}^{n}(j_i-1).$$

Proof : Let us prove by induction. For $n=1$, let $J=(j)$. Then $\theta_{(j)} = x_1^{j-1}$. Therefore $\theta_{(j)}$ is homogeneous and $|\theta_{(j)}| = j-1$. Suppose the lemma holds for any admissible $(n-1)$-tuple J where $n \geq 2$. Then for any admissible n-tuple J, we get that $\theta_{\widetilde{J}}$ is homogeneous with :
$$|\theta_{\widetilde{J}}| = \sum_{i=1}^{n-1}(j_i - 2^{n-i}(j_n-1)-1) = \left(\sum_{i=1}^{n-1} j_i - 1\right) - (2^n-2)(j_n-1).$$

Since ω_n is an homogeneous element of degree 2^n-1, we get that $\theta_J = \omega_n^{j_n-1}(\theta_{\widetilde{J}} I_{n-1,n}\overline{T}_n)$ is homogeneous and :
$$\begin{aligned}
|\theta_J| &= |\omega_n^{j_n-1}| + |\theta_{\widetilde{J}}| \\
&= (2^n-1)(j_n-1) + \left(\sum_{i=1}^{n-1}(j_i-1)\right) - (2^n-2)(j_n-1) \\
&= (j_n-1) + \left(\sum_{i=1}^{n-1}(j_i-1)\right) \\
&= \sum_{i=1}^{n}(j_i-1).
\end{aligned}$$

This proves the induction. $\square$

Notice now that if J is positive, then θ_J is a polynomial, as $j_i - 1 \geq 0$. For $n \geq 2$, for any admissible J, as $\widetilde{J}$ is positive, then $\theta_{\widetilde{J}}$ is a polynomial in $\mathbb{F}_2[x_1,\ldots,x_{n-1}]$ and so θ_J is an element in $\mathbb{F}_2[x_1,\ldots,x_n]\cdot\{\omega_n^{j_n-1}\}$.

Lemma 2.2.7.
For any admissible n-tuple J in $\mathbb{Z}$, the equality $\theta_J e_n = \theta_J$ holds. In particular, θ_J is an element in $\mathbb{F}_2[x_1,\ldots,x_n]\cdot\{\omega_n^{j_n-1}\} \cong H^*(L(n, j_n-1))$.

Proof : Let us prove by induction. The case $n = 1$ is vacuous. Suppose the lemma holds for any $(n-1)$-tuple where $n \geq 2$. Let J be a n-tuple, then :

$$\begin{aligned}
\theta_J e_n &= \Big(\omega_n^{j_n-1}(\theta_{\widetilde{J}} I_{n-1,n}\overline{T}_n)\Big)e_n \\
&= \omega_n^{j_n-1}(\theta_{\widetilde{J}} I_{n-1,n}\overline{T}_n e_n), \text{ as } \omega_n \text{ is } \mathrm{GL}_n(\mathbb{F}_2)\text{-invariant,} \\
&= \omega_n^{j_n-1}(\theta_{\widetilde{J}} e_{n-1} I_{n-1,n}\overline{T}_n), \text{ by Lemma 1.3.7,} \\
&= \omega_n^{j_n-1}(\theta_{\widetilde{J}} I_{n-1,n}\overline{T}_n), \text{ by induction hypothesis,} \\
&= \theta_J.
\end{aligned}$$

This proves the induction. □

Lemma 2.2.8.
For any n-tuple $J = (j_1, \ldots, j_n)$ in $\mathbb{Z}$, and $k \in \mathbb{Z}$, we have :

$$\omega_n^k \theta_J = \theta_{(j_1+2^{n-1}k, j_2+2^{n-2}k, \ldots, j_n+k)}.$$

Proof : Denote $J' = (j_1 + 2^{n-1}k, j_2 + 2^{n-2}k, \ldots, j_n + k)$. Notice that $\widetilde{J} = \widetilde{J'}$, and so we get :

$$\theta_{J'} = \omega_n^{j_n-1+k}(\theta_{\widetilde{J'}} I_{n-1,n}\overline{T}_n) = \omega_n^k \omega_n^{j_n-1}(\theta_{\widetilde{J}} I_{n-1,n}\overline{T}_n) = \omega_n^k \theta_J.$$

This finishes the proof. □

Proposition 2.2.9.
The set of element $\{\theta_J \mid J \text{ admissible and } j_n \geq k+1\}$ is a basis of $H^*(L(n,k))$.

Proof : As $H^*(L(n,k))$ can be regarded as a fractional ideal of $H^(BV_n)$ we get an isomorphism :

$$\begin{array}{rcl}
H^*(L(n,0)) & \longrightarrow & H^*(L(n,k)) \\
a & \longmapsto & \omega_n^k a,
\end{array}$$

which sends θ_J to $\omega_n^k\theta_J$. The previous lemma shows that it is enough to prove the case $k = 0$. Notice now that if J is positive, then $|\theta_J| = |\mathrm{Sq}^J((x_1 \cdots x_n)^{-1})|$, by Lemma 2.2.6. Thus Theorem 2.2.3 implies it is enough to show that the elements θ_J for positive admissible n-tuples J are linearly independant. For this matter, it will be enough to show that for J positive and admissible, we have :

$$\theta_J = x_1^{j_1-1}x_2^{j_2-1}\cdots x_n^{j_n-1} + \text{lower order terms.}$$

Let us prove this above equality by induction on n. As usual, the case $n = 1$ is vacuous. Now suppose that it is true for $(n-1)$-tuple, where $n \geq 2$. Notice first that in general :

$$\omega_n = x_1^{2^{n-1}}x_2^{2^{n-2}}\cdots x_n + \text{lower order terms.}$$

So we obtain by induction hypothesis :

$$\theta_J = \omega_n^{j_n-1}(\theta_{\widetilde{J}} I_{n-1,n}\overline{T}_n) = \Big(x_1^{2^{n-1}(j_n-1)}x_2^{2^{n-2}(j_n-1)}\cdots x_n^{j_n-1}\Big)\Big(x_1^{\widetilde{j}_1-1}\cdots x_{n-1}^{\widetilde{j}_{n-1}-1}\Big) + \text{lower terms.}$$

This finishes the proof. □

Remark 2.2.10.
For J positive and admissible, one can prove that $\theta_J = \mathrm{Sq}^J((x_1\cdots x_n)^{-1})$, thus the basis $\{\theta_J\}$ generalizes the basis of Mitchell and Priddy.

2.2.4 The Long Exact Sequence

In the following argument, let $n \geq 1$. For $k \in \mathbb{Z}$, let us denote σ_k the generator of $H^k(S^k)$, where S^k is the sphere spectrum of dimension k. For simplicity, we denote $M \otimes a$ for $M \otimes \mathbb{F}_2\{a\}$, where $\mathbb{F}_2\{a\}$ is the free $\mathbb{F}_2$-module generated by a. We are interested in the homomorphism :

$$i_{n,k}^* : H^*(L(n,k)) \longrightarrow H^*(\Sigma^k L(n-1,2k)) \cong H^*(L(n-1,2k)) \otimes_{\mathbb{F}_2} \sigma_k.$$

Notation 2.2.11.
For an admissible n-tuple $J = (j_1, \ldots, j_n)$, we denote $\overline{J}$ the $(n-1)$-tuple obtained by :

$$\overline{J} \; = \; (j_1, \ldots, j_{n-1}).$$

As J is admissible, if $j_n \geq k+1$, then $j_{n-1} \geq 2k+2$. Thus if θ_J is in $H^*(L(n,k))$, then $\theta_{\overline{J}}$ is in $H^*(L(n-1,2k+1)) \overset{j_{n-1,2k}^*}{\hookrightarrow} H^*(L(n-1,2k))$.

