NONSTANDARD ANALYSIS AND THE LOCAL HILBERT'S FIFTH PROBLEM

MIHAIL HURMUZOV

Abstract. We follow Isaac Goldbring's solution, which employs nonstandard analysis, to the local Hilbert's fifth problem - that every locally euclidean local group is locally isomorphic to a Lie group. The discussion develops and uses the nonstandard machinery in a similar way to how it is developed and used in Joram Hirschfeld's treatment of the global version of Hilbert's fifth problem.

1. Introduction

David Hilbert poses his problem of giving a classification of Lie groups in publications following his 1900 speech at the Paris conference of the International Congress of Mathematicians. His original formulation has consequently been seen as somewhat vague. One interpretation is the question whether a locally euclidean topological group is isomorphic to a Lie group (call this H5). This question was answered in the affirmative in 1952 by Andrew Gleason, Deane Montgomery and Leo Zippin in [1] and [2]. However, it was still held at a symposium on Hilbert's problems in 1976, that the proof of H5 is too complicated and technical to be clearly presented to a wide audience. It was also considered a challenge to find a simpler proof.

A significant simplification of the proof was achieved with the use of nonstandard analysis. This new methodology was introduced by Abraham Robinson in the early 60s [3] and was initially developed through a model theoretic approach. Others, such as Wilhelmus Luxemburg, showed that the same results could be achieved using ultrafilters, which made Robinson's work more accessible to mathematicians who lacked training in formal logic. (It is the ultrafilter construction that will be presented here in 2.1.) Robinson's nonstandard setting made possible the presentation of the solution to H5 to be fitted in a 22 page paper - that by Joram Hirschfeld in 1990 [4].

Some have found another formulation of Hilbert's fifth problem more natural - since the original conception of Lie groups by Sophus Lie was that they were not groups but 'local groups' where multiplication and inversion can be carried out only near the identity, it makes sense to ask whether a locally euclidean local group is locally isomorphic to a Lie group (call this local version LH5). This is the question tackled by Isaac Goldbring [5], the solution to which will be presented here. The approach resembles that of dealing with H5, particularly that of Hirschfeld, with the modification demanded by the move from a global to a local setting.

In section 2 we introduce nonstandard analysis and local groups, so as to ensure that a relatively unfamiliar with these concepts person is well-equipped to read the following discussion. Section 3 describes different ways infinitesimals can grow out of neighbourhoods of the identity and how these can be used to build local one-parameter subgroups. In 4 we explore the relevant version of the exponential map and some of its properties. Sections 5 and 6 are the technical heart of the solution. 7 gives the first substantial result - that locally compact NSS groups are locally euclidean. Section 8 shows further that NSS local groups are locally isomorphic to Lie groups. Finally, in 9 we show that locally euclidean local groups are NSS, which completes the proof of LH5.

2. Preliminaries

We assume familiarity with Lie groups and Lie algebras. Introductions to the nonstandard component of the paper and to local groups are included in this chapter.

2.1. **The nonstandard setting.**

2.1.1. *General approach and construction.* We start with a mathematical universe V containing all relevant mathematical objects - $\mathbb{N}$, $\mathbb{R}$, various groups G, various topological spaces X, cartesian products of the above, their powersets, etc.

We then want to extend to a nonstandard universe V^* such that:

- To each basic set A we have its nonstandard extension $A^* \supseteq A$;
- To each function $f : A \to B$ between basic sets, we have the nonstandard extension $f^* : A^* \to B^*$, such that $f^*|_A = f$.

Our main tool for communication between V and V^* is to be the transfer principle:

Theorem. *(Transfer Principle) If S is a bounded first-order statement about objects in V, then it is true in V if and only if it is true in V^*.*

We can use Model Theory to show that there is an elementary extension V^* of V obeying the above, but one (arguably) more intuitive approach is to employ ultrapowers.

The idea is the following. Given a set X and a set I, we want to define a notion of 'closeness' on X^I, and then take the quotient over that relation. We will say that $f \in X^I$ **agrees with** $g \in X^I$ **on a set** $A \subseteq I$ if $f(i) = g(i)$ for all $i \in A$. Two elements of X^I will be considered 'close' if they agree on a 'big enough' subset of I. Several intuitive consequences of that idea become immediate:

(1) If a and b do not agree on any subset of I they shouldn't be considered 'close'.
(2) If a is 'close' to b, because they agree on a set $\rho \subseteq I$, then if c agrees with b on a larger set $\rho \subseteq \tau \subseteq I$, then c should also be considered 'close' to b.
(3) If a is 'close' to b and b is 'close' to c, then a should be 'close' to c (after all we want 'closeness' to be an equivalence relation so we would be able to quotient it out later). This means that if a and b agree on $\rho \subseteq I$ and b and c agree on $\tau \subseteq I$, then it is sufficient that a and c agree on $\rho \cap \tau \subseteq I$ for them to be considered 'close'.
(4) For every a and b in X^I they should either be considered 'close' or not.

These considerations now make the following definition a lot less mysterious.

Definition 2.1. *Given a set I, a* ***filter*** *on I is a set $\mathcal{F} \subseteq \mathcal{P}(I)$ such that:*

(1) $\emptyset \notin \mathcal{F}$.
(2) $A \in \mathcal{F}, A \subseteq B \implies B \in \mathcal{F}$.
(3) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$.

An ***ultrafilter*** *is a filter that cannot be enlarged, i.e. a filter that satisfies the further condition:*

(4) $\forall A \subseteq \mathcal{P}(I)$ *either* $A \in \mathcal{F}$ *or* $I \backslash A \in \mathcal{F}$.

Now we can make precise the ultrapower construction.

Definition 2.2. *Given sets X and I and an ultrafilter $\mathcal{F}$ of I, the* ***ultrapower*** *of X with respect to $\mathcal{F}$ is $X^{\mathcal{F}} = X^I / \sim_{\mathcal{F}}$, where for $f, g \in X^I$*

$$f \sim_{\mathcal{F}} g \text{ if and only if } \{i \in I | f(i) = g(i)\} \in \mathcal{F}.$$

We can now identify X as a subset of $X^{\mathcal{F}}$ by identifying X with

$$\{f \in X^I | f(i) = f(j) \forall i, j \in I\},$$

where $x \in X$ is identified with $f_x \in X^I$ if $f_x(i) = x$ for all $i \in I$. And since with this identification $f_x \sim_{\mathcal{F}} f_y$ if and only if $x = y$, then we can consider $X \subseteq X^{\mathcal{F}}$. Moreover, a function $f : X \to Y$ naturally extends to a function $f : X^{\mathcal{F}} \to Y^{\mathcal{F}}$ by $f([(i)]_{\mathcal{F}}) := [f(i)]_{\mathcal{F}}$. Therefore $X^{\mathcal{F}}$serves as a nonstandard extension of X in the sense defined above.

The transfer principle in this setting is known as **Łoś's Theorem** and is due to Jerzy Łoś. It states that any first-order formula is true in the ultraproduct if and only if the set of indices i, such that the formula is true in the copy of X in the ultraproduct corresponding to the index i, is in $\mathcal{F}$.

Definition 2.3. *A set $A \subseteq X^*$ is said to be* ***internal*** *if there exists a set $B \subseteq X$ such that $A = B^*$. Otherwise A is said to be* ***external****.*

The Internal Definition Principle states that A^* for any A in X is internal, and any set defined from internal parameters in a first-order way is internal.

Definition 2.4. *Let κ be an infinite cardinal. We say that X^* is* ***κ-saturated*** *if whenever $\{\mathcal{O}_i | i < \kappa\}$ is a family of internal sets such that any intersection of a finite number of them is nonempty, then the intersection of all of them is nonempty.*

What the largest cardinal κ such that X^* is κ-saturated depends both on the cardinality of I and on the particular ultrafilter $\mathcal{F}$ we choose in the ultrapower construction. However, since we can always choose a larger I and a 'finer' ultrafilter, it is usually the case that we do not belabour the point and we assume that our nonstandard extension is saturated for a large enough cardinal κ as is required by our arguments.

2.1.2. *Particular constructions in examples.*

Hyperreals $\mathbb{R}^*$. Calculus, as originally conceived by Gottfried Wilhelm Leibniz, involved infinitesimals. For reasons of mathematical rigour, however, this approach to analysis has been substituted for one using limits that is due largely to Cauchy and Weierstrass. Almost 300 years after the invention of calculus a rigourous way to deal with infinitesimals emerges in Robinson's hyperreal system. The key fact that allows infinitesimal quantities in the hyperreal extension $\mathbb{R}^*$ is the fact that $\mathbb{R}^*$ is taken to be at least $\aleph_0$-saturated. Therefore, when we take the family of internal sets $\{(0, \frac{1}{n})^* \in \mathbb{R}^* | n \in \mathbb{N}\}$ saturation tells us that (since clearly every finite intersection of these sets in nonempty) there is an element (which we sometimes will call ϵ) that is in the intersection of all of them, and is hence smaller than any representative from $\mathbb{R}$.

But we can construct this extension explicitly. There are many ways to do this, of course. We show the (arguably) simplest one:

Take $I = \omega$. We now want to pick an ultrafilter $\mathcal{F}$ on ω such that $\mathbb{R}^{\mathcal{F}}$ is $\aleph_0$-saturated.

Proposition 2.5. *If there is a finite set $\{a_1, a_2, \ldots, a_n\} \in \mathcal{F}$, then $\mathcal{F}$ is a principal ultrafilter, i.e. $\mathcal{F} = \{S \subseteq \omega | a \in S\}$ for some $a \in \omega$, abbreviated $\mathcal{F} = \uparrow a$.*

Proof. We proceed by induction on n.

Base case. If $\{a_1\} \in \mathcal{F}$, then $\mathcal{F}$ is clearly principal.

Inductive hypothesis. If $\{a_1, a_2, \ldots, a_n\} \in \mathcal{F}$, then $\mathcal{F}$ is a principal ultrafilter, for all $n < k$.

Inductive step. Suppose $\{a_1, a_2, \ldots, a_k\} \in \mathcal{F}$. Consider $\{a_1, a_2, \ldots, a_{k-1}\}$. Either $\{a_1, a_2, \ldots, a_{k-1}\} \in \mathcal{F}$ or $\{a_1, a_2, \ldots, a_{k-1}\} \notin \mathcal{F}$. The first case implies that $\mathcal{F}$ is a principal ultrafilter by IH. The second case implies (by property (4) of ultrafilters) that $\omega \backslash \{a_1, \ldots, a_{k-1}\} \in \mathcal{F}$. But then, by property (3) of ultrafilters $\{a_k\} = (\omega \backslash \{a_1, \ldots, a_{k-1}\}) \cap (\{a_1, a_2, \ldots, a_k\}) \in \mathcal{F}$ and so $\mathcal{F}$ is a principal filter. □

We want to avoid $\mathcal{F}$ being a principal ultrafilter. This is because in the case where $\mathcal{F} = \uparrow a$ for some $a \in \omega$, when we construct the ultraproduct $\mathbb{R}^{\mathcal{F}}$ as above the equivalence classes of the relation $\sim_{\mathcal{F}}$ would correspond to the values of $f(a)$ in $\mathbb{R}$, and so $\mathbb{R}^{\mathcal{F}}$ will be isomorphic to $\mathbb{R}$. $\mathbb{R}$, however, is as we know *not* $\aleph_0$-saturated, which is what we are aiming for.

So we want for all finite sets S, S not to be in $\mathcal{F}$. This means (by property (4) of ultrafilters) that our ultrafilter $\mathcal{F}$ has to be a superset of

$$\mathcal{C} := \{A \subseteq \omega \,|\, \omega \backslash A \text{ is finite}\},$$

the set of all cofinite sets in ω. We have to show that such an ultrafilter exists. We do that by showing that $\mathcal{C}$ is a filter and that any filter can be extended to an ultrafilter.

Proposition 2.6. *$\mathcal{C}$ is a filter.*

Proof. We check the three requirements in the filter definition:

(1) $\emptyset$ is a finite set, so $\emptyset \notin \mathcal{C}$.
(2) Suppose $A \in \mathcal{C}$. So $A = \omega \backslash S$ for some finite S. If $B \supseteq A$, we have $B = (\omega \backslash S) \cup T = \omega \backslash (S \cap T)$ for some $T \subseteq \omega$. Since S is finite, then $S \cap T$ is also finite, so $B \in \mathcal{C}$.
(3) Suppose $A = \omega \backslash S \in \mathcal{C}$ and $B = \omega \backslash T \in \mathcal{C}$. Then

$$A \cap B = (\omega \backslash S) \cap (\omega \backslash T) = \omega \backslash (S \cup T).$$

Since both S and T are finite, then $S \cup T$ is finite. So $A \cap B \in \mathcal{C}$. □

Lemma 2.7. *(The Ultrafilter Lemma) If $\mathcal{K}$ is a filter, then there exists $\mathcal{F} \supseteq \mathcal{K}$ – an ultrafilter.*

Proof. Let $\mathcal{K}$ be a filter on a set I.

Let $\Omega = \{\mathcal{W} \subseteq \mathcal{P}(I) | \mathcal{W} \text{ is a filter on } I\}$.

Now, considering the subset relation as an ordering, we can see that that gives us Ω as a partially ordered set.

If $N \subseteq \Omega$ is a non-empty chain, then we check that $\bigcup N$ is a filter as well (and so is in Ω):

(1) $\emptyset \notin \mathcal{W}$ for all $\mathcal{W} \in N$. So $\emptyset \notin \bigcup N$.
(2) Suppose $A \in \bigcup N$. Then $A \in \mathcal{W}_k$ for some $\mathcal{W}_k \in N$. If $B \supseteq A$, then $B \in \mathcal{W}_k$ (by $\mathcal{W}_k$'s being a filter). So $B \in \bigcup N$.
(3) Suppose $A, B \in \bigcup N$. Then, by N's being a chain, $A, B \in \mathcal{W}_k$ for some $\mathcal{W}_k \in N$. Then $A \cap B \in \mathcal{W}_k \subseteq \bigcup N$.

So by Zorn's Lemma, there is a maximal element in Ω for every $\mathcal{W} \in \Omega$. So there is a maximal element $\mathcal{F}$ of Ωsuch that $\mathcal{K} \subseteq \mathcal{F}$. In this context this means that $\mathcal{F}$ is an ultrafilter extending $\mathcal{K}$. □

The previous two results show that there is an ultrafilter $\mathcal{F}$ that contains all cofinite sets. We can now construct the ultraproduct $\mathbb{R}^{\mathcal{F}}$. Choosing $\mathcal{F}$ to be nonprincipal is enough to ensure that we have at least $\aleph_0$-saturation. We shall thus not endeavour to make the choice of ultrafilter any more precise, for this would be enough for the present purposes. We denote the set $\mathbb{R}^{\mathcal{F}}$ by $\mathbb{R}^*$ from now on and we will be concerned with the properties shared by all ultrapowers of $\mathbb{R}$ with respect to a nonprincipal ultrafilter.

We can give a concrete example of saturation. Take $[f] \in \mathbb{R}^*$ for $f \in \mathbb{R}^\omega$ such that $f(n) = \frac{1}{n}$. It is clear that $-x < [f] < x$ for all $x \in \mathbb{R}$, because for any such x $x \geq f(n)$ for only finitely many $n \in \omega$. So we have an infinitesimal element of $\mathbb{R}^*$. Call this element ϵ.

Definition 2.8. *We call numbers e in $\mathbb{R}^*$, such that $-x < e < x$ for all $x \in \mathbb{R}$,* ***infinitesimal.*** *We call numbers f in $\mathbb{R}^*$ with infinitesimal multiplicative inverse* ***infinite.*** *We further call all non-infinite numbers* ***finite****.*

Consider the ring $\tilde{\mathbb{R}}^*$ of all finite numbers in $\mathbb{R}^*$. It is clear that $e.x$ is infinitesimal for all e – infinitesimal and x – finite. Therefore the set $(\epsilon) = \{e \text{ -infinitesimal}\}$ is an ideal. Moreover, it is a maximal ideal. So we can now take the quotient $\tilde{\mathbb{R}}^*/(\epsilon)$. It is obvious that $\tilde{\mathbb{R}}^*/(\epsilon) \cong \mathbb{R}$.

Definition 2.9. *We denote by $st(.) : \tilde{\mathbb{R}}^* \to \mathbb{R}$ the* ***standard part function*** *– the composition of the natural map $\tilde{\mathbb{R}}^* \to \tilde{\mathbb{R}}^*/(\epsilon)$ with the isomorphism alluded to above $\tilde{\mathbb{R}}^*/(\epsilon) \to \mathbb{R}$. The $st(.)$ map is thus an additive and multiplicative homomorphism.*

We can now do some simple analysis in $\mathbb{R}$ by simply doing algebra in $\tilde{\mathbb{R}}^*$.

Example 2.10. (The derivative of x^2.) We can calculate the derivative of the function $x \mapsto x^2$ by considering the infinitesimal change of x^2 when x changes infinitesimally and taking the standard part:

$$\begin{aligned} st\left(\frac{(x+e)^2 - x^2}{e}\right) &= st\left(\frac{x^2 + 2xe + e^2 - x^2}{e}\right) \\ &= st\left(\frac{2xe + e^2}{e}\right) \\ &= st(2x + e) = 2x \end{aligned}$$

Another useful idea is to identify $\mathbb{Z}^*$ as a subset of $\mathbb{R}^*$. We take the floor function

$$(\lfloor\cdot\rfloor) : \mathbb{R} \to \mathbb{Z} \text{ defined by } (\lfloor\cdot\rfloor) : x \mapsto max\{n \in \mathbb{Z} | n \leq x\}$$

We know this map is onto $\mathbb{Z}$. We can take its nonstandard extension in $\mathbb{R}^*$ and identify $image(\lfloor\cdot\rfloor^*) = \mathbb{Z}^*$. This way we actually have some information as to what kind of set $\mathbb{Z}^*$ actually is.

Since $\mathbb{R}^*$ inherits its order from $\mathbb{R}$, then it is totally ordered. It is clear that $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\}$. Therefore $\mathbb{N}^* = \{n \in \mathbb{Z}^* | n \geq 0\}$.

If we apply the transfer principle to the process of induction on $\mathbb{N}$ we get the following analogue for $\mathbb{N}^*$:

Theorem 2.11. *(Internal Induction) For any internal subset A of $\mathbb{N}^*$ if*

(1) 1 is an element of A, and

(2) for every $n \in A$, $n+1 \in A$,

then $A = \mathbb{N}^*$.

This allows us to prove the overflow principle, which we will need at several points in our solution.

Theorem 2.12. *(Overflow Principle) Assume $A \subseteq \mathbb{N}^*$ is internal and $\mathbb{N} \subseteq A$. Then there is a $\nu \in \mathbb{N}^* \backslash \mathbb{N}$ such that $\nu \in A$.*

Proof. Assume for a contradiction that $A = \mathbb{N} \subseteq \mathbb{N}^*$ is internal. Then, clearly, for all $n \in A$, $n+1 \in A$. But by Theorem 2.11 this implies that $\mathbb{N} = A = \mathbb{N}^*$. However, we are assuming sufficient saturation for this to not be true (even $\aleph_0$-saturation is enough here). □

A Hausdorff space $\mathbf{X}^*$. The nonstandard extension of a Hausdorff space has interesting and useful properties. We will be using many of them in what is to come.

Again we assume enough saturation of the extension as to make the infinite intersections we are concerned with nonempty.

Definition 2.13. *For any $a \in X$ we define* ***the monad of*** *a to be the set*

$$\mu(a) = \bigcap \{\mathcal{O}^* | \mathcal{O} \textit{ is a neighbourhood of } a\},$$

i.e. $\mu(a)$ is the set of elements of X^* infinitely close to a.

Our assumptions about the saturation of X^* ensure that $\mu(a) \neq \{a\}$, because the intersection

$$\bigcap \{\mathcal{O}^* \backslash \{a\} \,|\, \mathcal{O} \text{ is a neighbourhood of } a\}$$

has the finite intersection property and thus has a nonempty intersection.

The analog to the set $\tilde{\mathbb{R}}^*$ of finite elements of $\mathbb{R}^*$ here is the set

$$X_{ns} := \bigcup_{a \in X} \mu(a)$$

of ***nearstandard*** elements of X^*. The Hausdorff axioms gives us $\mu(a) \cap \mu(b) = \emptyset$ for all $a \neq b \in X$, so we can write $st(\alpha)$ (the ***standard part*** of α) for the unique standard $a \in X$ such that $\mu(a) \cap \{\alpha\} \neq \emptyset$. Since $\mathbb{R}$ is a Hausdorff space with the usual topology, we can see how this definition of the standard part function for general spaces X coincides with the above definition of $st(.)$ for the particular case of $\tilde{\mathbb{R}}^*$.

We write $\alpha \sim \beta$ if $\alpha, \beta \in X_{ns}$ and $st(\alpha) = st(\beta)$.

Note that if a is not isolated in X, then $\mu(a)$ is external. For otherwise $\mathcal{O}^* \backslash \mu(a)$ would be an internal set for all neighbourhoods $\mathcal{O}$ of a in X. But then we can consider

$$\bigcap \{\mathcal{O}^* \backslash \mu(a) \,|\, \mathcal{O} \text{ is a neighbourhood of } a \text{ in } X\},$$

which will have the finite intersection property, but an empty intersection. From our assumptions about saturation this is impossible.

Remark 2.14. We can now give useful characterisations of various concepts:

Continuity: Suppose X and Y are Hausdorff spaces, a is an element of X. A function $f : X \to Y$ is continuous at a if and only if $f(\mu(a)) \subseteq \mu(f(a))$.

Convergence: A sequence (a_n) from X converges if and only if $a_\sigma \sim a_\nu$ for all $\sigma, \nu \in \mathbb{N}^* \backslash \mathbb{N}$.

Interior and Closure: Suppose $F \subseteq X$, $a \in F$. Then

(1) $a \in int(F)$ if and only if $\mu(a) \subseteq F^*$.
(2) $a \in \bar{F}$ if and only if $\mu(a) \cap F^* \neq \emptyset$.

Compactness: $F \subseteq X$ is compact if and only if $F is closed$ and $F^* \subseteq X_{ns}$.

2.2. **Local groups and technical tools.** In this subsection we introduce the main objects in the paper - the local groups, and the relevant "local" analogues to subgroups and group morphisms. We also define the key notion of a one-parameter subgroup and at the end we give some further assumptions about local groups, explaining why we can take them without loss of generality for our purpose.

Definition 2.15. *A group that is also a topological space, such that the group operations of multiplication and inversion are continuous functions, is called a* ***topological group.***

Definition 2.16. *A* ***local group*** *is a tuple* $(G, 1, \imath, \rho)$ *where:*

(1) G is a Hausdorff topological space with distinguished element $1 \in G$.
(2) $\imath : \Lambda \to G$ *is continuous, where* $\Lambda \subseteq G$ *is open.*
(3) $\rho : \Omega \to G$ *is continuous, where* $\Omega \subseteq G \times G$ *is open.*
(4) $1 \in \Lambda, \{1\} \times G \subseteq \Omega, G \times \{1\} \subseteq \Omega$.
(5) $\rho(1, x) = \rho(x, 1) = x$.
(6) If $x \in \Lambda$, *then* $(x, \imath(x)), (\imath(x), x) \in \Omega$, *and*
$$\rho(x, \imath(x)) = \rho(\imath(x), x) = 1$$
(7) If $(x, y), (y, z), (\rho(x, y), z), (x, \rho(y, z)) \in \Omega$, *then*
$$\rho(\rho(x, y), z) = \rho(x, \rho(y, z))$$

From now on, unless otherwise stated, by G we will mean a local group $(G, 1, \iota, \rho)$. We usually write x^{-1} instead of $\iota(x)$ and xy or $x.y$ instead of $\rho(x, y)$. Note that (5) and (6) above give us $1^{-1} = 1$ and just (6) gives us that if $x, y \in \Lambda$, $(x, y) \in \Omega$ and $xy = 1$, then $x = y^{-1}$ and $y = x^{-1}$.

We can now define several important notions at once:

Definition 2.17.

- ***The restriction of*** G ***to*** U *is the local group* $U_G := (U, 1, \iota|_{\Lambda_U}, \rho|_{\Omega_U})$, *where* U *is an open neighbourhood of* 1 *in* G, $\Lambda_U := \Lambda \cap U \cap \iota^{-1}(U)$, *and* $\Omega_U := \Omega \cap (U \times U) \cap \rho^{-1}(U)$.
- *G is* ***locally euclidean*** *if there is an open neighbourhood of* 1 *homeomorphic to an open subset of* $\mathbb{R}^n$ *for some* n.
- *G is a* ***local Lie group*** *if G admits a* C^ω *structure such that the maps* ι *and* ρ *are* C^ω.
- *G is* ***globalisable*** *if there is a topological group* H *and an open neighbourhood* U *of* 1 *in* H *such that* $G = U_H$.

- *A* ***subgroup*** *H of G is a set with $H \subseteq G$ that is an actual group with the inherited inversion and multiplication from G; i.e. $1 \in H, H \subseteq \Lambda$, $H \times H \subseteq \Omega$ and for all $x, y \in H, x^{-1} \in H$ and $xy \in H$.*
- *G has* ***no small subgroups*** *(abbreviated G is NSS), if there is a neighbourhood U of 1 in G, such that U does not contain a subgroup of G other than $\{1\}$; equivalently, such that U_G has no nontrivial subgroups.*

We can now state the result that we will be aiming for.

Theorem. *(The Local Hilbert's Fifth Problem) If G is a locally euclidean local group, then some restriction of G is a local Lie group.*

The Category of Local Groups. To reflect the local nature of local groups we need to develop specific to the current debate notions of subgroup, normal subgroup, morhism, isomorphism and quotient group. We do this by investigating the category of local groups ***LocGrp***.

Definition 2.18. *Let us put $G = (G, 1, \iota, \rho)$, $G' = (G', 1', \iota', \rho')$, $domain(\iota) = \Lambda$, $domain(\iota') = \Lambda'$, $domain(\rho) = \Omega$, $domain(\rho') = \Omega'$. Then a* ***morphism*** *between G and $G\prime$ is a contonuous function $f : G \to G'$ such that:*

(1) $f(1) = 1', f(\Lambda) \subseteq \Lambda'$ and $(f \times f)(\Omega) \subseteq \Omega'$,
(2) $f(\iota(x)) = \iota'(f(x))$ for $x \in \Lambda$, and
(3) $f(\rho(x, y)) = \rho'(f(x), f(y))$ for $(x, y) \in \Omega$.

We define an equivalence relation on the set of all morphisms $f : U_G \to G'$, where U ranges over all open neighbourhoods of 1 in G:
For $f_1 : U_G \to G'$ and $f_2 : U\prime_G \to G'$:

$$f_1 \sim f_2 \text{ if and only if there exists } U'' \text{ - a neighbourhood of 1 in } G \text{ such that } U'' \subseteq U' \cap U \text{ and } f_1|_{U''} = f_2|_{U''}.$$

Definition 2.19. *The equivalence classes of morphisms in this relation we will call* ***local morphisms****.*

Instead of specifying a concrete morphism f and then considering $[f]$ we will say that $f : G \to G'$ is a local morphism to mean the equivalence class of partial functions from G to G'.

We can now show that the local groups with the so-defined local morphisms form a category ***LocGrp.***

Objects: All local groups $G, H \dots$

Morphisms: All local morphisms $[f], [g], \dots$

Composition: Let $f : G \to G'$ and $g : G' \to G''$ be local morphisms. We can choose a representative from f with image in the domain of a representative from g (because every morphism is a continuous map from an open set around 1_G into an open set around $1_{G'}$). We define $g \circ f$ as the equivalence class of the composition of the above two representatives.

Identity: The identity local morphism is the equivalence class of the identity morphism on a particular local group.

It is an easy check to verify that these satisfy the category axioms. An isomorphism in this category we will call a ***local isomorphism.*** Hence if $f : G \to G'$ is a local isomorphism, then there is a representative $f : U_G \to U'_{G'}$, where U and U' are open neighbourhoods of 1_G and $1_{G'}$ respectively, $f : U \to U'$ is a homeomorphism, and $f : U_G \to U'_{G'}$ and $f^{-1} : U'_{G'} \to U_G$ are morphisms.

If there is a morphism between G and G' we will say that G and G' are ***locally isomorphic***.

We now need to deal with the potential lack of generalised associativity in local groups.

The sets $\mathcal{U}_n$.

Definition 2.20. *Let $a_1, a_2, \ldots, a_n, b \in G$ with $n \geq 1$. We will say that $(a_1, \ldots, a_n)$ represents b, denoted $(a_1, \ldots, a_n) \to b$ if*

(1) $(a) \to b$ if and only if $a = b$;
(2) $(a_1, \ldots, a_{n+1}) \to b$ if and only if for every $i \in 1, \ldots, n$, there exists $b'_i, b''_i \in G$ such that $(a_1, \ldots, a_i) \to b'_i, (a_{i+1}, \ldots, a_{n+1}) \to b''_i$, $(b'_i, b''_i) \in \Omega$ and $b'_i.b''_i = b$.

By convention, we say that $(a_1, \ldots, a_n)$ represents 1 if $n = 0$.

We will say that $(a_1, \ldots, a_n)$ is defined if $(a_1, \ldots, a_n) \to b$ for some b. From the definition above and the fact that ρ is a function it is easy to see that if $(a_1, \ldots, a_n) \to b_1$ and $(a_1, \ldots, a_n) \to b_2$, then $b_1 = b_2$.

If $(a_1, \ldots, a_n)$ is defined, then we have ensured that all possible combinations of products are defined and that there is full associativity in the product $a_1 \cdots a_n$. We can thus write $(a_1, \ldots, a_n) \to a_1 \cdots a_n$, because $a_1 \cdots a_n$ is now a well-defined product. Therefore, $\underbrace{(a, \ldots, a)}_{n \text{ times}} \to a^n$.

From now on $A^{\times n} := \underbrace{A \times \ldots \times A}_{n \text{ times}}$.

Definition 2.21. *A subset $W \subseteq G$ is **symmetric** if $W \subseteq \Lambda$ and $W = \iota(W)$.*

We now prove that as we get closer to the identity in G, we can multiply more and more elements unambiguously.

Lemma 2.22. *There are open symmetric neighbourhoods $\mathcal{U}_n$ of 1 for $n > 0$ such that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$ and for all $(a_1, \ldots, a_n) \in \mathcal{U}_n^{\times n}, a_1 \cdots a_n$ is defined.*

Proof. We proceed by induction on n.

Base case. Let $\mathcal{U}_1$ be any open symmetric neighbourhood of 1. Since Ω is open in $G \times G$ and $\mathcal{U}_1$ is open in G, we can choose an open symmetric neighbourhood of 1 $\mathcal{U}_2$ such that $\mathcal{U}_2 \times \mathcal{U}_2 \subseteq \Omega$ and $\mathcal{U}_2 \subseteq \mathcal{U}_1$. Then, the map $\phi_2 : \mathcal{U}_2^{\times 2} \to G$ defined by $\phi_2 : (a_1, a_2) \mapsto a_1.a_2$ is continuous, because it coincides with ρ.