Proposition 2.2.12.
Let $J = (j_1, \ldots, j_n)$ be an admissible n-tuple in $\mathbb{Z}$ such that $j_n \geq k+1$. Then :

$$i_{n,k}^*(\theta_J) = \begin{cases} 0, & \text{if } j_n \geq k+2, \\ \theta_{\overline{J}} \otimes \sigma_k, & \text{if } j_n = k+1. \end{cases}$$

Proof : Let us first do the case $k = 0$ and then the Thom isomorphism will allow us to generalize it.

Case $k = 0$. Recall that in that case the map $i_{n,0} : L(n-1,0) \to L(n,0)$ corresponds to $B\iota : BV_{n-1} \to BV_n$ where $\iota : V_{n-1} \hookrightarrow V_n$ is the first $(n-1)$-terms inclusion. Thus, for $f(x_1, \ldots, x_n) \in H^*(L(n,0))$, we have :

$$i_{n,0}^*(f) = f(x_1, \ldots, x_{n-1}, 0) \cdot e_n \in H^*(L(n-1,0)).$$

For an admissible J, the element θ_J is divisible by x_n if $j_n > 1$ because it is of the form $\omega_n^{j_n-1}$ times a polynomial. Thus $i_{n-1,0}^*(\theta_J) = 0$. If $j_n = 1$, then we get $\overline{J} = \widetilde{J}$. So $\theta_J = \theta_{\overline{J}} I_{n-1,n} \overline{T}_n$, by definition of θ_J. Now from the definition of the operation $I_{n-1,n}$, we get $i_{n,0}^*(\theta_{\overline{J}} I_{n-1,n}) = \theta_{\overline{J}}$. Let $t \in T_n$ be an element which is not the unit, since $\theta_{\overline{J}}$ is divisible by $x_1 \cdots x_{n-1}$, then we get $i_{n,0}^*(\theta_{\overline{J}} I_{n-1,n} t) = 0$. Thus $i_{n,0}^*(\theta_J) = \theta_{\overline{J}}$ when $j_n = 1$.

General Case $k \in \mathbb{Z}$. From the embedding $H^*(BV_n^{k\overline{\rho}_n}) \hookrightarrow H^*(BV_n)$, the Thom isomorphism :

$$H^*(BV_n) \xrightarrow{\cong} H^{*+k(2^n-1)}(BV_n^{k\overline{\rho}_n}),$$

sends θ_J to $\omega_n^k \theta_J = \theta_{(j_1+2^{n-1}k, j_2+2^{n-2}k, \ldots, j_n+k)}$. Now we have the commutative diagram :

$$\begin{tikzcd}
H^{*+k(2^n-1)}(BV_n^{k\overline{\rho}_n}) \arrow[d, "i_{n,k}^*"'] & H^*(BV_n) \arrow[l, "\cong"', "\text{Thom}"] \arrow[r, "\cong", "\text{Thom}"'] \arrow[d, "(B\iota)^* \smile \sigma_k"] & H^*(BV_n^{0\cdot\overline{\rho}_n}) \arrow[d, "i_{n,0}^* \smile u_{\varepsilon k}"] \\
H^{*+k(2^n-1)+k}\left(\Sigma^k BV_{n-1}^{2k\overline{\rho}_{n-1}}\right) \arrow[d, "\cong"'] & H^{*+k}(\Sigma^k BV_{n-1}) \arrow[l, "\cong"', "\text{Thom}"] \arrow[r, "\cong", "\text{Thom}"'] \arrow[d, "\cong"] & H^{*+k}(\Sigma^k BV_{n-1}^{0\cdot\overline{\rho}_n}) \arrow[d, "\cong"] \\
H^{*+k(2^n-1)}\left(BV_{n-1}^{2k\overline{\rho}_{n-1}}\right) \otimes \sigma_k & H^*(BV_{n-1}) \otimes \sigma_k \arrow[l, "\cong"', "\text{Thom}"] \arrow[r, "\cong", "\text{Thom}"'] & H^*(BV_{n-1}^{0\cdot\overline{\rho}_n}) \otimes \sigma_k,
\end{tikzcd}$$

which implies directly the desired result.

Therefore we have finish the proof. $\square$

THEOREM 2.2.13.
There is a long exact sequence :

$$\begin{tikzcd} 0 \arrow[r] & H^*(L(n,k+1)) \arrow[r, "j^*_{n,k}"] & H^*(L(n,k)) \arrow[r, "i^*_{n,k}"] & H^*(\Sigma^k L(n-1,2k)) \arrow[dlll, rounded corners, "\Sigma^k i^*_{n-1,2k}"] \\ H^*(\Sigma^{3k}L(n-2,4k)) \arrow[r, "\Sigma^{3k} i^*_{n-2,4k}"] & H^*(\Sigma^{7k}L(n-3,8k)) \arrow[r, "\Sigma^{7k} i^*_{n-3,8k}"] & \cdots . \end{tikzcd}$$

PROOF : It is straightfoward from last proposition. We proved already that $j^*_{n,k}$ is injective, so the basis of its image is :

$$\{\theta_J \mid J = (j_1, \ldots, j_n) : \text{ admissible and } j_n \geq k+2\},$$

which is the same as the basis of the kernel of $i^*_{n,k}$ by previous proposition. Its image has basis :

$$\{\theta_J \mid J = (j_1, \ldots, j_{n-1}) : \text{ admissible and } j_{n-1} \geq 2k+2\},$$

but this is the basis of the kernel of $\Sigma^k i^*_{n-1,2k}$. $\square$

2.3. TAKAYASU COFIBRE SEQUENCE

LEMMA 2.3.1.
The map :

$$j_{n,k} : L((n,n-1),k) \longrightarrow L((n,n-1),k+1),$$

is nullhomotopic.

PROOF : First notice that the $\mathrm{GL}_n(\mathbb{F}_2)$-map $j_{n,k} : X^k_{n,n-1} \Longrightarrow X^k_{n,n-1}$ obviously restricts to a $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-map :

$$j_{n,k} : (X^k_{n,n-1})\big|_{\mathbf{C}(n-1)} \Longrightarrow (X^{k+1}_{n,n-1})\big|_{\mathbf{C}(n-1)},$$

in the sense of Definition A.11. This natural transformation extends obviously into a $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-map :

$$j_{n,k} : \mathscr{M}\left((X^k_{n,n-1})\big|_{\mathbf{C}(n-1)}\right) \Longrightarrow \mathscr{M}\left((X^{k+1}_{n,n-1})\big|_{\mathbf{C}(n-1)}\right),$$

using Notation A.14 for the mapping cone diagram. Now recall the homotopy equivalence γ in the proof of Theorem 2.1.7. It fits in the commutative diagram :

$$\begin{tikzcd} L((n,n-1),k) \arrow[r, equal] \arrow[d, "j_{n,k}"] & e_n \cdot \mathrm{hocolim}(X^k_{n,n-1}) \arrow[r, "\gamma", "\simeq"'] \arrow[d, "j_{n,k}"] & e_{n-1} \cdot \mathrm{hocolim}\left(\mathscr{M}\left((X^k_{n,n-1})\big|_{\mathbf{C}(n-1)}\right)\right) \arrow[d, "j_{n,k}"] \\ L((n,n-1),k+1) \arrow[r, equal] & e_n \cdot \mathrm{hocolim}(X^{k+1}_{n,n-1}) \arrow[r, "\gamma", "\simeq"'] & e_{n-1} \cdot \mathrm{hocolim}\left(\mathscr{M}\left((X^{k+1}_{n,n-1})\big|_{\mathbf{C}(n-1)}\right)\right). \end{tikzcd}$$

Therefore, in order to prove that the left vertical map $j_{n,k}$ is nullhomotopic, it suffices to show that the natural transformation :

$$j_{n,k} : (X^k_{n,n-1})\big|_{\mathbf{C}(n-1)} \Longrightarrow (X^{k+1}_{n,n-1})\big|_{\mathbf{C}(n-1)}$$

is nullhomotopic, as a $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-map. But using Theorem 1.4.10, it fits in the commutative diagram of $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-maps :

$$\begin{tikzcd}
(X^k_{n,n-1})\big|_{\mathbf{C}(n-1)} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "j_{n,k}"'] & \Sigma^k X^{2k}_{n-1} \arrow[d, Rightarrow] \\
(X^{k+1}_{n,n-1})\big|_{\mathbf{C}(n-1)} \arrow[r, Rightarrow, "\cong"] & \Sigma^{k+1} X^{2(k+1)}_{n-1}.
\end{tikzcd}$$

The right vertical natural transformation is defined for all W in $\mathbf{C}(n-1)$ as the maps :

$$\mathbb{S}^k \wedge BW^{2k\overline{\rho}_{n-1}|_W} = BW^{2k\overline{\rho}_{n-1}|_W \oplus \varepsilon^k} \longrightarrow BW^{2(k+1)\overline{\rho}_{n-1}|_W \oplus \varepsilon^{k+1}} = \mathbb{S}^{k+1} \wedge BW^{2(k+1)\overline{\rho}_{n-1}|_W},$$

induced by the inclusion $\mathbb{S}^k \hookrightarrow \mathbb{S}^{k+1}$ and the bundle inclusions :

$$2k\overline{\rho}_{n-1}|_W \hookrightarrow (2k+1)\overline{\rho}_{n-1}|_W \hookrightarrow 2k\overline{\rho}_{n-1}|_W.$$

The inclusion $\mathbb{S}^k \hookrightarrow \mathbb{S}^{k+1}$ is nullhomotopic, as the standard map of spaces $S^m \hookrightarrow S^{m+1}$ is nullhomotopic for all $m \geq 0$ since it factors through $S^{m+1} - \{*\}$. Since this inclusion is induced by $\varepsilon^k \hookrightarrow \varepsilon^{k+1}$, the nullhomotopy is compatible with the $\mathrm{GL}_{n-1}(\mathbb{F}_2)$-action. The commutativity of the diagram in $\mathscr{S}^{\mathbf{C}(n-1)}$ proves that $j_{n,k}$ is nullhomotopic. □

Proposition 2.3.2 (A Second Cofibre Sequence).
There is a cofibre sequence :

$$L((n,n-1),k) \longrightarrow L(n,k) \xrightarrow{j_{n,k}} L(n,k+1).$$

Proof : The left map is induced by the inclusion $\mathbf{C}(n,n-1) \hookrightarrow \mathbf{C}(n)$. Let us prove that the sequence induces an exact sequence in cohomology :

$$0 \longrightarrow H^*(L(n,k+1)) \xrightarrow{j^*_{n,k}} H^*(L(n,k)) \longrightarrow H^*(L((n,n-1),k) \longrightarrow 0.$$

Recall the cohomological spectral sequence of $L((n,n-1),k)$:

$$\overline{E}^{p,q}_r = E^{p,q}_r(X^k_{n,n-1}) \cdot e_n \Rightarrow H^{p+q}(L((n,n-1),k)).$$

Its $\overline{E}_2$-page is given, according to Lemma 2.1.4 **(ii)** :

$$\overline{E}^{p,q}_2 = H^p(K^q_\bullet(X^k_{n,n-1})[-1]).$$

Here the cochain complex $K^q_\bullet(X^k_{n,n-1})[-1]$ is simply :

$$0 \longrightarrow H^q(\Sigma^k L(n-1,2k)) \xrightarrow{\Sigma^k i^*_{n-1,2k}} H^q(\Sigma^{3k} L(n-2,4k)) \longrightarrow \cdots,$$

where used repeatdly Theorem 1.4.10 and Lemma 1.4.6. We recognize the long exact sequence of Theorem 2.2.13. Therefore, the $\overline{E}^{p,q}_2$ is trivial for every colunms $p > 0$, and for $p = 0$, we get :

$$\overline{E}^{0,q}_2 = \ker(\Sigma^k i^*_{n-1,2k}) = \mathrm{im}(i^*_{n,k}),$$

by exactness. Thus the convergence of the spectral sequence gives $H^*(L((n,n-1),k) = \mathrm{im}(i^*_{n,k})$ and so :

$$0 \longrightarrow H^*(L(n,k+1)) \xrightarrow{j^*_{n,k}} H^*(L(n,k)) \longrightarrow \underbrace{H^*(L((n,n-1),k)}_{\mathrm{im}(i^*_{n,k})} \longrightarrow 0$$

is exact. Notice that the right map can indeed be regarded as the map induced by the inclusion $\mathbf{C}(n,n-1) \hookrightarrow \mathbf{C}(n)$. Notice now that the diagram :

$$\begin{tikzcd} L((n,n-1),k) \arrow[r] \arrow[d, "j_{n,k}"'] & L(n,k) \arrow[d, "j_{n,k}"] \\ L((n,n-1),k+1) \arrow[r] & L(n,k+1), \end{tikzcd}$$

is commutative, where the horizontal maps are induced by the inclusion $\mathbf{C}(n,n-1) \hookrightarrow \mathbf{C}(n)$. But previous Lemma 2.3.1 says that the left map is nullhomotopic, whence the composite :

$$L((n,n-1),k) \longrightarrow L(n,k) \xrightarrow{j_{n,k}} L(n,k+1),$$

is nullhomotopic. Using the fact that the sectra are 2-completed and of finite type (their mod 2 cohomology are finite-dimensional in each degree) proves that we obtained the sequence is a cofibre sequence. $\square$