Inductive hypothesis. For all $m \leq n$ (where $n \geq 2$) the following hold

- $\mathcal{U}_m$ is a symmetric open neighbourhood of 1;
- $\mathcal{U}_{m+1} \subseteq \mathcal{U}_m$ if $m < n$;
- For all $(a_1, \ldots, a_m) \in \mathcal{U}_m^{\times m}, a_1 \cdots a_m$ is defined;
- The map $\phi_m : \mathcal{U}_m^{\times m} \to G$ defined by $\phi_m : (a_1, \cdots, a_m) \mapsto a_1 \cdots a_m$ is continuous.

Inductive step. Since ϕ_n is continuous and $\mathcal{U}_2$ is open, then $\phi_n^{-1}(\mathcal{U}_2)$ is open in $\underbrace{G \times \ldots \times G}_{n \text{ times}}$. Thus, we can choose $\mathcal{U}_{n+1}$ to be an open symmetric neighbourhood of 1 such that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$ and $\mathcal{U}_{n+1}^{\times n} \subseteq \phi_n^{-1}(\mathcal{U}_2)$. We claim that this choice works.

We want to show that $a_1 \cdots a_{n+1}$ is defined.

For all $i \in 1, \ldots, n \quad a_1 \cdots a_i$ and $a_{i+1} \cdots a_{n+1}$ are defined. This means that however we place the brackets on writing out the product properly, we still get a well-defined expression and the same result.

Let $k \in 1, \ldots, n-1$. We will show that

$$\phi_k(a_1, \ldots, a_k).\phi_{n+k-1}(a_{k+1}, \ldots, a_{n+1}) = \\ = \phi_{k+1}(a_1, \ldots, a_{k+1}).\phi_{n+k-2}(a_{k+2}, \ldots, a_{n+1})$$

Therefore we will have that $(a_1, \ldots, a_{n+1}) \to b$ for some b in G by the definition of representation. But this follows easily from the simple associativity of ρ, the definedness of all products shorter than $n+1$ and the fact that we can put the brackets however we desire:

$$(\ldots((a_1.a_2)\cdots a_k).(a_{k+1}.(a_{k+2}\cdots(a_n.a_{n+1})\ldots) = \\ = (\ldots(a_1.a_2)\cdots a_k).a_{k+1}).(a_{k+2}\cdots(a_n.a_{n+1})\ldots)$$

Moreover, ϕ_{n+1} as defined above is continuous, because it is the continuous ρ product of two continuous maps -

$$\phi_{n+1}(a_1, \ldots, a_{n+1}) = \phi_n(a_1, \ldots, a_n).id(a_{n+1})$$

This completes the induction. □

From now on we will refer to the above sets always as $\mathcal{U}_n$. Also, for $A \subseteq \mathcal{U}_n$, we define

$$A^n := \{a_1 \cdots a_n | (a_1, \ldots, a_n) \in A^{\times n}\}$$

We can now define the local analogues of a subgroup and of a normal subgroup.

Definition 2.23. *A **sublocal group** of G is a set $H \subseteq G$ containing 1 for which there exists an open neighbourhood V of 1 in G such that*

(1) $H \subseteq V$ and H is closed in V;
(2) if $x \in H \cap \Lambda$ and $x^{-1} \in V$, then $x^{-1} \in H$;
(3) if $(x, y) \in (H \times H) \cap \Omega$ and $xy \in V$, then $xy \in H$;

*A set $H \subseteq G$ is a **normal sublocal group** of G if it is a sublocal group with associated **normalizing** neighbourhood V and*

(4) if $y \in V$ and $x \in H$ are such that yxy^{-1} is defined and $yxy^{-1} \in V$, then $yxy^{-1} \in H$.

We can see that if H is a sublocal group of G with associated neighbourhood V, then

$$(H, 1, \iota|_{H\cap\Lambda\cap\iota^{-1}(V)}, \rho|_{(H\times H)\cap\Omega\cap\rho^{-1}(V)})$$

is a local group, which we will usually denote simply by H, and that the inclusion $H \hookrightarrow G$ is a morphism. We say that two sublocal groups H and H' are ***equivalent*** if there is an open neighbourhood U of 1 in G such that $H \cap U = H' \cap U$.

Lemma 2.24. *Suppose H is a normal sublocal group of G with associated normalizing neighbourhood V. Suppose $U \subseteq H$ is open in H and symmetric. Let $U' \subseteq V$ be a symmetric open neighbourhood of 1 in G such that $U = H \cap U'$. Then U_H is a normal sublocal group of G with associated normalizing neighbourhood U'.*

Proof. We want to show that $U_H = (U, 1, \iota|_{\Lambda_U}, \rho|_{\Omega_U})$ obeys the normal sublocal group axioms:

(1) $U \subseteq U'$ and U is closed in U'. $U = H \cap U'$, so clearly $U \subseteq U'$. H is closed in V (by the definition of a normal sublocal group). So H is closed in $U' \subseteq V$. Therefore $H \cap U' = U$ is closed in U'.
(2) If $x \in U \cap \Lambda$ and $x^{-1} \in U'$, then $x^{-1} \in U$. Assuming $x \in U \cap \Lambda$, since U is symmetric, then $x^{-1} \in U$.
(3) If $(x, y) \in (U \times U) \cap \Omega$ and $xy \in U'$, then $xy \in U$. Now we assume that $(x, y) \in (U \times U) \cap \Omega$ and $xy \in U'$. Then $(x, y) \in (H \times H) \cap \Omega$ and $xy \in V$. So $xy \in H$, by the normality of H. But then $xy \in H \cap U' = U$.
(4) If $y \in U'$, $x \in U$ and yxy^{-1} is defined and in U', then $yxy^{-1} \in U$. Assume $y \in U'$, $x \in U$ and yxy^{-1} is defined and in U'. So, again, $y \in V$, $x \in H$, $yxy^{-1} \in V$, so $yxy^{-1} \in H$. Therefore $yxy^{-1} \in H \cap U' = U$.

□

We now investigate quotients in the category ***LocGrp***. For the remainder of this section we will take H to be a normal sublocal group of G with associated neighbourhood V.

Lemma 2.25. *Let W be a symmetric open neighbourhood of* 1 *in G such that $W \subseteq \mathcal{U}_6$ and $W_6 \subseteq V$. Then*

(1) The binary relation E_H on W is defined by

$$E_H(x, y) \text{ if and only if } x^{-1}y \in H$$

is an equivalence relation on W.

*(2) For $x \in W$, let $xH := \{xh | h \in H \text{ and } (x, h) \in \Omega\}$. Then for $x, y \in W$, $E_H(x, y)$ if and only if $(xH) \cap W = (yH) \cap W$. In other words, if $E_H(x)$ denotes the equivalence class of x, then $E_H(x) = (xH) \cap W$. We call the equivalence classes **local cosets** of H.*

Proof.

(1) We check the equivalence relation axioms.
Reflexivity: Trivial.
Symmetry: Assume $E_H(x, y)$. Now, note that $x^{-1}y(x^{-1}y)^{-1} = 1$ and multiplying by x and then y^{-1} on the left gives us $(x^{-1}y)^{-1} = y^{-1}x$. Since we are defining E_H on W, then $x, y \in W$. But W is symmetric, so $y^{-1} \in W$. Also, $W^6 \subseteq V$, so $y^{-1}x = (x^{-1}y)^{-1} \in V$. Therefore $(x^{-1}y)^{-1} = y^{-1}x \in H$ and $E_H(y, x)$.
Transitivity: Assume $E_H(x, y)$ and $E_H(y, z)$. Again, $x, y, z \in W$, so $x^{-1}, y^{-1}, z^{-1} \in W$. Then $x^{-1}yy^{-1}z \in W^6 \subseteq V$, so $x^{-1}yy^{-1}z \in V$. Therefore $x^{-1}yy^{-1}z = x^{-1}z \in H$, so $E_H(x, z)$.
(2) *(=>)*: Suppose $E_H(x, y)$. Suppose further that $w = xh \in (xH) \cap W$, for some $w \in W, h \in H$. Then $h = x^{-1}w \in W^2 \cap H$, and so

$$(y^{-1}x)h = (y^{-1}x)(x^{-1}w) \in W^4 \subseteq V$$

and also the product $(y^{-1}x, x^{-1}w) \in (H \times H)$. So

$$(y^{-1}x)h = y^{-1}w \in H, w = y(y^{-1}x)h \in (yH) \cap W$$

So $(xH) \cap W \subseteq (yH) \cap W$. Analogously, using the fact that W is symmetric, we get $(yH) \cap W \subseteq (xH) \cap W$. Therefore we have that $(xH) \cap W = (yH) \cap W$.

(<=): Assume $(xH) \cap W = (yH) \cap W$. Since $y \in (yH) \cap W$, we know that $y \in (xH) \cap W$. Let h be such that $y = xh$. Now, x^{-1} and y are both in W and so $x^{-1}y = x^{-1}(xh)$ is defined. By associativity of ρ and because all of $(x^{-1}, xh), (x^{-1}x, h), (x, h), (x^{-1}, x) \in \Omega$, we have

$$x^{-1}y = x^{-1}(xh) = (x^{-1}x)h = h \in H$$

Therefore $x^{-1}y \in H$, i.e. $E_H(x, y)$.

□

Let $\pi_{H,W} : W \to W/E_H$ be the canonical map. Give W/E_H the quotient topology. Then $\pi_{H,W}$ becomes an open continuous map.

We want to define maps $\iota_{H,W}$ and $\rho_{H,W}$ on W/E_H to make it a local group.

(1) Let $\iota_{H,W} : W/E_H \to W/E_H$ be defined by $\iota_{H,W} : E_H(x) \mapsto E_H(x^{-1})$ Since x in this setting is from W and W is symmetric, then $\iota_{H,W}$ can be defined globally on the whole group. So here $\Lambda_{H,W} = W/E_H$.
(2) Let $\Omega_{H,W} := (\pi_{H,W} \times \pi_{H,W})((W \times W) \cap \rho^{-1}(W))$. Now we can define $\rho_{H,W} : \Omega_{H,W} \to W/E_H$ by $\rho_{H,W} : (E_H(x), E_H(y)) \mapsto E_H(xy)$, where x and y are representatives from their respective local cosets, chosen so that $xy \in W$.

We thus get the following lemma.

Lemma 2.26. *With the notations as above,*

$$(G/H)_W := (W/E_H, E_H(1), \iota_{H,W}, \rho_{H,W})$$

is a local group and $\pi_{H,W} : W_G \to (G/H)_W$ is a morphism.

We need to show that if we had made different choices in the construction of the above local group, then we would have ended up with a locally isomorphic one.

Lemma 2.27. *Suppose that H' is also a normal sublocal group of G with associated normalizing neighbourhood V' such that H is equivalent to H'. Let W' be a symmetric open neighbourhood of 1 in G used to construct $(G/H')_{W'}$. Then $(G/H)_W$ and $(G/H')_{W'}$ are locally isomorphic.*

Proof. Since H is equivalent to H', we can take U-open neighbourhood of 1 in G such that $H \cap U = H' \cap U$. Now choose an open $U_1 \subseteq \mathcal{U}_2$ containing 1 such that $U_1^2 \subseteq U$. Let $U_2 := W \cap W' \cap U_1$. Let $\phi : \pi_{H,W}(U_2) \to W'/E_{H'}$ be defined by $\phi : E_H(x) \mapsto E_{H'}(x)$ for $x \in U_2$. Then $\phi : \pi_{H,W}(U_2)_{(G/H)_W} \to (G/H')_{W'}$ is a representative of the desired local isomorphism. □

The above lemma allows us to write the quotient of G by H as G/H, meaning the local group $(G/H)_W$ for any appropriate open neighbourhood of 1 W. We will also write $\pi : G \to G/H$ for the local morphism represented by $\pi_{H,W} : W_G \to (G/H)_W$. This is no loss of generality, because wherever confusion may occur we have locally isomorphic local groups.

We now define a key notion in the discussion - that of a local one-parameter subgroup.

Definition 2.28. *A* ***local*** 1***-parameter subgroup*** *of* G*, henceforth abbreviated* ***an LPS of*** G*, is a continuous map* $X : (-r, r) \to G$*, for some* $r \in (0, \infty]$*, such that*

(1) image$(X) \subseteq \Lambda$*, and*
(2) if $r_1, r_2, r_1 + r_2 \in (-r, r)$*, then* $(X(r_1), X(r_2)) \in \Omega$ *and*

$$X(r_1 + r_2) = X(r_1).X(r_2).$$

Remark 2.29. (2) above is a very involved requirement in the local group setting - it allows extensive multiplication. Note that it straightforwardly implies that if we have an LPS of G $\quad X : (-r, r) \to G$ and $s \in (-r, r)$, then $ns \in (-r, r)$ implies that $X(s)^n$ is defined and $X(ns) = X(s)^n$.

We now define the space of LPS - the local analogue of the space which plays a key role in the solution of the H5.

Definition 2.30. *Let* X, Y *be LPSs of* G*. We say that* X *is* ***equivalent to*** Y *if there is* $r \in \mathbb{R}^+$ *such that*

$$r \in domain(X) \cap domain(Y) \text{ and } X|_{(-r,r)} = Y|_{(-r,r)}$$

We let $[X]$ *denote the equivalence class of* X *with respect to this equivalence relation. We also let* $L(G) := \{[X] \,|\, X$ *is an LPS of* $G\}$*.*

We will usually write $\mathbb{X}$ for an element of $L(G)$, and X for an LPS $X \in \mathbb{X}$.

Remark 2.31. Firstly, put $\mathbb{O} \in L(G)$ to be the equivalence class of the trivial LPS; i.e. $\mathbb{O} = [O]$, where $O : \mathbb{R} \to G$ is defined by $O : t \mapsto 1_G$. We want to define an expression of the form $s \cdot \mathbb{X}$. We put $0 \cdot \mathbb{X} = \mathbb{O}$ for all $\mathbb{X} \in L(G)$.

Let $X \in \mathbb{X}$ be an LPS with $X : (-r, r) \to G$. Define

$$sX : \left(\frac{-r}{|s|}, \frac{r}{|s|} \right) \to G \text{ by } sX : t \mapsto X(st).$$

Then sX is an LPS of G and we set $s \cdot \mathbb{X} = [sX]$. It is easy to see that for all $\mathbb{X} \in L(G)$ and $s, s' \in \mathbb{R}$, $1 \cdot \mathbb{X} = \mathbb{X}$ and $s \cdot (s' \cdot \mathbb{X}) = (ss') \cdot \mathbb{X}$. We can now define scalar multiplication on $L(G)$.

$$(. \cdot .) : \mathbb{R} \times L(G) \to L(G) \text{ is defined by } (. \cdot .) : (s, \mathbb{X}) \mapsto s \cdot \mathbb{X}$$

Let X_1, X_2 be LPSs of G with $[X_1] = [X_2]$, $t \in domain(X_1) \cup domain(X_2)$. Since X_1 is equivalent to X_2, then we can find a $r \in \mathbb{R}^+$ such that we have $X_1|_{(-r,r)} = X_2|_{(-r,r)}$. Now choose $n \in \mathbb{Z}^+$ such that $\frac{t}{n} \in (-r, r)$. Then

$$X_1(t) = \left(X_1 \left(\frac{t}{n} \right) \right)^n = \left(X_2 \left(\frac{t}{n} \right) \right)^n = X_2(t)$$

Hence, we can meaningfully define, for $\mathbb{X} \in L(G)$,

$$domain(\mathbb{X}) := \bigcup_{X \in \mathbb{X}} domain(X)$$

and for $t \in domain(\mathbb{X})$, we will write $\mathbb{X}(t)$ to denote $X(t)$ for any $X \in \mathbb{X}$ with $t \in domain(X)$.