Theorem 2.3.3 (Takayasu's Cofibre Sequence).
(i) *There exists a (non-unique) homotopy equivalence :*

$$\mu_{n,k} : \Sigma^k L(n-1,2k+1) \xrightarrow{\simeq} L((n,n-1),k).$$

(ii) *There exist maps :*

$$i'_{n,k} : \Sigma^k L(n-1,2k+1) \longrightarrow L(n,k),$$

such that the diagram commutes :

$$\begin{tikzcd} \Sigma^k L(n-1,2k) \arrow[r, "i_{n,k}"] \arrow[d, "\Sigma^k j_{n-1,2k}"'] & L(n,k) \\ \Sigma^k L(n-1,2k+1), \arrow[ur, "i'_{n,k}"'] & \end{tikzcd}$$

which induce the cofibre sequence :

$$\Sigma^k L(n-1,2k+1) \xrightarrow{i'_{n,k}} L(n,k) \xrightarrow{j_{n,k}} L(n,k+1).$$

Proof : We relate the two cofibre sequences. Consider the second cofibre sequence (Proposition 2.3.2) and substitute $2k$ for k and $n-1$ for n, and consider with k-fold suspension, i.e., consider the map $\Sigma^k j_{n-1,2k}$ instead of $j_{n,k}$. Then we get the commutative diagram :

$$\begin{tikzcd} \cdots \arrow[r] & \Sigma^k L((n-1,n-2),2k) \arrow[r] \arrow[d, equal] & \Sigma^k L(n-1,2k) \arrow[r, "\Sigma^k j_{n-1,2k}"] \arrow[d, equal] & \Sigma^k L(n-1,2k+1) \arrow[r] & \cdots \\ \cdots \arrow[r] & \Sigma^k L((n-1,n-2),2k) \arrow[r] & \Sigma^k L(n-1,2k) \arrow[r, "i_{n,k}"] & L((n,n-1),k) \arrow[r] & \cdots , \end{tikzcd}$$

where the bottom row is the first cofibre sequence (Corollary 2.1.8). Since $\mathscr{S}$ is a triangulated category (or see Lemma 8.31 in [SWITZER, 1975]), there exists a non-unique homotopy equivalence, say :

$$\mu_{n,k} : \Sigma^k L(n-1, 2k+1) \xrightarrow{\simeq} L((n, n-1), k),$$

such that the previous diagram is homotopy commutative. This proves **(i)**. For **(ii)**, define the map :

$$i'_{n,k} : \Sigma^k L(n-1, 2k+1) \longrightarrow L(n,k),$$

as the composite :

$$\Sigma^k L(n-1, 2k+1) \xrightarrow[\simeq]{\mu_{n,k}} L((n, n-1), k) \longrightarrow L((n,n), k) \simeq L(n,k),$$

where the right map is the usual map induced by the inclusion $\mathbf{C}(n, n-1) \hookrightarrow \mathbf{C}(n)$. Now the commutativity of the above diagram proves that $i'_{n,k} \circ \Sigma^k j_{n-1,2k} = i_{n,k}$. Now the second cofibre sequence (Proposition 2.3.2) induces the cofibre sequence :

$$\begin{tikzcd}
\Sigma^k L(n-1, 2k+1) \arrow[r, "i'_{n,k}"] \arrow[d, "\mu_{n,k}"', "\simeq"] & L(n,k) \arrow[r, "j_{n,k}"] & L(n, k+1) \\
L((n, n-1), k), \arrow[ur] & &
\end{tikzcd}$$

which is the desired cofibre sequence. $\square$

For instance, if we pick $n = 1$, we get the cofiber sequence :

$$\mathbb{S}^k \longrightarrow \Sigma^\infty \mathbb{R}P^\infty_k \longrightarrow \Sigma^\infty \mathbb{R}P^\infty_{k+1}.$$

Very recently, the paper [HAI NGUYEN and SCHWARTZ, 2015] proves the Takayasu cofibre sequence for $k \geq 0$ in a different way using the machinery of the category of unstable modules over the Steenrod algebra.

APPENDIX A

Grothendieck Construction

We denote by **Cat** the *category of small categories* whose objects are small categories and whose morphisms are functors between categories.

Definition A.1.
Let $\mathscr{J}$ be any category. Given a functor $F : \mathscr{J} \to \mathbf{Cat}$, the *Grothendieck construction of* F is the category denoted $\mathscr{J} \displaystyle\int F$ defined as follows :

- $\mathrm{Ob}\left(\mathscr{J} \displaystyle\int F\right) = \{(j, v) \mid j \in \mathrm{Ob}\mathscr{J},\, v \in \mathrm{Ob}F(j)\}$,
- $\mathrm{Mor}\left(\mathscr{J} \displaystyle\int F\right) = \left\{(j, v) \xrightarrow{(\varphi, f)} (j', v') \mid \left(j \xrightarrow{\varphi} j'\right) \in \mathrm{Mor}\mathscr{J},\, \left(F(\varphi)(v) \xrightarrow{f} v'\right) \in \mathrm{Mor}F(j')\right\}$,
- and the composition of morphisms $(v, j) \xrightarrow{(\varphi, f)} (v', j') \xrightarrow{(\varphi', j')} (v'', j'')$ is defined as :
$$(\varphi', f') \circ (\varphi, f) = (\varphi' \circ \varphi, f' \circ F(\varphi')(f)).$$

In other words, the Grothendieck construction is a way to index small categories by a category $\mathscr{J}$ via a functor F.

For any object $j \in \mathrm{Ob}\mathscr{J}$, define a functor :

$$\begin{aligned} \iota_j : F(j) &\longrightarrow \mathscr{J} \int F \\ v &\longmapsto (j, v) \\ \left(v \xrightarrow{f} w\right) &\longmapsto (\mathrm{id}_j, f). \end{aligned}$$

For any morphism $\left(j \xrightarrow{\varphi} j'\right) \in \mathrm{Mor}\mathscr{J}$, define a natural transformation :

$$\iota_\varphi : \iota_j \Longrightarrow \iota_{j'} \circ F(\varphi),$$

which is given, for any $v \in \mathrm{Ob}F(j)$, by :

$$(\iota_\varphi)_v : (\varphi, \mathrm{id}_{F(\varphi)(v)}) : (j, v) \longrightarrow (j', F(\varphi)(v)).$$

Lemma A.2.
These above natural transformations are compatible with composition, meaning that for any two morphisms $\varphi : j \to j'$ and $\varphi' : j' \to j''$ in $\mathscr{J}$, we have : $\iota_{\varphi'\circ\varphi} = (\iota_{\varphi'})_{F(\varphi)} \circ \iota_{\varphi}$, where $(\iota_{\varphi'})_{F(\varphi)}$ is the natural transformation obtained by restricting $\iota_{\varphi'}$ along the functor $F(\varphi) : F(j) \to F(j')$.

Proof : This follows directly from the functoriality of $F : \mathscr{J} \to \mathbf{Cat}$. □

$(\mathscr{J} \int F)$-Shaped Diagrams Consider any category $\mathscr{C}$ and a functor $X : \mathscr{J} \int F \to \mathscr{C}$. To any object $j \in \mathrm{Ob}\mathscr{J}$, we define a functor :

$$X_j := X \circ \iota_j : F(j) \longrightarrow \mathscr{C}.$$

To any morphism $\left(j \overset{\varphi}{\to} j'\right) \in \mathrm{Mor}\mathscr{J}$, we define a natural transformation :

$$X_\varphi : X_j \Longrightarrow X_{j'} \circ F(\varphi),$$

via ι_φ : for any $v \in \mathrm{Ob}F(j)$, we have $(X_\varphi)_v := X((\iota_\varphi)_v)$. From Lemma A.2, we get directly the following result.

Lemma A.3.
These above natural transformations are compatible with compositions : $X_{\varphi'\circ\varphi} = (X_{\varphi'})_{F(\varphi)} \circ X_\varphi$.

The next proposition tells us that each functor $X : \mathscr{J} \int F \to \mathscr{C}$ is a collection of functors X_j from a small category to $\mathscr{C}$ which fit naturally.

Proposition A.4.
A functor $X : \mathscr{J} \int F \to \mathscr{C}$ is uniquely determined by functors $\{X_j\}_{j\in\mathrm{Ob}\mathscr{J}}$ and natural transformations $\{X_\varphi\}_{\varphi\in\mathrm{Mor}\mathscr{J}}$ which are compatible with composition in $\mathrm{Mor}\mathscr{J}$.

Proof : Given $\{X_j\}_{j\in\mathrm{Ob}\mathscr{J}}$ and $\{X_\varphi\}_{\varphi\in\mathrm{Mor}\mathscr{J}}$ as above, we recover X by defining :

$$\begin{array}{rcl} X : \mathscr{J} \displaystyle\int F & \longrightarrow & \mathscr{C} \\ (j,v) & \longmapsto & X_j(v), \\ \left((j,v) \overset{(\varphi,f)}{\to} (j',v')\right) & \longmapsto & X_{j'}(f) \circ (X_\varphi)_v, \end{array}$$

as $X(\varphi, f) = X(\mathrm{id}_{j'}, f) \circ X(\varphi, \mathrm{id}_{F(\varphi)(v)})$. Let us check that this definition of X defines indeed a functor. It is easy to see that $X(\mathrm{id}_j, \mathrm{id}_v) = \mathrm{id}_{X(j,v)}$ for any object $(j,v) \in \mathrm{Ob}(\mathscr{J} \int F)$. Now, given two morphisms $(j,v) \overset{(\varphi,f)}{\to} (j',v') \overset{(\varphi',f')}{\to} (j'',v'')$ in $\mathscr{J} \int F$, we have :

$$\begin{array}{rcl} X((\varphi',f') \circ (\varphi,f)) & = & X(\varphi' \circ \varphi, f' \circ F(\varphi')(f)) \\ & = & X_{j''}(f' \circ F(\varphi')(f)) \circ (X_{\varphi'\circ\varphi})_v \\ & = & X_{j''}(f') \circ X_{j''}(F(\varphi')(f)) \circ (X_{\varphi'\circ\varphi})_v, \text{ by functoriality of } X_{j''}, \\ & = & X_{j''}(f') \circ X_{j''}(F(\varphi')(f)) \circ (X_{\varphi'})_{F(\varphi)(v)} \circ (X_\varphi)_v, \text{ by compatibility,} \\ & = & X_{j''}(f') \circ (X_{\varphi'})_{v'} \circ X_{j'}(f) \circ (X_\varphi)_v, \text{ by naturality of } X_{\varphi'}, \\ & = & X(\varphi',f') \circ X(\varphi,f). \end{array}$$

Thus X is indeed a functor. □

We are interested in computing the homotopy colimit of a ($\mathscr{J} \int F$)-diagram. In what follows, the result are also true for colimits. We will work with the category $\mathscr{S}$, but the results remain valid for any category where homotopy colimits make sense.

Proposition A.5.
Let $X : \mathscr{J} \int F :\to \mathscr{S}$ be a functor. Then there is a functor :

$$\begin{array}{rcl} \mathscr{J} & \longrightarrow & \mathscr{S} \\ j & \longmapsto & \mathrm{hocolim}_{F(j)} X_j \\ \left(j \xrightarrow{\varphi} j'\right) & \longmapsto & \left(\mathrm{hocolim}_{F(j)} X_j \to \mathrm{hocolim}_{F(j)} \left(X'_j \circ F(\varphi)\right) \to \mathrm{hocolim}_{F(j')} X_{j'}\right). \end{array}$$

Proof : The morphism $\mathrm{hocolim}_{F(j)} X_j \to \mathrm{hocolim}_{F(j)} \left(X'_j \circ F(\varphi)\right)$ is induced by the natural transformation X_φ. The second morphism stems from the following Lemma A.6. Identity and compositions will be simple to verify. □

Lemma A.6.
Suppose $\mathscr{I} \xrightarrow{F} \mathscr{K}$ is a functor between small categories. Let $X : \mathscr{K} \to \mathscr{S}$ be a functor. Then there is a natural morphism :

$$\mathrm{hocolim}_{\mathscr{I}}(X \circ F) \longrightarrow \mathrm{hocolim}_{\mathscr{K}} X.$$

Sketch of the Proof : This is a generalization of the morphism :

$$\mathrm{colim}_{\mathscr{I}}(X \circ F) \to \mathrm{colim}_{\mathscr{K}} X.$$

Let us give a concrete definition. As in [Bousfield and Kan, 1972], for simplicity replace $\mathscr{S}$ by the category of based spaces. For $i \in \mathrm{Ob}\mathscr{I}$, denote $\mathscr{I}/i$ the over category as defined in Chapter XI in [Bousfield and Kan, 1972], and $B(\mathscr{I}/i)$ its underlying space (geometric realization). Recall we have :

$$\mathrm{hocolim}_{\mathscr{K}} X := \mathrm{coequ}\left(\coprod_{\left(k \xrightarrow{\gamma} k'\right) \in \mathrm{Mor}\mathscr{K}} B(\mathscr{K}/k')^{\mathrm{op}} \ltimes X(k) \rightrightarrows \coprod_{k \in \mathrm{Ob}\mathscr{K}} B(\mathscr{K}/k)^{\mathrm{op}} \ltimes X(k) \right),$$

where the morphisms are induced respectively by $X(\gamma)$ and $B(\mathscr{K}/\gamma)$. In other words, an element in $\mathrm{hocolim}_{\mathscr{K}} X$ can be regarded as an element :

$$[k, (k \to k_0 \to \cdots \to k_s) \wedge x],$$

in $\coprod_{k \in \mathrm{Ob}\mathscr{K}} B(\mathscr{K}/k)^{\mathrm{op}} \ltimes X(k)$, under some relations. Then the morphism :

$$\mathrm{hocolim}_{\mathscr{I}}(X \circ F) \to \mathrm{hocolim}_{\mathscr{K}} X,$$

is given by :

$$[i, (i \to i_0 \to \cdots \to i_s) \wedge x] \to [F(i), (F(i) \to F(i_0) \to \cdots \to F(i_s)) \wedge x],$$

where $x \in X(F(i))$, and $i \in \mathrm{Ob}\mathscr{I}$. □

If $\mathscr{J}$ is also a small category, one can take also the homotopy colimit of the functor $\mathscr{J} \to \mathscr{S}$ defined in Proposition A.5. It turns out that this gives a way to compute the homotopy colimit of $X : \mathscr{J} \int F \to \mathscr{S}$, as we see in the following theorem.