Further assumptions on our group G. Since we are concerned with local groups up to local isomorphism, if we show that *any* local group has a restriction with certain properties, then we can incorporate that property into our assumptions. The idea is that since any group is locally isomorphic to one with a particular property, then it would be beneficial to consider exactly the groups in the isomorphism class with that property.

Remark 2.32. Let $U := \Lambda \cap \iota^{-1}(\Lambda)$, an open neighbourhood of 1 in G. Then, if $g \in U$, we have $\iota(g) \in \Lambda$ and

$$\begin{aligned} &\iota(g).\iota(\iota(g)) = 1 && \text{(by the definition of local group)} \\ \Longrightarrow\ & g.(\iota(g).\iota(\iota(g))) = g && \\ \Longrightarrow\ & (g.\iota(g)).\iota(\iota(g)) = g && \text{(by direct associativity)} \\ \Longrightarrow\ & 1.\iota(\iota(g)) = \iota(\iota(g)) = g \in \Lambda && \end{aligned}$$

Therefore $\Lambda_U = U$, where $\Lambda_U = \Lambda \cap U \cap \iota^{-1}(U)$ as before.

The above remark shows that every local group has a restriction U satisfying $\Lambda_U = U$. We can therefore, from here on, consider our local group of choice G as one for which $\Lambda = G$.

Remark 2.33. With this further assumption on G, if $(x, y) \in \Omega$ and $xy = 1$, then $x = y^{-1}, y = x^{-1}, (x^{-1})^{-1} = 1$ for all $x \in G$ and G is symmetric.

A corollary of the following lemma will provide another useful further assumption we can take on G.

Lemma 2.34. *(Homogeneity)*

1. *For any $g \in G$, there are open neighbourhoods V and W of 1 and g respectively such that $\{g\} \times V \subseteq \Omega, gV \subseteq W, \{g^{-1}\} \times W \subseteq \Omega, g^{-1}W \subseteq V$, and the maps*
$$v \mapsto gv : V \to W \quad \text{and} \quad w \mapsto g^{-1}w : W \to V$$
are each other's inverses (and hence homeomorphisms).
2. *G is locally compact if and only if there is a compact neighbourhood of 1.*

Proof. (1) clearly implies (2). So it suffices to show (1).

For any $g \in G$ define
$$\Omega_g := \{h \in G \mid (g, h) \in \Omega\}.$$

Ω_g is thus an open subset of G and we define $L_g : \Omega_g \to G$ by $L_g : h \mapsto gh$, which is continuous. Let $V := (L_g)^{-1}(\Omega_{g^{-1}})$. Then V is open and $1 \in V$. Let $W := L_g(V) \subseteq \Omega_{g^{-1}}$. Then W is open since $W = L_{g^{-1}}^{-1}(V)$. Therefore, the maps $L_g|_V$ and $L_{g^{-1}}|_W$ are inverses of each other and thus satisfy the requirements of the lemma. □

Corollary 2.35. *Let $U = \mathcal{U}_3$. Then for any $g, h \in U$ such that $(g, h) \in \Omega_U$, one has $(h^{-1}, g^{-1}) \in \Omega_U$ and $(gh)^{-1} = h^{-1}g^{-1}$.*

Proof. Assume $g, h \in U$ and $(g, h) \in \Omega_U$. The latter assumption implies that $gh \in U$. Also, since $\mathcal{U}_3$ is symmetric, $g^{-1}, h^{-1} \in U$. Therefore $h^{-1} \cdot g^{-1} \cdot (gh)$ is defined and
$$h^{-1} \cdot g^{-1} \cdot (gh) = h^{-1} \cdot (g^{-1} \cdot (gh)) = h^{-1}((g^{-1}g) \cdot h) = h^{-1}h = 1,$$

since $(g^{-1}, g), (g, h), (g^{-1}, gh), (g^{-1}g, h) \in \Omega$ and we can use direct associativity. But we can write the same product as $(h^{-1}g^{-1}) \cdot (gh)$. So $h^{-1}g^{-1} = (gh)^{-1}$. Moreover, again by the symmetry of $\mathcal{U}_3$, $h^{-1}g^{-1} = (gh)^{-1} \in U$ and therefore $(h^{-1}, g^{-1}) \in \Omega_U$. $\square$

The above corollary allows us to make another further assumption on G, because it proves that any local group has a restriction satisfying it. We assume that in G

$$\text{if } (g, h) \in \Omega, \text{ then } (h^{-1}, g^{-1}) \in \Omega \text{ and } (gh)^{-1} = h^{-1}g^{-1}.$$

We can now derive several useful results in G with these new assumptions.

Lemma 2.36. *Let $a, a_1, \ldots, a_n \in G$. Then*

(1) If $a_1 \cdots a_n$ is defined and $1 \leq i \leq j \leq n$, then $a_i \cdots a_j$ is defined. In particular, if a^n is defined and $m \leq n$, then a^m is defined.

(2) If a^m is defined and $i, j \in \{1, \ldots, m\}$ are such that $i + j = m$, then we have $(a^i, a^j) \in \Omega$ and $a^i \cdot a^j = a^m$.

(3) If a^n is defined and, for all $i, j \in \{1, \ldots, n\}$ with $i + j = n + 1$, one has $(a^i, a^j) \in \Omega$, then a^{n+1} is defined. More generally, if $a_1 \cdots a_n$ is defined, $a_i \cdots a_{n+1}$ is defined for all $i \in \{2, \ldots, n\}$ and

$$(a_1 \cdots a_i, a_{i+1} \cdots a_{n+1}) \in \Omega \textit{ for all } i \in \{1, \ldots, n\},$$

then $a_1 \cdots a_{n+1}$ is defined.

(4) If a^n is defined, then $(a^{-1})^n$ is defined and $(a^{-1})^n = (a^n)^{-1}$. (In this case, we denote $(a^{-1})^n$ by a^{-n}.) More generally, if $a_1 \cdots a_n$ is defined, then $a_n^{-1} \cdots a_1^{-1}$ is defined and $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$.

(5) If $k, l \in \mathbb{Z}, l \neq 0$, and $a^{k.l}$ is defined, then a^k is defined, $(a^k)^l$ is defined and $(a^k)^l = a^{k.l}$.

Proof.

(1) From the definition of representation '$a_1 \cdots a_n$ – defined' implies '$a_1 \cdots a_j$ – defined', implies '$a_i \cdots a_j$ – defined'.

(2) This follows directly from the definition of representation.

(3) The proof of this mimics word for word the proof of Lemma 2.22.

(4) We do this one by induction on n with the base cases $n = 1, 2$ being obvious from our assumptions about local groups.

Inductive hypothesis: We assume that $n > 2$ and for any $m < n$, if $a_1 \cdots a_m$ is defined, then $a_m^{-1} \cdots a_1^{-1}$ is defined and $(a_1 \cdots a_m)^{-1} = a_m^{-1} \cdots a_1^{-1}$.

Inductive step: Suppose $a_1 \cdots a_n$.

First note that all the various products of the form $a_j^{-1} \cdots a_i^{-1}$ are defined for all $1 \leq i \leq j \leq n$, except when $i = 1$ and $j = n$. We just need to show that for all $i, j \in \{1, \ldots, n-1\}$ we have

$$(a_n^{-1} \cdots a_{i+1}^{-1}) \cdot (a_i^{-1} \cdots a_1^{-1}) = (a_n^{-1} \cdots a_{j+1}^{-1}) \cdot (a_j^{-1} \cdots a_1^{-1})$$

We can do this by showing that every two consequetive i and j in the above the equality holds. But this is simple:

$$\begin{aligned}(a_n^{-1} \cdots a_{i+1}^{-1}) \cdot (a_i^{-1} \cdots a_1^{-1}) &= ((a_n^{-1} \cdots a_{i+2}^{-1}) \cdot a_{i+1}^{-1}) \cdot (a_i^{-1} \cdots a_1^{-1}) \\ &= (a_n^{-1} \cdots a_{i+2}^{-1}) \cdot (a_{i+1}^{-1} \cdot (a_i^{-1} \cdots a_1^{-1})) \\ &= (a_n^{-1} \cdots a_{i+2}^{-1}) \cdot (a_{i+1}^{-1} \cdots a_1^{-1})\end{aligned}$$

So $a_m^{-1} \cdots a_1^{-1}$ is defined. We now show the second claim of this part:

$$\begin{aligned}
(a_1 \cdots a_n)^{-1} &= ((a_1 \cdots a_{n-1}) \cdot a_n)^{-1} && \text{(by definedness)} \\
&= (a_n^{-1} \cdot (a_1 \cdots a_{n-1})^{-1}) && \text{(by Corollary 2.35)} \\
&= (a_n^{-1} \cdot (a_{n-1}^{-1} \cdots a_1^{-1})) && \text{(by IH)} \\
&= (a_n^{-1} \cdots a_1^{-1}) && \text{(by definedness)}
\end{aligned}$$

(5) Let $k, l \in \mathbb{N}$, $k, l \neq 0$. This restriction is without loss of generality by the previous part of this lemma. Assume $a^{k/l}$ is defined. By part (1) a^k is defined. For the rest we give another induction, this time on l. The claim clearly holds for $l = 1$. Suppose it holds for all $i < l$. We need to show that $(a^k)^l$ is defined. If we put $b = a^k$ we can use part (2) to get $b^i.b^j = b^l$ for all $i, j \in \{1, \ldots, l-1\}$ such that $i + j = l$. But by our IH $b^i = (a^k)^i = a^{k.i}$ for all $i \in \{1, \ldots, l-1\}$. So $(a^k)^l$ is defined. Finally, from IH again, we have

$$(a^k)^l = (a^k)^{l-1} \cdot a^k = a^{k.(l-1)} \cdot a^k = a^{k.l}$$

□

Corollary 2.37. *Suppose $i, j \in \mathbb{Z}$ and $i.j < 0$. If a^i and a^j are defined and $(a^i, a^j) \in \Omega$, then a^{i+j} is defined and $a^i \cdot a^j = a^{i+j}$.*

Proof. The fact that a^{i+j} is defined follows directly from the previous lemma. Also, the case $i = -j$ is trivial. Again by the lemma above we can assume without loss of generality that $i > 0, j < 0$ and $i > |j|$. The lemma also implies that here a^{-j} is defined. Finally, we can see that, since all powers are positive, $a^i = a^{i+j} \cdot a^{-j}$. By multiplying on the right by a^j we get $a^i \cdot a^j = a^{i+j}$. □

Nonstandard results in G. The nonstandard extension G^* of our local group of choice G will play a crucial role in our proof of LH5. Since our local group is a Hausdorff space, we can use all the results on nonstandard extensions of Hausdorff spaces described before - the monad of a point, the set of nearstandard points, the standard part map, etc.

For the sake of readability we will only denote with a star the nonstandard extensions of basic sets and not the extensions of functions or relations. For example, we will write Ω^* for the set of allowed product pairs in G^*, but we will usually write $\rho : \Omega^* \to G^*$ for the product map in G^*. Sometimes we will add the star sign on functions when it is particularly crucial to the proof that we are using the nonstandard extension of the function.

We fix some notation: $\nu, \sigma, \tau, \eta, N$ will range over $\mathbb{N}^*$; i, j will range over $\mathbb{Z}^*$; m, n will range over $\mathbb{N}$. We let $\mu := \mu(1)$ be the monad of 1 in G^*. Note that by transfer and Lemma 2.22 μ is an actual group with product map $(x, y) \mapsto xy := \rho(x, y)$.

We use some asymptotical (Landau) notation:

(1) $i \in o(\nu)$ means that i is '***much smaller than***' ν; formally, $|i| < \frac{\nu}{n}$ for all $n > 0$.
(2) $i \in O(\nu)$ means that i is '***not much bigger than***' ν; formally, there is an $n > 0$ such that $|i| < n\nu$.

As remarked in the subsection on the nonstandard setting, in our arguments we will assume that G^* is sufficiently saturated (κ-saturated for a sufficiently large cardinal κ) so that out deductions make sense.

Lemma 2.38. *G is NSS if and only if there are no internal subgroups of μ other than $\{1\}$.*

Proof. First note that if S is an internal subgroup of G^*, then if $a \in S$ we have that $a^i \in S$ for all $i \in \mathbb{N}^*$.

(<=): Suppose G is not NSS. Therefore, for all neighbourhoods V of 1 in G there is a nontrivial subgroup H_V of G in V.

The transfer principle then gives us that for all internal neighbourhoods V^* of 1 in G^* there is a nontrivial internal subgroup H_V^* of G^*.

By saturation, we can intersect all such internal neighbourhoods V^* and still get a nontrivial subgroup in the intersection. So μ contains a nontrivial internal subgroup of G^*.

(=>): Suppose G is NSS. Then there exists a neighbourhood S of 1 such that $S \subseteq G$ and there are no subgroups of G in S except $\{1\}$.

Transfer gives us that S^* contains no nontrivial internal subgroups of G^*. But $\mu \subseteq S^*$, so μ contains no internal subgroups of G^* other than $\{1\}$. □

We have the internal versions of the notions defined up until now, so it makes sense to say that $(a_1, \ldots, a_n) \to b$ for $a_1, \ldots, a_n, b \in G^*$. With that in mind, the following lemma is trivial.

Lemma 2.39.

(1) Suppose $a, b \in G$, $a' \in \mu(a)$ and $b' \in \mu(b)$. If $(a, b) \in \Omega$, then we have $(a', b') \in \Omega^*, a' \cdot b' \in G_{ns}^*$, and $st(a' \cdot b') = a \cdot b$.

(2) For any $a \in G_{ns}^*$ and $b \in \mu$, $(a, b), (b, a) \in \Omega^*, a \cdot b, b \cdot a \in G_{ns}^*$, and $st(a \cdot b) = st(b \cdot a) = st(a)$.

(3) For any $a, b \in G_{ns}^*$, if $(a, b^{-1}) \in \Omega^*$ and $ab^{-1} \in \mu$, then a b.

(4) For any $a \in G, a' \in \mu(a)$, and any n, if a^n is defined, then $(a')^n$ is defined and $(a')^n \in \mu(a^n)$.

Lemma 2.40. *Suppose U is a neighbourhood of 1 in G and $a \in \mu$. Then there is a $\nu > \mathbb{N}$ such that a^σ is defined and $a^\sigma \in U^*$ for all $\sigma \in \{a, \ldots, \nu\}$.*

Proof. Let $X := \{\sigma \in \mathbb{N}^* | a^\sigma$ is defined and $a^\sigma \in U^*\}$. Then X is an internal subset of $\mathbb{N}^*$ (because it is defined in a first-order way). Moreover, $\mu \subseteq U^*$ and so we have that a^m is defined and $a^m \in U^*$ for all m. Therefore, by the overflow principle (Theorem 2.12) we have that there is a $\nu \in \mathbb{N}^* \backslash \mathbb{N}$ such that $\{0, 1, \ldots, \nu\} \subseteq X$. □

3. Connectedness and Purity

Let us recap the assumptions on our local group G. We are considering G to be a local group as defined in Definition 2.16 with

(1) $\Lambda = G$, and
(2) if $(x, y) \in \Omega$, then $(y^{-1}, x^{-1}) \in \Omega$ and $(xy)^{-1} = y^{-1}x^{-1}$.

We also assume that G is locally compact. Since in the end we are concerned with locally euclidean local groups and it is clear that locally euclidean implies locally compact, then for our purposes this produces no loss of generality.