Theorem A.7.
Let $\mathscr{J}$ be a small category, and $F : \mathscr{J} \to \mathbf{Cat}$ a functor. Then for any functor $X : \mathscr{J} \int F \to \mathscr{S}$, we have :

$$\mathrm{hocolim}_{\mathscr{J}} \left(\mathrm{hocolim}_{F(j)} X_j \right) \simeq \mathrm{hocolim}_{\mathscr{J} \int F} X.$$

Sketch of the Proof[1] : We make use of the left Kan extension. Recall that given a functor $G : \mathscr{I} \to \mathscr{K}$ and a category $\mathscr{D}$, the left Kan extension of a functor $X : \mathscr{I} \to \mathscr{D}$ is the functor $LX : \mathscr{K} \to \mathscr{D}$, where L is the left adjoint functor in :

$$L : \mathrm{Fun}(\mathscr{I}, \mathscr{D}) \leftrightarrows \mathrm{Fun}(\mathscr{K}, \mathscr{D}) : G^*.$$

A concrete construction of LX can be given as follows. For any k in $\mathrm{Ob}\mathscr{K}$, we define :

$$(LX)(k) := \mathrm{colim} \left(G/k \to \mathscr{I} \xrightarrow{X} \mathscr{D} \right),$$

where G/k is the category with objects given by an object i in $\mathscr{I}$ and a morphism $\alpha_i : G(i) \to k$ in $\mathscr{K}$. The morphisms in G/k are defined from morphisms in $\mathscr{I}$. The functor $G/k \to \mathscr{I}$ is the obvious projection. Now, each morphism $\varphi : k \to k'$ in $\mathrm{Ob}\mathscr{K}$ defines an obvious functor $\varphi^* : G/k \to G/k'$ such that it commutes with the projection :

$$\begin{tikzcd} G/k \arrow[rr, "\varphi_*"] \arrow[dr] & & G/k' \arrow[dl] \\ & \mathscr{I}, & \end{tikzcd}$$

whence this defines the morphism $(LX)(k) \to (LX)(k')$. Notice that if $\mathscr{K} = *$, then we get $LX = \mathrm{colim}_{\mathscr{I}} X$. One can generalize this construction to a functor L^h where one replaces colimits by homotopy colimits.
Returning to our problem, we make use of the property that left Kan extensions commute, so that if we define the functor :

$$P : \mathscr{J} \int F \to \mathscr{J},$$

to be the obvious projection, and denote $L^h_P X : \mathscr{J} \to \mathscr{S}$ the left Kan extension :

$$\begin{tikzcd} \mathscr{J} \int F \arrow[r, "X"] \arrow[d, "P"'] & \mathscr{S} \\ \mathscr{J}, \arrow[ur, dashed, "L^h_P X"'] & \end{tikzcd}$$

then the commutativity of the diagram :

$$\begin{tikzcd} & \mathscr{J} \int F \arrow[r, "X"] \arrow[dl, "P"'] \arrow[d] & \mathscr{S} \\ \mathscr{J} \arrow[r] & *, \arrow[ur, dashed, "L^h X"'] & \end{tikzcd}$$

gives the equivalence :

$$L^h X = \mathrm{hocolim}_{\mathscr{J} \int F} X \simeq \mathrm{hocolim}_{\mathscr{J}} (L^h_P X).$$

[1] A full detailed proof can be found in [Dotto and Moi, 2014].

It remains to prove that for each object j in $\mathscr{J}$, we have the equivalence :

$$(L_P^h X)(j) \simeq \mathrm{hocolim}_{F(j)} X_j.$$

For this, recall that $(L_P^h X)(j) = \mathrm{hocolim}\left(P/j \to \mathscr{J} \int F \xrightarrow{X} \mathscr{S}\right)$. We define a functor :

$$\Phi : P/j \to F(j),$$

as follows. We have :

$$\mathrm{Ob}(P/j) = \left\{ \left((j',v), j' \xrightarrow{\alpha} j\right) \mid j' \in \mathrm{Ob}\mathscr{J}, v \in \mathrm{Ob}(F(j')), \alpha \in \mathrm{Mor}\mathscr{J} \right\},$$

so we define $\Phi(j',v,\alpha) = F(\alpha)(v)$, and define the morphisms in an obvious way, so that we have the following commutative diagram of functors :

$$\begin{tikzcd} P/j \arrow[r] \arrow[d, "\Phi"'] & \mathscr{J} \int F \arrow[d, "X"] \\ F(j) \arrow[ur, "\iota_j"] \arrow[r, "X_j"'] & \mathscr{S}. \end{tikzcd}$$

This defines a morphism $\Phi^* : (L_P^h X)(j) \longrightarrow \mathrm{hocolim}_{F(j)} X_j$ as in Lemma A.6. Now the cofinality Theorem (see in [Bousfield and Kan, 1972] in Chapter XI) states that the map Φ^* is an equivalence if the underlying space $B(\Phi/v)$ is contractible, for any object v in $F(j)$. This happens when the category Φ/v admits a terminal object. Since :

$$\mathrm{Ob}(\Phi/v) = \left\{ \left(\left((j',v'), j' \xrightarrow{\alpha} j\right), v' \xrightarrow{\beta} v \right) \mid (j',v',\alpha) \in \mathrm{Ob}(P/j), \beta \in \mathrm{Mor}(F(j)) \right\},$$

it can be verified that the element $(j, v, \mathrm{id}_j, \mathrm{id}_v)$ is the desired terminal object. □

Monoid Diagrams We apply the Grothendieck construction for $\mathscr{J}$ a monoid acting on a poset $(F, \leq)$. This means that $\mathscr{J}$ is some category with one object, say $*$. If we require the morphism in $\mathscr{J}$ to be all invertible, then $\mathscr{J}$ is a group in the usual sense. Recall also that a poset can be regarded as a small category with objects its elements such that there is a morphism between a and b in F if and only if $a \leq b$. Since $\mathscr{J}$ acts on F, this defines a functor, denoted by abuse of notation also F, as follow :

$$\begin{array}{rcl} F : \mathscr{J} & \longrightarrow & \mathbf{Cat} \\ * & \longmapsto & F \\ \left(* \xrightarrow{g} *\right) & \longmapsto & \begin{pmatrix} F \to F \\ v \mapsto gv \end{pmatrix}. \end{array}$$

In this case, we get :

$$\mathrm{Ob}\left(\mathscr{J} \int F\right) = \mathrm{Ob}(F),$$

$$\mathrm{Mor}\left(\mathscr{J} \int F\right) = \left\{ v \xrightarrow{g} w \mid v, w \in F,\ g \in \mathrm{Mor}\mathscr{J},\ va \leq w \right\}.$$

We denote $1_{\mathscr{J}} = \mathrm{id}_* \in \mathrm{Mor}\mathscr{J}$. Subsequently, we will write simply $g \in \mathscr{J}$ instead of $g \in \mathrm{Mor}\mathscr{J}$.

Definition A.8 ($\mathscr{J}$-Diagram).
Let $\mathscr{J}$ and F be as above. For any category $\mathscr{C}$, a functor $X : \mathscr{J} \int F \to \mathscr{C}$ is called a *monoid diagram in* $\mathscr{C}$. If $\mathscr{J}$ is a group, then X is called a *group diagram in* $\mathscr{C}$. If the structure of $\mathscr{J}$ is understood, we will say that X is a $\mathscr{J}$*-diagram.*

Proposition A.9.
A monoid diagram $X : \mathscr{J} \int F \to \mathscr{C}$ is a functor $X : F \to \mathscr{C}$ together with a collection of natural transformations $\{g_* : X \Rightarrow X \circ F(g)\}_{g\in\mathscr{J}}$, such that :

(i) $(1_{\mathscr{J}})_* = \mathrm{id}_X$,

(ii) $(gh)_* = (g_*)_{F(h)} \circ h_*$, *for all* $g, h \in \mathscr{J}$,

where $(g_*)_{F(h)}$ is the natural transformation obtained by restricting g_* along the functor $F(h)$.

Proof : This is Proposition A.4 for $\mathscr{J}$ a monoid acting on the poset F. The functor $X_* = X \circ \iota_*$ is the underlying functor $F \to \mathscr{C}$. The natural transformations $\{X_g\}$ are the natural transformations $\{g_*\}$, and the compatibiliy with composition is the statement **(ii)**. The statement **(i)** follows from the functoriality of ι_*. □

Corollary A.10.
For two $\mathscr{J}$-diagrams $X, Y : \mathscr{J} \int F \to \mathscr{C}$, a natural transformation $\eta : X \Longrightarrow Y$ is equivalent to a natural transformation between their underlying functors such that the following diagram commutes in $\mathscr{C}^F$ for all $g \in \mathscr{J}$:

$$\begin{tikzcd} X \arrow[rr, Rightarrow, "\eta"] \arrow[d, Rightarrow, "g_*"'] & & Y \arrow[d, Rightarrow, "g_*"] \\ X \circ F(g) \arrow[rr, Rightarrow, "(\eta)_{F(g)}"] & & Y \circ F(g). \end{tikzcd}$$

Definition A.11 ($\mathscr{J}$-Map).
Such a natural transformation η is called a $\mathscr{J}$*-map* of F-shaped diagrams.

Proposition A.12.
Let $X : \mathscr{J} \int F \to \mathscr{S}$ be a group diagram. The homotopy colimit of the underlying functor $\mathrm{hocolim}_F X$ *is endowed with a $\mathscr{J}$-action.*

Proof : This is Proposition A.5. □

Mapping Cone Diagrams Let $\mathscr{J} = \{b \leftarrow a \to c\}$. Let $\mathbf{C}$ be a small category, and let $\mathbf{C}'$ be a full subcategory of $\mathbf{C}$. Recall that the terminal object in $\mathbf{Cat}$ is the category $*$ with one object and one morphism. Define a functor F as follows :

$$\begin{aligned} F : \mathscr{J} &\longrightarrow \mathbf{Cat} \\ a &\longmapsto \mathbf{C}' \\ b &\longmapsto * \\ c &\longmapsto \mathbf{C} \\ (a \mapsto b) &\longmapsto \left(\mathbf{C}' \xrightarrow{!} *\right) \\ (a \mapsto c) &\longmapsto (\mathbf{C}' \hookrightarrow \mathbf{C}). \end{aligned}$$

The Grothendieck construction in this case is given by :

$$\mathrm{Ob}\left(\mathscr{J}\int F\right) = \mathrm{Ob}\mathbf{C} \sqcup *,$$

$$\mathrm{Mor}\left(\mathscr{J}\int F\right) = \mathrm{Mor}\mathbf{C} \cup \{v' \to * \mid v' \in \mathrm{Ob}\mathbf{C}'\}.$$

Definition A.13.
Let $\mathscr{J}$, $\mathbf{C}$, $\mathbf{C}'$ and F be as above. For any category $\mathscr{C}$, we say that a functor $X : \mathscr{J}\int F \to \mathscr{C}$ is a *mapping cone diagram in* $\mathscr{C}$ *of shape* $\mathbf{C}$ *over* $\mathbf{C}'$.

Notation A.14.
We refer to the category $\mathscr{J}\displaystyle\int F$ by $\mathbf{Cone}(\mathbf{C}', \mathbf{C})$.

Definition A.15.
Given a functor $X : \mathbf{C} \to \mathscr{C}$, where $\mathscr{C}$ admits a terminal object, we define its *mapping cone diagram* $\mathscr{M}X : \mathbf{Cone}(\mathbf{C}', \mathbf{C}) \to \mathscr{C}$ as follows :

$$\begin{array}{rcl}
\mathscr{M}X : \mathbf{Cone}(\mathbf{C}', \mathbf{C}) & \longrightarrow & \mathscr{C} \\
v & \longmapsto & X(v) \\
* & \longmapsto & * \\
\big(v \to w\big) & \longmapsto & \big(X(v) \to X(w)\big) \\
\big(v' \to *\big) & \longmapsto & \big(X(v') \xrightarrow{!} *\big).
\end{array}$$

Proposition A.16.
Let X be as previous definition. Let $\mathbf{C}$ be a poset such that a monoid $\mathscr{I}$ acts on $\mathbf{C}$. If X is a $\mathscr{I}$-diagram, then so is $\mathscr{M}X$.

Proof : The natural transformations $\{g_* : X \Longrightarrow X \circ \mathbf{C}(g)\}_{g \in \mathscr{I}}$ extends naturally into natural transformations $\{g_* : \mathscr{M}X \Longrightarrow \mathscr{M}X \circ \mathbf{C}(g)\}_{g \in \mathscr{I}}$ □

Theorem A.17.
Let X be as previous definition. Let $\mathscr{C} = \mathscr{S}$ be the stable homotopy category. Then the homotopy colimit hocolim $\mathscr{M}X$ *is the mapping cone of the map* hocolim $X_{|\mathbf{C}'} \to$ hocolim X*, which is induced by the inclusion $\mathbf{C}' \hookrightarrow \mathbf{C}$.*

Proof : This is Theorem A.7 which states that the diagram :

$$\begin{tikzcd}
\mathrm{hocolim}\ X_{|\mathbf{C}'} \arrow[r] \arrow[d] & \mathrm{hocolim}\ X \arrow[d] \\
* \arrow[r] & \mathrm{hocolim}\ \mathscr{M}X,
\end{tikzcd}$$

is a homotopy pushout. □

APPENDIX B

2-Completion via the Bousfield Localization

We gather in this Appendix all the results needed in this paper. We shall not make any proofs as they are not enlightening in our case. We refer the reader to [Bousfield, 1979] for any details. We present the 2-completion via the Bousfield localisation with the Moore spectrum $S\mathbb{Z}/2$. We present the results in a more general way for p-completion, for any p a prime number.

Definition B.1.
Let E_* be a generalized homology theory. A spectrum X is called E_**-acyclic* if $E_*(X) = 0$, i.e., we have the homotopy equivalence $E \wedge X \simeq *$. A spectrum X is called E_**-local* if $[A, X]_* = 0$, for every E_*-acyclic spectrum A. A map of spectra $f : A \to B$ in $\mathscr{S}$ is called an E_**-equivalence* if $E_*(f) : E_*(A) \overset{\cong}{\to} E_*(B)$ is an isomorphism.