In this chapter we start with all the relevant definitions first, so as to make the following arguments flow more naturally. We will investigate the different ways infinitesimals can 'grow out' of μ and also how we can build LPSs from a certain kind of infinitesimals.

From now on, unless otherwise stated, consider $a, b \in G^*$ and U to be a compact symmetric neighbourhood of 1 such that $U \subseteq \mathcal{U}_2$.

Definition 3.1. *For $\nu > \mathbb{N}$, we set:*

$$\mathbf{G}(\nu) : = \{a \in \mu \,|\, a^i \textit{ is defined and } a^i \in \mu \textit{ for all } i \in o(\nu)\},$$
$$\mathbf{G^o}(\nu) : = \{a \in \mu \,|\, a^i \textit{ is defined and } a^i \in \mu \textit{ for all } i \in O(\nu)\}.$$

Note that $G^o(\nu) \subseteq G(\nu)$ and, since the definitions we gave of $o(\nu)$ and $O(\nu)$ involved only the absolute value of i, we see that both sets $G(\nu)$ and $G^o(\nu)$ are symmetric.

We let Q range over symmetric internal neighbourhoods $Q \subseteq \mu$ of 1 in G^*. If ν is such that for all internal sequences $a_1, \dots, a_\nu$ from Q, $a_1 \cdots a_\nu$ is defined, then we define Q^ν to be the following internal subset of G^*:

$$Q^\nu := \{a_1 \cdots a_\nu \,|\, a_1, \dots, a_\nu \text{ is an internal sequence from } Q\}.$$

In this situation, we say that Q^ν is defined.

Definition 3.2.

(1) For $a \in G^*$, if for all i, a^i is defined and $a^i \in U^*$, define $\mathbf{ord}_U(a) = \infty$. Otherwise, define

$$\mathbf{ord}_U(a) := max\{\nu \,|\, \textit{for all } i \textit{ with } |i| \leq \nu,\ a^i \textit{ is defined and } a^i \in U^*\}.$$

$Q \subseteq G^*$, if for all ν, Q^ν is defined and $Q^\nu \subseteq U^*$, define $\mathbf{ord}_U(Q) = \infty$. Otherwise, define

$$\mathbf{ord}_U(Q) := max\{\nu \,|\, \textit{for all } i \textit{ with } |i| \leq \nu,\ Q^i \textit{ is defined and } Q^i \subseteq U^*\}.$$

The above maximums exist by overflow.

*(2) We say that an element $a \in \mu$ is **degenerate** if for all i, a^i is defined and $a^i \in \mu$. We say that a set $Q \subseteq \mu$ is **degenerate** if for all ν, Q^ν is defined and $Q^\nu \subseteq \mu$.*

*(3) We say that $a \in \mu$ is U-**pure** if it is nondegenerate and $a \in G(ord_U(a))$. We say that $a \in \mu$ is **pure** if it is V-pure for some compact neighbourhood V of 1 such that $V \subseteq \mathcal{U}_2$.*

*We say that $Q \subseteq \mu$ is U-**pure** if it is nondegenerate and if $Q^\nu \subseteq \mu$ for all $\nu = o(ord_U(Q))$.. We say that $Q \subseteq \mu$ is **pure** if it is V-pure for some compact neighbourhood V of 1 such that $V \subseteq \mathcal{U}_2$.*

(4) *G **has no small connected subgroups**, abbreviated as G is NSCS, if there is a neighbourhood of 1 in G that contains no connected subgroup of G other than $\{1\}$. We say that G is **pure** if every nondegenerate Q as above is pure.*

Remark 3.3. We know (from Lemma 2.38) that G is NSS is equivalent to there being no nontrivial internal subgroups of μ. It is clear that the existence of a nontrivial internal subgroup of μ is equivalent to the existence of a such that a^i is defined for all i and $\{a^i \mid i \in \mathbb{Z}^*\} \subseteq \mu$. Therefore, G is NSS is equivalent to there being no degenerate elements in μ.

Lemma 3.4. *Let $a \in \mu$ and $\nu > \mathbb{N}$. Then the following are equivalent:*

(1) *a^i is defined and $a^i \in \mu$ for all $i \in \{1, \ldots, \nu\}$;*
(2) *$a \in G^o(\nu)$;*
(3) *there is $\tau \in \{1, \ldots, \nu\}$ such that $\nu \in O(\tau)$ and a^i is defined and $a^i \in \mu$ for all $i \in \{1, \ldots, \tau\}$.*

Proof.

((2) => (1)): Since for all $i \in \{1, \ldots, \nu\}$ we have $i \in O(\nu)$, then by the definition of $G^o(\nu)$ if $a \in G^o(\nu)$, then (1) holds.

((1) => (3)): Assume (1) Then there certainly is a τ that satisfies (3), namely $\tau = \nu$.

((3) => (1)): Assume $\tau \in \{1, \ldots, \nu\}$ is such that $\nu \in O(\tau)$ and a^i is defined and $a^i \in \mu$ for all $i \in \{1, \ldots, \tau\}$. We first show that a^j is defined for all $j \in \{1, \ldots, \nu\}$ by internal induction.

The base case clearly holds (a is defined), so suppose a^j is defined for $j < \nu$. To prove that a^{j+1} is defined it suffices to show (by Lemma 2.36) that $(a^k, a^l) \in \Omega^*$ for all $k, l \in \{1, \ldots, j\}$ with $k + l = j + 1$. However, $(a^k, a^l) \in \mu \times \mu \subset \Omega^*$. Therefore, by internal induction, a^j is defined for all $j \in \{1, \ldots, \nu\}$.

By our assumptions $a^j \in \mu$ for $j \in \{1, \ldots, \tau\}$. Now let $i \in \{\tau, \ldots, \nu\}$ (if this set is empty there is nothing to prove). We can now write i as $i = n\tau + \eta$ for $n > 0, \eta < \tau$. By Lemma 2.36, again, we have

$$a^i = (a^\tau)^n \cdot a^\eta \in \mu \cdot \mu \subset \mu,$$

which completes the proof. □

Lemma 3.5. *Let $a_1, \ldots a_\nu$ be an internal sequence of elements of G^* with $\nu > \mathbb{N}$ such that for all $i \in \{1, \ldots, \nu\}$ we have $a_i \in \mu, a_1 \cdots a_i$ is defined and $a_1 \cdots a_i \in G^*_{ns}$. Then the set*

$$S := \{st(a_1 \cdots a_i) \,:\, 1 \leq i \leq \nu\} \subseteq G$$

is compact, connected and contains 1.

Proof. Clearly, S contains 1.

Since $S^* = \{a_1 \cdots a_i \mid 1 \leq i \leq \nu\}$ and $a_1 \cdots a_i$ is defined with $a_1 \cdots a_i \in G^*_{ns}$, then $S^* \subseteq G^*_{ns}$. Also, $st(a) \in S$ for all elements $a \in S^*$ implies that S is closed. By Remark 2.14 S is compact.

Suppose S is not connected. Then, there exist $V, W \subseteq G$ such that V, W are open, $V \cap W = \emptyset, S \subseteq V \cap W, S \cap V \neq \emptyset$ and $S \cap W \neq \emptyset$. Assume $1 \in V$, so $a_1 \in V^*$. Let

$$i = min\{j \in \{1, \ldots, \nu\} \mid a_1 \cdots a_j \in W^*\}.$$

(This minimum exists by overflow.) Clearly $i \geq 2$. Consider $a_1 \cdots a_{i-1}$.

By assumption $a_1 \cdots a_{i-1} \in V^*$, so $st(a_1 \cdots a_{i-1}) \in V$. Also we have that $st(a_1 \cdots a_i) \in W$, since $a_1 \cdots a_i \in W^*$. But this is impossible since $a_i \in \mu$.

So S is connected. □

Lemma 3.6. *Suppose $a \in \mu$ and $a^i \neq \mu$ for some $i \in o(ord_U(a))$. Then U contains a nontrivial connected subgroup of G.*

Proof. For each $\sigma \in o(ord_U(a))$ set

$$S_\sigma := \{st(a^i) \mid 1 \leq i \leq \sigma\} \subseteq G,$$

as in the previous lemma. Then we set

$$G_U(a) := \bigcup_{\sigma \in o(ord_U(a))} S_\sigma.$$

Since for all $\sigma \in o(ord_U(a))$ $\sigma < ord_U(a)$, we have that a^i is defined and $a^i \in U^*$ for all $i \in \{1, \ldots, \sigma\}$. Therefore, for all σ $S_\sigma \subseteq U$. Hence, $G_U(a)$ is a union of connected subsets of U, all containing 1 (by Lemma refstandardimage.lemma). So $G_U(a)$ is connected.

$G_U(a)$ is also a subgroup of G by Lemma 2.36 and Corollary 2.37. □

The previous lemma also shows that if U contains no nontrivial connected subgroups of G, then every $a \in \mu$ which is nondegenerate is U-pure.

Lemma 3.7. *Let $a \in \mu$. Then a is pure if and only if there is a $\nu > \mathbb{N}$ such that a^ν is defined, $a^\nu \neq \mu$ and $a \in G(\nu)$.*

Proof. Assume $a \in \mu$.

(=>): Assume a is V-pure for some compact neighbourhood V of 1 with $V \subseteq \mathcal{U}_2$. Put $\nu = ord_V(a)$. Then by the definition of V-pure, $a \in G(\nu)$. Also, by the definition of $ord_V(a)$, a^ν is defined and $a^\nu \in V^*$. It is clear that $a^{\nu+1}$ is defined as well, so $a^\nu \notin \mu$, because otherwise $a^\nu \cdot a \in \mu \cdot \mu \subseteq \mu \subseteq V^*$ contradicting $a^{\nu+1} \notin V^*$ from the definition of $ord_V(a)$. Finally, since μ is a subgroup of G^* and $a^\nu \notin \mu$, then certainly $\nu > \mathbb{N}$.

(<=): Assume $\nu > \mathbb{N}, a^\nu$ is defined, $a^\nu \notin \mu$ and $a \in G(\nu)$. We want to find a compact neighbourhood V of 1 with $V \subseteq \mathcal{U}_2$ such that a is V-pure, i.e. $a \in G(ord_V(a))$.

Since $a^\nu \notin \mu$ we can pick a compact neighbourhood V of 1 with $V \subseteq \mathcal{U}_2$ such that $a^\nu \notin V^*$. Set $\tau = ord_V(a)$. Here the definition of $ord_V(a)$ directly implies that $\tau < \nu$. Therefore, $a \in G(\tau)$ and so a is V-pure.

□

Lemma 3.8. *Suppose $Q^\nu \nsubseteq \mu$ for some $\nu \in o(ord_U(Q))$. Then U contains a nontrivial connected subgroup of G.*

Proof. Let $(a_\nu) := (a_1, \ldots, a_\nu)$ be a special kind of internal sequence - one which is such that $\nu \in o(ord_U(Q))$ and for all $i \in \{1, \ldots, \nu\}$ $a_1 \cdots a_i$ is defined, $a_i \in Q$, and $a_1 \cdots a_i \in G^*_{ns}$. Now set

$$S_{(a_\nu)} := \{st(a_1 \cdots a_i) \mid 1 \leq i \leq \nu\} \subseteq G,$$

as in Lemma 3.5. It is clear that $Q^\nu = \bigcup_{(a_\nu)} \{a_1 \cdots a_i \mid 1 \leq i \leq \nu\}$. Now put

$$G_U(Q) := \bigcup_{(a_\nu)} S_{(a_\nu)}.$$

Since $\nu \in o(ord_U(Q))$ $\nu < ord_U(Q)$, we have that $S_{(a_\nu)} \subseteq U$ for all sequences (a_ν). Then, by Lemma 3.5 $G_U(Q)$ is a union of connected subsets of U, each containing 1. So $G_U(Q)$ is connected.

To see that $G_U(Q)$ is a group, let $(a_\eta), (b_\tau)$ be two sequences as defined above. Then $\eta + \tau \in o(ord_U(Q))$ and by Lemma 2.36 $(c_{\eta+\tau}) = (a_1, \ldots, a_\eta, b_1, \ldots, b_\tau)$ is a sequence as defined above and the product of the members of (a_η) and (b_τ) is necessarily the product of the elements of $(c_{\eta+\tau})$. Therefore, $S_{(c_{\eta+\tau})} \subseteq G_U(Q)$. Again by Lemma 2.36 it follows easily that $G_U(Q)$ is closed under inverses. □

The previous lemma also shows that if U contains no nontrivial connected subgroups of G, then every Q which is nondegenerate is U-pure.

Lemma 3.9. *Q is pure if and only if there is some $\nu > \mathbb{N}$ such that Q^ν is defined, $Q^\nu \nsubseteq \mu$ and $Q^\tau \subseteq \mu$ for all $\tau \in o(\nu)$.*

Proof. The proof of this lemma is *exactly* the same as the proof of Lemma 3.7. □

Corollary 3.10. *If G is NSCS, then G is pure.*

We can now prove the main theorem of the chapter.

Theorem 3.11. *Suppose $a \in \mu$ is U-pure and $\tau = ord_U(a)$. Then there is a nontrivial LPS of G $X : (-r_{\tau,a,U}, r_{\tau,a,U}) \to G$ such that if we denote by $a_s \in G^*$ the point $X^*(\frac{1}{\tau})$, then $a^i \sim a_s^i$ for all $i \in O(\tau)$.*

Proof. Note that a being U-pure with $\tau = ord_U(a)$ means that if $\frac{i}{\tau} \sim 0$, then $a^i \in \mu$. More generally, if a^i, a^j are defined, then $\frac{i}{\tau} \sim \frac{j}{\tau}$ implies $\frac{i-j}{\tau} \sim 0$ and so $a^{i-j} \in \mu$. Hence $a^i \sim a^j$. Let

$$r_{\tau,a,U} := sup\{r \in \mathbb{R}^+ \mid \text{there exists } i \text{ with } \frac{i}{\tau} \sim r,$$
$$a^i \text{ is defined and } a^j \in U^* \text{ if } |j| \leq i\}.$$

If this supremum does not exist put $r_{\tau,a,U} = \infty$. We know that a^i is defined for all $a \in O(\tau)$. Moreover, since $a^\tau \in U^*$, then $a^\tau \in G^*_{ns}$ and $a^i \in G^*_{ns}$ for all $i \in O(\tau)$.

We now describe the function X. Let

$$X : (-r_{\tau,a,U}, r_{\tau,a,U}) \to G \text{ be defined by}$$
$$X : t \mapsto st(a^i) \text{ for } i \text{ such that } \frac{i}{\tau} \sim t.$$

By what we noted at the beginning of the proof, this gives a well-defined standard map X. Since $U \subseteq \mathcal{U}_2$, we have that $image(X) \subseteq \mathcal{U}_2$. We now prove that X is an LPS of G.

Let $s_1, s_2, s_1 + s_2 \in (0, r_{\tau,a,U})$. Let $\frac{i_1}{\tau} \sim s_1, \frac{i_2}{\tau} \sim s_2$. We then clearly have $\frac{i_1+i_2}{\tau} \sim s_1 + s_2$. But also $a^{i_1} \sim X(s_1), a^{i_2} \sim X(s_2)$ and $a^{i_1} \cdot a^{i_2} \sim X(s_1) \cdot X(s_2)$. So we also have

$$a^{i_1} \cdot a^{i_2} = a^{i_1+i_2} \sim X(s_1 + s_2).$$

Thus $X(s_1+s_2) \sim X(s_1)\cdot X(s_2)$, which means equality for standard group elements. The above result, combined with Lemma 2.36 and Corollary 2.37 give that X is a homomorphism.