Lemma B.2.
A spectrum X is E_*-local if and only if every E_*-equivalence $A \to B$ induces an isomorphism of abelian groups : $[B, X]_* \cong [A, X]_*$.

Proof : An E_*-equivalence is a map with an E_*-acyclic cofiber. □

Definition B.3.
An E_**-localization* is a functor $L_E : \mathscr{S} \to \mathscr{S}$ together with a natural equivalence $1_{\mathscr{S}} \Rightarrow L_E$, with the property that for any spectrum X, the natural map $X \to L_E X$ is an E_*-equivalence, such that $L_E X$ is E_*-local.

Theorem B.4.
If $f : X \to Y$ is an E_*-equivalence of E_*-local spectra in $\mathscr{S}$, then f is a homotopy equivalence.

Proof : Since X and Y are E_*-local, we obtain that $[X, Y]_* \cong [Y, Y]_*$ and $[Y, X]_* \cong [X, X]_*$. The isomorphisms provide an isomorphism $[\mathbb{S}, X]_* \xrightarrow{\cong} [\mathbb{S}, Y]_*$. □

The localization L_E exists for any spectrum E (see Theorem 1.1 in [Bousfield, 1979]). The functor L_E is unique in the following way. If $f : A \to B$ is any E_*-equivalence in $\mathscr{S}$ with B E_*-local, then f is canonically equivalent to the functor L_E. Moreover, for any spectrum X, we have $L_E(L_E(X)) \simeq L_E(X)$.

The Moore Spectrum Let G be an abelian group. Then there is a free resolution of G, i.e. free abelian groups F_0 and F_1 that fit in the exact sequence of abelian groups :

$$0 \hookrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0$$

Let us consider $\bigvee_{\alpha\in A} S^0$ and $\bigvee_{\beta\in B} S^0$ such that :

$$\pi_0\left(\bigvee_{\alpha\in A}\mathbb{S}\right) = F_1,\ \pi_0\left(\bigvee_{\beta\in B}\mathbb{S}\right) = F_0$$

Then there exists a map $f : \bigvee_{\alpha\in A}\mathbb{S} \to \bigvee_{\beta\in B}\mathbb{S}$ inducing the inclusion $F_1 \hookrightarrow F_0$. Define the *Moore spectrum* SG of G to be the cofibre of f :

$$\bigvee_{\alpha\in A}\mathbb{S} \xrightarrow{f} \bigvee_{\beta\in B}\mathbb{S} \longrightarrow SG.$$

Notice that the above construction is completely similar to the Moore spaces. In particular, we get :

$$\pi_r(SG) = 0, \text{ for } r < 0$$
$$\pi_0(SG) = (H\mathbb{Z})_0(SG) = G,$$
$$(H\mathbb{Z})_r(SG) = 0, \text{for } r > 0.$$

Definition B.5 (p-Completion).
Let p be a prime number in $\mathbb{Z}$. The *p-completion* of a spectrum X is its $(S\mathbb{Z}/p)_*$-localization, i.e., $L_p := L_{S\mathbb{Z}/p}(X)$.

Proposition B.6.
For any spectrum X, its p-completion is given by :

$$L_p := L_{S\mathbb{Z}/p}(X) = \operatorname{holim}\left(S\mathbb{Z}/p \wedge X \xleftarrow{q_1\wedge \mathrm{id}_X} S\mathbb{Z}/p^2 \wedge X \xleftarrow{q_2\wedge \mathrm{id}_X} S\mathbb{Z}/p^3 \wedge X \longleftarrow \cdots \right),$$

where $q_i \wedge \mathrm{id}_X : S\mathbb{Z}/p^{i+1} \wedge X \longrightarrow S\mathbb{Z}/p^i \wedge X$ is given by the quotient homomorphism $\mathbb{Z}/p^{i+1} \longrightarrow \mathbb{Z}/p^i$. Moreover, if the groups $\pi_*(X)$ are finitely generated abelian groups, then $\pi_*(L_{S\mathbb{Z}/p}(X)) \cong \widehat{\mathbb{Z}}_p \otimes \pi_*(X)$.

Proof : See Proposition 2.5 in [Bousfield, 1979]. □

The p-completion $L_p = L_{S\mathbb{Z}/p}$ can be regarded as a functor :

$$L_p : \mathscr{S} \longrightarrow \mathscr{S}_p,$$

where $\mathscr{S}_p$ is the full subcategory of $\mathscr{S}$ consisting of $(S\mathbb{Z}/p)_*$-localized spectra. We can work just as well with the category $\mathscr{S}_p$ as we can in $\mathscr{S}$, meaning for instance that the functor L_p sends cofibre sequences to cofibre sequences.

Theorem B.7.
Let E and X be connective spectra. Suppose $\pi_0(E) = G$. Then $L_E(X) \simeq L_{SG}(X)$.

Proof : See Theorem 3.1 in [Bousfield, 1979]. □

If one take $E = H\mathbb{Z}/p$, then $L_{H\mathbb{Z}/p}(X) \simeq L_{S\mathbb{Z}/p}(X)$, we obtain the following result.

Corollary B.8.
An $(H\mathbb{Z}/p)_*$-equivalence $f : X \to Y$ between connective p-completed spectra is a homotopy equivalence. If the spectra X and Y are moreover locally finite type (i.e. with finitely generated homology $H\mathbb{Z}/p$), then if a map $f : X \to Y$ induces an isomorphism on cohomology $(H\mathbb{Z}/p)^*$, then f is also a $(H\mathbb{Z}/p)_*$-equivalence, and thus f is a homotopy equivalence.

Proof : When X and Y are locally of finite type, then if f induces an isomorphism on cohomology, then its cofibre has trivial cohomology. But a usual universal coefficient argument implies that its homology is also trivial, as the homology of X and Y are finitely generated. Thus f induces also an isomorphism on homology. $\square$

We finish this appendix with the following remark. The E_*-localization functor need not to preserve smash products, but there is a canonical map $L_E(X) \wedge L_E(Y) \to L_E(X \wedge Y)$. If G is a commutative ring spectrum (see page 246 in [Adams, 1974] for a definition), then so is $L_E(G)$. In particular the localized sphere spectrum $L_E(\mathbb{S})$ is a commutative ring spectrum and each E_*-local spectrum is canonically a module spectrum over $L_E(\mathbb{S})$. Moreover, for each spectrum X in $\mathscr{S}$, there is a canonical map $L_E(\mathbb{S}) \wedge L_E(X) \longrightarrow L_E(X)$. The Kunneth map then gives a structure of $\pi_0(L_E(\mathbb{S}))$-module for the homotopy class $[L_E(X), L_E(X)]$. In particular, by Proposition B.6, we have $\pi_0(L_p(\mathbb{S})) = \widehat{\mathbb{Z}_p}$, so we obtain the following result.

Proposition B.9.
For any p-complete spectrum X, the abelian group of all self maps $[X, X]$ is a $\widehat{\mathbb{Z}_p}$-module.

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