Finally, we need to show that X is continuous. By Lemma 2.34 it is sufficient to prove that X is continuous at 0. By Remark 2.14 it is enough to show that $X^*(\mu(0)) \subseteq \mu(X(0)) = \mu$.

Let $s \in \mu(0)$. Then there is i with $\frac{i}{\tau} \sim s \sim 0$. But then, $i \in o(\tau)$ and so $a^i \in \mu$.

So X is an LPS of G.

Therefore, if we set $a_s = X^*(\frac{1}{\tau})$, then clearly, $a^i \sim a_s^i$ for all $i \in O(\tau)$. □

At the end of this chapter we prove an extension lemma that will be useful in a later argument. We denote $I := [-1,1] \subseteq \mathbb{R}$.

Lemma 3.12. *Suppose $X : I \to G$ is a continuous function such that for all $r, s \in I$ with $r+s \in I$ we have $(X(r), X(s)) \in \Omega$ and $X(r+s) = X(r)\cdot X(s)$. Assume further that image$(X) \subseteq \mathcal{U}_4$. Then there exists $\epsilon \in \mathbb{R}^+$ and an LPS $\overline{X} : (-1-\epsilon, 1+\epsilon) \to G$ of G such that $\overline{X}|_I = X$.*

Proof. Fix $\epsilon \in (0, \frac{1}{2})$. Then define $\overline{X} : (-1-\epsilon, 1+\epsilon) \to G$ by

$$\overline{X} : t \mapsto \begin{cases} X(t) & \text{for } t \in I \\ X(1)\cdot X(t-1) & \text{for } t \in (1, 1+\epsilon) \\ X(t+1)\cdot X(-1) & \text{for } t \in (-1-\epsilon, -1) \end{cases}$$

We want to show that for all $r_1, r_2 \in (-1-\epsilon, 1+\epsilon)$, if $r_1 + r_2 \in (-1-\epsilon, 1+\epsilon)$, then $(\overline{X}(r_1), \overline{X}(r_2)) \in \Omega$ and $\overline{X}(r_1+r_2) = \overline{X}(r_1)\cdot\overline{X}(r_2)$.

Since $\overline{X}(r_1), \overline{X}(r_2) \in \mathcal{U}_4^2$ for all $r_1, r_2 \in (-1-\epsilon, 1+\epsilon)$, the $(\overline{X}(r_1), \overline{X}(r_2)) \in \Omega$ part is clear.

For $\delta \in (-\epsilon, \epsilon)$ one of $X(\delta+1), X(\delta-1)$ might not be defined. For that reason we will need to first prove a small claim before we can show the additivity property of $\overline{X}$. We claim that if $\delta \in (-\epsilon, \epsilon)$, then $X(1)\cdot X(\delta) = X(\delta)\cdot X(1)$. For $\delta \geq 0$ we have

$$\begin{aligned} X(1)\cdot X(\delta) &= X(1)\cdot X(\delta-1)\cdot X(1) \\ &= X(\delta)\cdot X(1) \end{aligned}$$

For $\delta < 0$ we have

$$\begin{aligned} X(1)\cdot X(\delta) &= X(1-\delta)\cdot X(\delta)\cdot X(\delta) \\ &= X(\delta)\cdot X(1-2\delta)\cdot X(2\delta) \\ &= X(\delta)\cdot X(1) \end{aligned}$$

With this the claim is verified. Since $\epsilon \in (0, \frac{1}{2})$ and we have $\overline{X}(r) = (\overline{X}(-r))^{-1}$ for all $r \in (-1-\epsilon, -1) \cup (1, 1+\epsilon)$, we just need to prove the "positive" cases of additivity.

Case 1: Let $r_1, r_2, r_1+r_2 \in I$. Then $\overline{X}(r_1+r_2) = \overline{X}(r_1)\cdot\overline{X}(r_2)$ is clear by $\overline{X}|_I = X$ and the properties of X.

Case 2: Let $r_1, r_2 \in I, r_1+r_2 \in (1, 1+\epsilon)$. Then at least one of r_1 and r_2 is strictly positive. WLOG let $r_1 > 0$. Then

$$\begin{aligned}\overline{X}(r_1+r_2) &= X(1)\cdot X(r_1+r_2-1)\\ &= X(1)\cdot X(r_1-1)\cdot X(r_2)\\ &= X(r_1)\cdot X(r_2)\\ &= \overline{X}(r_1)\cdot \overline{X}(r_2)\end{aligned}$$

Case 3: Let $r_1 \in I, r_2 \in (1, 1+\epsilon), r_1+r_2 \in I$. So $r_1 < 0$. Then

$$\begin{aligned}\overline{X}(r_1+r_2) &= X(r_1+r_2)\\ &= X(r_1+1)\cdot X(r_2-1)\\ &= X(r_1)\cdot X(1)\cdot X(r_2-1)\\ &= \overline{X}(r_1)\cdot \overline{X}(r_2)\end{aligned}$$

Case 4: Let $r_1 \in I, r_2 \in (1, 1+\epsilon), r_1+r_2 \in (1, 1+\epsilon)$. Then

$$\begin{aligned}\overline{X}(r_1+r_2) &= X(1)\cdot X(r_1+r_2-1)\\ &= X(1)\cdot X(r_1)\cdot X(r_2-1)\\ &= X(r_1)\cdot X(1)\cdot X(r_2-1) \quad \text{by the claim above}\\ &= \overline{X}(r_1)\cdot \overline{X}(r_2)\end{aligned}$$

Case 5: Let $r_1 \in (-1-\epsilon, -1), r_2 \in (1, 1+\epsilon)$, so $r_1+r_2 \in I$. Then

$$\begin{aligned}\overline{X}(r_1+r_2) &= X(r_1+r_2)\\ &= X(r_1+1)\cdot X(r_2-1)\\ &= X(r_+1)\cdot 1\cdot X(r_2-1)\\ &= X(r_1+1)\cdot X(-1)\cdot X(1)\cdot X(r_2-1)\\ &= \overline{X}(r_1)\cdot \overline{X}(r_2)\end{aligned}$$

$\square$

4. Consequences of NSS

For this section assume G is NSS.

Definition 4.1. *A* ***special neighbourhood*** *of G is a compact symmetric neighbourhood $\mathcal{U}$ of 1 in G such that $U \subseteq \mathcal{U}$, $\mathcal{U}$ contains no nontrivial subgroup of G, and for all $x, y \in \mathcal{U}$, if $x^2 = y^2$, then $x = y$.*

Lemma 4.2. *G has a special neighbourhood.*

Proof. We pick three symmetric neighbourhood of 1 $U \supseteq W \supseteq V$ in the following way:

- $U \subset \mathcal{U}_6$, U is compact and U contains no subgroups of G other than $\{1\}$;
- $W^3 \subseteq U$;
- $V^2 \subseteq W$, V is compact and $gVg^{-1} \subseteq W$ for all $g \in U$.

We can make the last choice, because, intuitively, since U is compact, shrinking V is an effective means of shrinking gVg^{-1}.

We now claim that V is a special neighbourhood. We will prove this by picking $x, y \in V$ with $x^2 = y^2$ and showing that $a := x^{-1}y$ generates a subgroup of U. By our choice of U that would mean that $x^{-1}y = 1$, i.e. $x = y$.

We see by induction that a^n is defined, $a^n \in U$ and $a^n = xa^{-n}x^{-1}$ for all n:

Base case.: $x^{-1}y$ is defined, because $V \subseteq \mathcal{U}_2$, and $x^{-1}y \in U$, because V is symmetric with $V^2 \subseteq W \subseteq U$. Also, since $x^2 = y^2$ and $V \subseteq \mathcal{U}_4$, then $1 = x^{-2}y^2$. Then

$$\begin{aligned} xa^{-1}x^{-1} &= x(x^{-1}y)^{-1}x^{-1} && (V \subseteq \mathcal{U}_4) \\ &= xy^{-1}xx^{-1} \\ &= x \cdot 1 \cdot y^{-1} \\ &= xx^{-2}y^2y^{-2} && (V \subseteq \mathcal{U}_6) \\ &= x^{-1}y = a \end{aligned}$$

Inductive hypothesis.: For all $m < n$, a^m is defined, $a^m \in U$ and $a^m = xa^{-m}x^{-1}$.

Inductive step.: To show that a^n is defined we need to show, by Lemma 2.36, that $(a^i, a^j) \in \Omega$ for all $i, j \in \{1, \ldots, n-1\}$ such that $i + j = n$. But by IH $(a^i, a^j) \in U \times U \subseteq \Omega$, so a^n is defined. Now,

$$\begin{aligned} a^n = a^{n-1} \cdot a &= (xa^{-n+1}x^{-1}) \cdot (xa^{-1}x^{-1}) \quad (\text{since } U \subseteq \mathcal{U}_6) \\ &= xa^{-n}x^{-1}. \end{aligned}$$

Remains to show that $a^n \in U$.

If $n = 2k$ for $k \in \mathbb{N}$, then

$$a^n = a^k \cdot a^k = xa^{-k}x^{-1}a^k \in xW \subseteq W^2 \subseteq U.$$

If $n = 2k + 1$ for $k \in \mathbb{N}$, then

$$a^n = a^k a^k a = xa^{-k}x^{-1}a^k a \in xW \cdot W \subseteq W^3 \subseteq U.$$

By Lemma 2.36 we now have that a^k is defined for all $k \in \mathbb{Z}$. This produces a subgroup in U. So $x = y$. $\square$

For the rest of the section we fix a special neighbourhood $\mathcal{U}$ of G. Then, since G is NSS, every $a \in \mu \backslash \{1\}$ is $\mathcal{U}$-pure. For $a \in G^*$, we set $ord(a) := ord_{\mathcal{U}}(a)$.

Lemma 4.3. *Let $a \in G^*$. Then $a \in \mu$ if and only if $ord(a) > \mathbb{N}$.*

Proof. We have seen before that, since μ is a subgroup, if $a \in \mu$, then we have $ord_U(a) > \mathbb{N}$ for any compact symmetric neighbourhood U of 1 with $U \subseteq \mathcal{U}_2$. Now, assume $ord(a) > \mathbb{N}$. Then a^k is defined and $a^k \in U^*$ for all $k \in \mathbb{Z}$. By induction, it is easy to see that $st(a)^k$ is defined, and $st(a)^k \in \mathcal{U}$, since $\mathcal{U}$ is compact. But then $st(a)$ generates a subgroup of G in $\mathcal{U}$. This implies that $st(a) = 1$, i.e. $a \in \mu$. □

Lemma 4.4. *Suppose $\sigma > \mathbb{N}$ and $a \in G(\sigma)$. Then $\sigma \in O(ord(a))$ and a^i is defined with $a^i \in \mu$ for all $i \in o(\sigma)$.*

Proof. $a \in G(\sigma)$ means that a^i is defined and $a^i \in \mu$ for all $i \in o(\sigma)$.

Suppose for a contradiction that $\sigma \notin O(ord(a))$. This is equivalent to saying that $ord(a) \in o(\sigma)$. But then $a^{ord(a)} \in \mu$, which implies $a^{ord(a)+1} \in \mu$ and this contradicts the maximality of $ord(a)$. So $ord(a) \notin o(\sigma)$, i.e. $\sigma \in O(ord(a))$.

Since $\sigma \in O(ord(a))$, then if $i \in o(\sigma)$, we have $i \in o(ord(a))$. This implies that a^i is defined and $a^i \in \mu$. □

Let $r_a := r_{\sigma,a,\mathcal{U}}$. We then denote by $X_a : (-r_a, r_a) \to G$ the LPS of G formed by $(\sigma, a, \mathcal{U})$ as in Theorem 3.11.

Lemma 4.5. *Suppose G is not discrete. Then $L(G) \neq \mathbb{O}$.*

Proof. Since G is not discrete we can pick $a \in \mu \backslash \{1\}$. Let $\sigma := rd(a)$ and X_a be as above. We want to show that $[X_a] \neq \mathbb{O}$. Since we are in the local setting, we will have to prove that no neighbourhood of $0_{\mathbb{R}}$ is sent to 1_G by X_a. It will be enough to show that for all n, $X_a(\frac{1}{n}) \neq 1_G$, i.e. that for all n, $a^i \notin \mu$ for some (and therefore all) i with $\frac{i}{\sigma} \sim \frac{1}{n}$.

Fix $n \in \mathbb{N}$ such that $|\frac{1}{n}| < r_a$. Then $\frac{\frac{\sigma}{n}}{\sigma} \sim \frac{1}{n}$. Suppose for a contradiction that $a^{\frac{\sigma}{n}} \in \mu$. We know that a^σ is defined and so by Lemma 2.36 $(a^{\frac{\sigma}{n}})^n$ is defined. Thus $(a^{\frac{\sigma}{n}})^n \in \mu^n \subseteq \mu$. But then $a^{\sigma+1} = a^{ord(a)+1} \in \mu$, which is impossible. □

Lemma 4.6. *Let $\sigma > \mathbb{N}$ and let $a \in G(\sigma) \backslash \{1\}$. Then:*

(1) $a^{-1} \in G(\sigma)$ and $[X_{a^{-1}}] = (-1) \cdot [X_a]$, where the operation here is the scalar multiplication defined in Remark 2.31;
(2) $b \in \mu \implies bab^{-1} \in G(\sigma)$ and $[X_{bab^{-1}}] = [X_a]$;
(3) $[X_a] = \mathbb{O} \iff a \in G^o(\sigma)$;
(4) $L(G) = \{[X_a] \,|\, a \in G(\sigma)\}$.

Proof. Suppose $\sigma > \mathbb{N}$ and $a \in G(\sigma)$.

(1) If $a \in G(\sigma)$, then $a^{-1} \in G(\sigma)$ by the definition of $G(\sigma)$. Moreover, by the definition of the scalar multiplication in the vector space of LPSs, it is clear that $[X_{a^{-1}}] = (-1) \cdot [X_a]$.

(2) Let $b \in \mu$. Let $\tau := ord(a)$. By Lemma 4.4 $\sigma \in O(\tau)$.We show by internal induction on η that $(bab^{-1})^\eta$ is defined and $(bab^{-1})^\eta = ba^\eta b^{-1}$ for all $\eta \leq \tau$, the case $\eta = 1$ being clear. Now, suppose the claim is true for $i < \eta$ and consider $(bab^{-1})^\eta$.

We have a^i defined and $a^i \in \mathcal{U}^*$ for all $|i| \leq \tau = ord(a)$. Then

$$(a^i, a^j) \in \mathcal{U}^* \times \mathcal{U}^* \subseteq \mathcal{U}_2^* \times \mathcal{U}_2^* \subseteq \Omega^*$$

for all $i, j \in \{1, \dots, \eta - 1\}$ with $i + j = \eta$. So a^η is defined. Then, since $b \in \mu$, $ba^\eta b^{-1}$ is defined. Since all of $b^{-1}b, ba^{\eta-1}b^{-1}, bab^{-1}$ are defined, then

$$\begin{aligned} ba^\eta b^{-1} &= ba^{\eta-1}b^{-1}bab^{-1} \\ &= (bab^{-1})^{\eta-1}(bab^{-1}) \quad \text{(by IH)} \\ &= (bab^{-1})^\eta \end{aligned}$$

Therefore $(bab^{-1})^\eta$ is defined and equal to $ba^\eta b^{-1}$ for all $\eta \leq \tau$. Since $\sigma \in O(\tau)$, we have that $(bab^{-1})^i$ is defined and $(bab^{-1})^i \in \mu$ for all $i \in o(\sigma)$ and hence $bab^{-1} \in G(\sigma)$.

Take i such that $\frac{i}{\sigma} \sim r$ with $|r| < r_a$. Then we know that a^i is defined and $a^i \in \mathcal{U}^*$. Thus $i \leq ord(a) = \tau$. Therefore, by the above, $(bab^{-1})^i$ is defined and $(bab^{-1})^i = ba^ib^{-1}$.

Since $\mathcal{U}$ is a standard neighbourhood, there is an $r_{bab^{-1}} \leq r_a$ such that for all i with $\frac{i}{\sigma} \sim r < r_{bab^{-1}}$, $(bab^{-1})^i \in \mathcal{U}^*$. Because $b \in \mu$, this means that the standard functions X_a and $X_{bab^{-1}}$ agree on $(-r_{bab^{-1}}, r_{bab^{-1}})$. Therefore $[X_a] = [X_{bab^{-1}}]$.

(3) $[X_a] = \mathbb{O}$ means that there is an $r \leq r_a$, such that $X_a|_{(-r,r)} = O$. So for all i such that $\frac{i}{\sigma} \sim q < r, a^i \in \mu$. By Lemma 3.4, this means that for all $i \in O(\sigma), a^i \in \mu$. Thus $a \in G^o(\sigma)$. The other direction follows immediately.

(4) Let $\mathbb{X} \in L(G)$ and $X \in \mathbb{X}$. Denote $b = X^*(\frac{1}{\sigma})$. Suppose X and X_b are both defined on $(-r, r)$. Then, if i is such that $\frac{i}{\sigma} \sim q \in (-r, r)$ we have

$$\begin{aligned} X(q) &\sim X^*(\frac{i}{\sigma}) \\ &= X^*(\frac{1}{\sigma})^i \\ &= b^i \\ &\sim X_b(q) \quad \text{(by Theorem 3.11)} \end{aligned}$$

So $X(q) \sim X_b(q)$ for $q \in (-r, r)$, which means equality for standard functions. Therefore $[X] = [X_b]$ and so $L(G) = \{[X_a] \,|\, a \in G(\sigma)\}$.

□

We now investigate the local analogue of the exponential map. Consider the following sets

$$\mathcal{K} := \{\mathbb{X} \in L(G) \,|\, I \subseteq domain(\mathbb{X}) \text{ and } \mathbb{X}(I) \subseteq \mathcal{U}\}$$

and

$$K := \{\mathbb{X}(1) \,|\, \mathbb{X} \in \mathcal{K}\}.$$

Lemma 4.7. *For every $\mathbb{X} \in L(G)$, there is an $s \in (0, 1)$ such that $s \cdot \mathbb{X} \in \mathcal{K}$.*

Proof. Take $\mathbb{X} \in L(G)$ and $X \in \mathbb{X}$. Let $domain(X) = (-r, r)$. If $r \leq 1$, we pick s_1 to be such that $0 < s_1 < r$. Then $s_1 \cdot X : (-\frac{r}{s_1}, \frac{r}{s_1}) \to G$ has $domain(s_1 \cdot X) \supseteq I$. If $r > 1$, let $s_1 = 1$. The continuity of $s_1 \cdot X$ allows us to pick s_2 with $0 < s_2 < 1$ such that $(s_2 \cdot (s_1 \cdot X))(I) \subseteq \mathcal{U}$. So for $s := s_1 \cdot s_2, s \cdot \mathbb{X} \in \mathcal{K}$. □

Lemma 4.8. *The map $E : \mathcal{K} \to K$ defined by $\mathbb{X} \mapsto \mathbb{X}(1)$ is bijective.*

Proof. Let $\mathbb{X}_1, \mathbb{X}_2 \in \mathcal{K}$, $X_1 \in \mathbb{X}_1$, $X_2 \in \mathbb{X}_2$, $domain(X_1) \cap domain(X_2) \supseteq I$. Suppose $X_1(1) = X_2(1)$. Then $(X_1(\frac{1}{2}))^2 = (X_1(\frac{1}{2}))^2$, which, since $\mathcal{U}$ is a special neighbourhood, implies $X_1(\frac{1}{2}) = X_2(\frac{1}{2})$. Repeating this inductively we have

$X_1(\frac{1}{2^n}) = X_2(\frac{1}{2^n})$ for all n, and thus $X_1(\frac{k}{2^n}) = X_2(\frac{k}{2^n})$ for all $k \in \mathbb{Z}$ such that $|\frac{k}{2^n}| \leq 1$. Since X_1, X_2 are continuous with a dense domain, then the above implies $X_1|_I = X_2|_I$, i.e. $[X_1] = [X_2]$. This proves injectivity. The surjectivity part of the claim is trivial. □

Finally, we obtain a countable neighbourhood basis of the identity. Let the partial map $p_n : G \rightharpoonup G$, be defined by $p_n : a \mapsto a^n$ if a^n is defined. By Lemma 2.39 we have that p_n has an open domain with $domain(p_n) \supseteq \mathcal{U}_n$ and is continuous.

For Q as before, we set $ord(Q) := ord_{\mathcal{U}}(Q)$. We extend the $ord(.)$ notation to standard subsets of G analogously - for symmetric $P \subset G$ with $1 \in P$ we let

$$ord(P) := max\{n \,|\, P^n \text{ is defined and } P^n \subseteq \mathcal{U}\},$$

if this maximum exists, otherwise we put $ord(P) = \infty$ when P^n is defined and $P^n \subseteq \mathcal{U}$ for all n.

We now define $V_n := \{x \in G \,|\, ord(x) \geq n\}$. We note that for all n

$$p_1^{-1}(\mathcal{U}) \cap \ldots \cap p_n^{-1}(\mathcal{U}) \subseteq V_n \subseteq \mathcal{U}_n.$$

Lemma 4.9. $(V_n \,|\, n \geq 1)$ *is a decreasing sequence of compact symmetric neighbourhoods of* 1 *in* G*,* $ord(V_n) \to \infty$ *as* $n \to \infty$*, and* $\{V_n \,|\, n \geq 1\}$ *is a countable neighbourhood basis of* 1 *in* G*.*

Proof. For $\sigma > \mathbb{N}$, consider the internal set

$$V_\sigma := \{g \in G^* \,|\, ord(g) \geq \sigma\}.$$

Since $ord(g) \geq \sigma > \mathbb{N}$ for all $g \in V_\sigma$, then $V_\sigma \subseteq \mu$. So given any neighbourhood U of 1 in G, we have $x_1 \cdots x_m$ is defined and in U^* for all m and $x_1, \ldots, x_m \in V_\sigma$. It follows that for any neighbourhood U of 1 in G and any m we have $(V_n)^m$ is defined and contained in U for all sufficiently large n. This shows that $ord(V_n) \to \infty$ as $n \to \infty$.

It is clear from the definition of the sets V_n that $(V_n \,|\, n \geq 1)$ is a decreasing sequence of compact symmetric neighbourhoods of 1 in G. Then $\{V_n \,|\, n \geq 1\}$ gives a countable neighbourhood basis of the identity in G. □

5. Local Gleason-Yamabe Lemmas

The trickiest and most technical part of our solution is obtaining local analogues of the Gleason-Yamabe Lemmas. The particular proofs, however, are highly unenlightening and tedious. Seeing as diving into them would be a laborious detour from our discussion so far, we will just cite the direct results from the lemmas for further use. Anyone interested in the derivations of the results in this section is encouraged to read through section 5 of Goldbring's paper [5].

Lemma 5.1. *Suppose $\nu \in \mathbb{N}^*\backslash\mathbb{N}$ and $a_1, \ldots, a_\nu$ is a hyperfinite sequence such that $a_i \in G^o(\nu)$ for all $i \in \{1, \ldots, \nu\}$. Let $Q := \{1, a_1, \ldots, a_\nu, a_1^{-1}, \ldots, a_\nu^{-1}\}$. Then Q^ν is defined and $Q^\nu \subseteq \mu$.*

Lemma 5.2. *Suppose U is a compact symmetric neighbourhood of 1 in G with $U \subseteq \mathcal{U}_2$. Let $\nu > \mathbb{N}$ be such that for all $i \in \{1, \ldots, \nu\}, a^i$ and b^i are defined and $a^i \in U^*, b^i \in \mu$. Then for all $i \in \{1, \ldots, \nu\}$, we have that $(ab)^i$ is defined and $(ab)^i \sim a^i$.*

Lemma 5.3. *Let $\nu > \mathbb{N}$ and $a \in G(\nu)$ be such that a^ν is defined and $a^i \in G^*_{ns}$ for all $i \in \{1, \ldots, \nu\}$. Suppose also that $b \in \mu$ is such that b^ν is defined and $b^i \in G^*_{ns}$ and $a^i \sim b^i$ for all $i \in \{1, \ldots, \nu\}$. Then $a^{-1}b \in G^o(\nu)$.*

Theorem 5.4. *Suppose $\sigma > \mathbb{N}$. Then*

(1) $G(\sigma)$ and $G^o(\sigma)$ are normal subgroups of μ;
(2) if $a \in G(\sigma)$ and $b \in \mu$, then $aba^{-1}b^{-1} \in G^o(\sigma)$;
(3) $G(\sigma)/G^o(\sigma)$ is abelian.

6. Consequences of the Gleason-Yamabe Lemmas

In this section we use the results in the previous section to derive several key theorems. We begin with a theorem that allows us to equip the space of LPSs $L(G)$ with an abelian group operation.

Remark 6.1. Let $\mathbb{X} \in L(G)$ and $\sigma > \mathbb{N}$. Suppose $i \in o(\sigma)$. Then $(\mathbb{X}(\frac{1}{\sigma}))^i = \mathbb{X}(\frac{i}{\sigma})$ and since $\frac{i}{\sigma} \sim 0$, we have $\mathbb{X}(\frac{i}{\sigma}) = 0$. If we consider $\mathbb{X}$ as a nonstandard function, then clearly $\mathbb{X}(\frac{i}{\sigma} \in \mu$. Therefore $\mathbb{X}(\frac{i}{\sigma}) \in G(\sigma)$.

Theorem 6.2. *The map* $S : L(G) \to G(\sigma)/G^o(\sigma)$ *defined by*

$$S : \mathbb{X} \mapsto \mathbb{X}(\frac{1}{\sigma})G^o(\sigma)$$

is a bijection.

Proof. Suppose $S(\mathbb{X}) = S(\mathbb{Y})$. Then if $a = \mathbb{X}(\frac{1}{\sigma}), b = \mathbb{Y}(\frac{1}{\sigma})$ we have $a^{-1}b \in G^o(\sigma)$. Pick a neighbourhood of 1 in G such that $U \subseteq \mathcal{U}_2$. We put $\tau := min\{ord_U(a), \sigma\}$. Then we know that a^i is defined and in U^* and $(a^{-1}b)^i$ is defined and in μ for all $i \in \{1, \ldots, \tau\}$. By Lemma 5.2 this implies that $(a \cdot (a^{-1}b))^i$ is defined and $(a \cdot (a^{-1}b))^i \sim a^i$ for all $i \in \{1, \ldots, \tau\}$. However, $(a \cdot (a^{-1}b))^i = ((aa^{-1}) \cdot b)^i = b^i$, since $(a \cdot (a^{-1}b))^i$ is defined. Therefore $a^i \sim b^i$ for all $i \in \{1, \ldots, \tau\}$. We know that $\sigma \in O(\tau)$, so

$$st(domain(\mathbb{X}) \cap domain(\mathbb{Y}) \cap \{r \mid \frac{i}{\sigma} \sim r \text{ for some } |i| \in \{1, \ldots, \tau\}\}) \neq \{0\}.$$

Therefore $\mathbb{X}$ and $\mathbb{Y}$ agree on a standard neighbourhood of $0_{\mathbb{R}}$, hence $\mathbb{X} = \mathbb{Y}$. So S is injective.

Suppose $b \in G(\sigma)/G^o(\sigma)$. Consider X_b as in Theorem 3.11. Let τ with $\sigma \in O(\tau)$ be such that if $i \in \{1, \ldots, \tau\}$, $\frac{i}{\sigma} \sim r \in \mathbb{R}$, then $r \in domain(X_b)$. We know, again by Theorem 3.11, that if $b' := X_b(\frac{1}{\sigma}) \in \mu$, then $(b')^i \sim b^i$ for all $i \in O(\tau)$. By Lemma 5.3 this implies that $b^{-1}b' \in G^o(\tau) \subseteq G^o(\sigma)$ and hence $bG^o(\sigma) = b'G^o(\sigma)$. By Lemma 4.6 we have that S is onto. □

Remark 6.3. We can now use the bijection from the previous theorem to define an abelian group operation $+_\sigma$ on $L(G)$. We put $\mathbb{X} +_\sigma \mathbb{Y} := S_{-1}(S(\mathbb{X}) \cdot S(\mathbb{Y}))$. That means that if $t \in domain(\mathbb{X} +_\sigma \mathbb{Y})$ and ν is such that $\frac{\nu}{\sigma} \sim t$, then

$$(\mathbb{X} +_\sigma \mathbb{Y})(t) \sim [\mathbb{X}(\frac{1}{\sigma})\mathbb{Y}(\frac{1}{\sigma})]^\nu.$$

In [5, p. 35] Goldbring explains that in the local setting it is not required to prove that the above definition of addition in $L(G)$ is independent of σ. This is because nowhere in our proof of LH5 do we use this independence, and moreover, the independence itself is a consequence of LH5. With this in mind, we will write $\mathbb{X}+\mathbb{Y}$ instead of $\mathbb{X} +_\sigma \mathbb{Y}$.

The next result, that will occupy us until the end of this section, is that the image of the local exponential map (defined in Lemma 4.8) is a neighbourhood of 1 in G. Therefore, we will see that there is an open neighbourhood of 1 in G that is ***ruled by LPS***, by which we mean that every element of this neighbourhood lies on some LPS.

In the rest of the section, we assume that G is NSS and that $\mathcal{U}$ is a special neighbourhood of 1 in G.

Lemma 6.4. *Suppose $\sigma > \mathbb{N}$ and $a, b \in G(\sigma)$. Then $[X_a] + [X_b] = [X_{ab}]$.*

Proof. By the definition of addition in $L(G)$ given in Remark 6.3 we have

$$\begin{aligned} S([X_a] + [X_b]) &= S([X_a]) \cdot S([X_b]) \\ &= (aG^o(\sigma))(bG^o(\sigma)) \\ &= (ab)G^o(\sigma) \\ &= S([X_{ab}]) \end{aligned}$$

Since S in injective, then the above implies that $[X_a] + [X_b] = [X_{ab}]$. □

Lemma 6.5. *Suppose $\sigma > \mathbb{N}$ and $a \in G(\sigma)$. Then $a = bc^2$ with $b \in G^o(\sigma)$ and $c \in G(\sigma)$.*

Proof. Put $c := X_a(\frac{1}{2\sigma}) \in \mu$. Clearly $c \in G(\sigma)$. Now, we have

$$c^{-2} = \left(X_a\left(\frac{1}{2\sigma}\right)\right)^{-2} = X_a\left(-\frac{1}{\sigma}\right) = X_{a^{-1}}\left(\frac{1}{\sigma}\right).$$

Therefore, we have $[X_{c^{-2}}] = [X_{a^{-1}}]$. So

$$[X_{ac^{-2}}] = [X_a] + [X_{c^{-2}}] = [X_a] + [X_{a^{-1}}] = [X_1] = \mathbb{O}.$$

If we put $b := ac^{-2}$, then $[X_b] = \mathbb{O}$ and so $b \in G^o(\sigma)$ by Lemma 4.6. □

Lemma 6.6. *There exists $q \in \mathbb{Q}^+$ such that for all $a, b, c \in \mu$, if $a = bc^2$ and $ord(b) \geq ord(a)$, then $ord(c) \geq q \cdot ord(a)$.*

Proof. First note, that if such a q exists, then taking any q' with $0 < q' < q$ also satisfies the conditions of the lemma. So we are looking for a 'small' such q.

Suppose there is no such q. That is, for all n there exist $a_n, b_n, c_n \in \mu$ such that $a = bc^2, ord(b) \geq ord(a)$, but $ord(c) < \frac{1}{n} \cdot ord(a)$ (equivalently, we can say $n \cdot ord(c) < ord(a)$). By saturation, we can find $a, b, c \in \mu$ such that we get $a = bc^2, ord(b) \geq ord(a)$ and $n \cdot ord(c) < ord(a)$ for all n.

Let $\sigma := ord(c)$ and generate the LPSs $[X_a], [X_b]$ and $[X_c]$ with σ as defined. We clearly have $\sigma \in o(ord(a))$ and $\sigma \in o(ord(b))$. Moreover, G is NSS and hence pure. This means that $a \in G(ord(a))$ and $b \in G(ord(b))$. Therefore $a^i, b^i \in \mu$ for all $i \in O(\sigma)$. But $image(X_a) = \{a^i \mid i \in O(\sigma)\}$ and $image(X_b) = \{b^i \mid i \in O(\sigma)\}$. Hence $\mathbb{O} = [X_a] = [X_b]$. But then

$$\mathbb{O} = [X_a] = [X_{bc^2}] = [X_b] + [X_{c^2}] = \mathbb{O} + 2[X_c].$$

This implies $[X_c] = \mathbb{O}$, which is a contradiction. □

We fix q as in the previous lemma.

Definition 6.7. *For $a, b \in \mu$ let **the commutator** of a and b be $[a, b] := aba^{-1}b^{-1}$. Note that $ab = [a, b]ba$.*

Lemma 6.8. *Let $a \in \mu$. Then for all $\nu \geq 1$, there are $b_\nu, c_\nu \in G^*$ such that $b_\nu \cdot c_\nu \cdot c_\nu$ is defined, $a = b_\nu c_\nu^2$, $ord(b_\nu) \geq \nu ord(a)$ and $ord(c_\nu) \geq q.ord(a)$.*

Proof. First, suppose $a = 1$. Then we put $b_\nu = c_\nu = 1$ for all $\nu \geq 1$. Clearly this choice of b_ν, c_ν satisfies the requirements of the lemma.

So suppose $a \neq 1$. Note that if the claim is true for some ν, then for those b_ν, c_ν we will have $ord(b_\nu), ord(c_\nu) > \mathbb{N}$. Therefore, for the statement to be true we at least need $b_\nu, c_\nu \in \mu$.

$a \in G(\sigma)$ for some $\sigma > \mathbb{N}$. Then, by Lemma 6.5, $a = bc^2$ with $b \in G^o(\sigma)$, $c \in G(\sigma)$. So $\sigma \in o(ord(b))$. Also $a \in G(\sigma) \implies ord(a) \in O(\sigma)$. So therefore $ord(b) \geq ord(a)$. Then, by Lemma 6.6, $ord(c) \geq q.ord(a)$.

Moreover, since $ord(a) \in o(ord(b)), ord(a) \in O(\sigma)$ and $\sigma \in O(ord(a))$, then $ord(b) \geq \sigma.ord(a)$. Then the claim of the lemma holds for some $\sigma > \mathbb{N}$ with b, c as above. It is easy to see that then the claim holds for all $\sigma' < \sigma$ as well. We now prove that the lemma holds for $\sigma + 1$, which is enough, by internal induction.

If $b = 1$, then $ord(b) = \infty \geq \nu.ord(a)$ for all $\nu \in \mathbb{N}^*$ and we are done.

Suppose $b \neq 1$. By Lemma 6.5 again, we have $b', c' \in \mu$ with $b = b'(c')^2$ and $ord(b') \geq \tau.ord(b)$ for some $\tau > \mathbb{N}$. Then $ord(b') \geq \tau\sigma ord(a)$. By Lemma 6.6, we also know that $ord(c') \geq q.ord(b) \geq q.\sigma.ord(a)$. Since μ is an actual group, we have

$$\begin{aligned} a = b'(c')^2c^2 &= b'c'c'cc \\ &= b'c'[c', c]cc'c \\ &= b'[c', [c', c]][c', c]c'cc'c \\ &= b'[c', [c', c]][c', c](c'c)^2. \end{aligned}$$

Let $b'' := b'[c', [c', c]][c', c]$ and $c'' := c'c$. We show that $ord(b'') \geq (\sigma+1)ord(a)$ and $ord(c'') \geq q.ord(a)$.

Let $\alpha = c^{-1}(c')^{-1}, \beta = (c')^{-1}c^{-1}$. Then $ord(\alpha), ord(\beta) \in O(ord(c'))$ and $ord(c') \in O(ord(\alpha)) \cap O(ord(\beta))$.

Then $ord(\alpha) > \mathbb{N}, \alpha \in G(ord(\alpha))$ (because G is pure) and α^i is defined and nonstandard for all $i \in \{1, \ldots, ord(\alpha)\}$. Clearly, $\beta \in \mu$ and β^i is also defined and nonstandard for $i \in \{1, \ldots, ord(\alpha)\}$. Moreover, for i as above,

$$\alpha^i = c^{-1}(c')^{-1} \cdots c^{-1}(c')^{-1} = c^{-1}\beta^{i-1}(c')^{-1} \sim \beta^i.$$

Then, by Lemma 5.3, we have $\alpha^{-1}\beta = c'c(c')^{-1}c^{-1} = [c', c] \in G^o(ord(\alpha))$, i.e.

$$ord([c', c]) \geq \eta_1 ord(c') \geq \eta_1 q\sigma\, ord(a),$$

for $\eta_1 > \mathbb{N}$. Analogously we can find $\eta_2 > \mathbb{N}$ such that

$$ord([c', [c', c]] \geq \eta_1\eta_2 ord(c') \geq \eta_1\eta_2 q\sigma\, ord(a).$$

Therefore

$$\begin{aligned} ord(b'') &= ord(b'.[c', [c', c]].[c', c]) \\ &\geq min\{ord(b'), ord([c', [c', c]]), ord([c', c])\} \\ &\geq (\sigma + 1)ord(a). \end{aligned}$$

Finally, Lemma 6.6 directly implies $ord(c'') \geq q.ord(a)$, which completes the internal induction and the proof. □

Lemma 6.9. *For every $a \in \mu$, there is $b \in \mu$ with $a = b^2$.*

Proof. Let V_n be as in Lemma 4.9. Fix $\sigma > \mathbb{N}$.

Now, by the previous lemma, the internal set of all $\eta \in \mathbb{N}^*$ such that for each $x \in V_\eta^*$ there are $y, z \in \mathcal{U}^*$ such that $y \cdot z \cdot z$ is defined, $x = yz^2$, $ord(y) \geq \sigma.ord(x)$ and $ord(z) \geq q.ord(x)$ includes all infinite η. But we know that $\mathbb{N}^* \backslash \mathbb{N}$ is not an internal subset of $\mathbb{N}^*$ (because $\mathbb{N}$ is not), so there is some $n > 0$ in this set.

Now let $x \in V_n \subseteq V_n^*$. Then there are $y, z \in \mathcal{U}^*$ such that $y \cdot z \cdot z$ is defined, $x = yz^2$ and $ord(y) \geq \sigma.ord(x)$. Since $ord(y) > \mathbb{N}$, we have $y \in \mu$. Therefore $x = st(z)^2$. Hence every element of V_n has a square root in $\mathcal{U}$.

By transfer, the last statement implies that every $a \in V_n^*$ (and, in particular, every $a \in \mu$) has a square root $b \in \mathcal{U}^*$. Moreover, when $a \in \mu$, then $b \in \mu$, because otherwise $st(b^2) = st(b)^2 = 1$ and $\{1, st(b)\}$ would be a subgroup of G in $\mathcal{U}$. $\square$

Lemma 6.10. *Suppose $a \in \mu$, $b \in G^*$, $ord(b) \geq \nu > 0$ and $a = b^\nu$. Then $b^i \in \mu$ for all $i \in \{1, \ldots, \nu\}$.*

Proof. We first prove by induction on n that $b^{n\nu}$ is defined for all n, the case $n = 1$ being true by assumption.

Assume $b^{n\nu}$ is defined. Let $j \in \{1, \ldots, n\nu\}$. Then we can write $j = n'\nu + j'$ for $n' \in \mathbb{N}$, $j' \in \{1, \ldots, \nu - 1\}$. We have

$$b^j = (b^\nu)^{n'} \cdot b^{j'} \sim b^{j'} \in \mathcal{U}^* \subseteq \mathcal{U}_2^*,$$

by $b^\nu \in \mu$, μ being an actual group and $ord(b) \geq \nu$. Therefore $b^j \in \mathcal{U}_2^*$ for all $j \in \{1, \ldots, n\nu\}$. We know that $b^{n\nu} = a^n \in \mu \subseteq \mathcal{U}_2^*$, so if we can prove that '$b^{n\nu+i}$ – defined and in $\mathcal{U}_2^*$' implies '$b^{n\nu+i+1}$ – defined and in $\mathcal{U}_2^*$' for all $i \in \{1, \ldots, \nu - 1\}$, then we will have $b^{n\nu+\nu} = b^{(n+1)\nu}$ – defined and in $\mathcal{U}_2^*$ by internal induction.

Assume $b^{n\nu+i}$ – defined and $b^{n\nu+i} \in \mathcal{U}_2^*$ for $i \in \{1, \ldots, \nu - 1\}$. To prove that $b^{n\nu+i+1}$ is defined we have to show (by Lemma 2.36) that $(b^k, b^l) \in \Omega^*$ for all $k, l \in \{1, \ldots, n\nu + i\}$ with $k + l = n\nu + i + 1$. But for any such k, l, we already know that $(b^k, b^l) \in \mathcal{U}_2^* \times \mathcal{U}_2^* \subseteq \Omega^*$. Hence $b^{n\nu+i+1}$ is defined. Therefore,

$$b^{n\nu+i+1} = (b^\nu)^n \cdot b^{i+1} \sim b^{i+1} \in \mathcal{U}^* \subseteq \mathcal{U}_2^*.$$

This completes the internal induction on i, the result from which completes the induction on n. So $b^{n\nu}$ is defined for all n.

Now suppose for a contradiction that $b^i \notin \mu$ for some $i \in \{1, \ldots, \nu - 1\}$. Let $x := st(b^i) \neq 1$. Let $m \in \mathbb{Z}$. Consider $x^m = st(b^i)^m = st(b^{mi}$, where the equality holds because $st(\cdot)$ is a homomorphism. Also, by the above, b^{mi} is defined, because $i < \nu$. Then we can write $mi = s\nu + q$ for $s \in \mathbb{Z}$, $q < \nu$. So

$$b^{mi} = (b^\nu)^s \cdot b^q \sim b^q \in \mathcal{U}^*,$$

and so we have $st(b^{mi}) \in \mathcal{U}$. Therefore $x^{\mathbb{Z}} \subseteq \mathcal{U}$, which is a contradiction. $\square$

Lemma 6.11. *Given $a \in \mu$ and ν, there is $b \in G^*$ such that $ord(b) \geq 2^\nu$ and $a = b^{2^\nu}$.*

Proof. First note that the statement in the lemma is internal. We can thus use internal induction on ν.

Also note, that by Lemma 6.10 any $b \in G^*$ with $ord(b) \geq 2^\nu$ and $a = b^{2^\nu}$ is actually in μ.

By Lemma 6.9 we know that the statement of the lemma holds for $\nu = 1$. Now suppose it holds for ν, i.e. $a = b^{2^\nu}$ with $ord(b) \geq 2^\nu$. By Lemma 6.9 again we have $c \in \mu$ with $b = c^2$. Then $a = (c^2)^{2^\nu}$. We have to check that $c^{2^{\nu+1}}$ is defined to be able to show that $a = c^{2^{\nu+1}}$. We prove by (another) internal induction on i that c^{2i} is defined and $(c^2)^i = c^{2i}$ for all $i \in \{1, \ldots, 2^\nu\}$.

First note that $c^{2 \cdot 1} = c^2$ is defined, because $c \in \mu$, and $(c^2)^1 = c^2$. Now suppose c^{2i} is defined and equal to $(c^2)^i$ for $i \in \{1, \ldots, 2^\nu\text{-}1\}$.

To prove that c^{2i+1} is defined, by Lemma 2.36, we have to show $(c^k, c^l) \in \Omega^*$ for $k, l \in \{1, \ldots, 2i\}$ with $k + l = 2i + 1$. If k is even, we have

$$c^k = (c^2)^{\frac{k}{2}} = b^{\frac{k}{2}} \in \mu,$$

by Lemma 6.10 since $\frac{k}{2} \leq i \leq 2^\nu$. Similarly, if k is odd, then

$$c^k = c \cdot (c^2)^{\frac{k-1}{2}} = c \cdot b^{\frac{k-1}{2}} \in \mu \cdot \mu \subseteq \mu.$$

So c^{2i+1} is defined. By repeating the above steps exactly, we get that c^{2i+2} is defined. Then, by induction, we have

$$(c^2)^{i+1} = (c^2)^i \cdot c^2 = c^{2i} \cdot c^2 = c^{2i+2}.$$

This concludes the induction and we now have that c^{2i} is defined and equal to $(c^2)^i$ for all $i \in \{1, \ldots, 2^\nu\}$. In particular, $c^{2.2^\nu} = (c^2)^{2^\nu} = b^{2^\nu} = a$.

Finally, we want to show that $ord(c) \geq 2^{\nu+1}$. We know that c^i is defined for i up to $2^{\nu+1}$. Considering the parity of i and using Lemma 6.10 as above, we get that $c^i \in \mu$ for all $i \in \{1, \ldots, 2^{\nu+1}\}$. Hence $ord(c) \geq 2^{\nu+1}$ and the induction is complete, along with the proof. □

Theorem 6.12. *K is a neighbourhood of* 1.

Proof. We use the same trick as in Lemma 6.9. Let the sets V_n be as in Lemma 4.9.

Fix $\nu > \mathbb{N}$. Then the internal set of all η such that for each $x \in V_\eta^*$ there is a $y \in \mathcal{U}^*$ such that $ord(y) \geq 2^\nu$ and $x = y^{2^\nu}$ contains all infinite η. Since we know that $\mathbb{N}^* \backslash \mathbb{N}$ is not an internal subset of $\mathbb{N}^*$, then there is some $n > 0$ in this set. So now, given $x \in V_n \subseteq V_n^*$, there is an $a \in \mathcal{U}^*$ such that $ord(a) \geq 2^\nu$ and $x = a^{2^\nu}$.

Let $\sigma := 2^\nu$. Now let i be such that $\frac{i}{\sigma} \sim t \in I = [-1, 1]$. Since $t \in I$, for all t we can choose i such that $|i| \leq \sigma$. Then a^i is defined and in $\mathcal{U}^*$ for all such i. Therefore, $st(a^i) \in \mathcal{U}$ for all i as above. Then $X_a(t) = st(a^i) \in \mathcal{U}$ and $X_a^*(1) = x$. Thus $[X_a] \in \mathcal{K}$ and $x = [X_a](1) \in K$. This shows that $V_n \subseteq K$, which means that K is neighbourhood of 1 in G, since V_n itself is one. □

7. The Space $L(G)$

In this section, our goal is to equip $L(G)$ with a topology in such a wat that it becomes a locally compact finite dimensional real topological vector space. We have already previously defined the operations of scalar multiplication and addition on the vector space $L(G)$ and we will continue using them as the operations of the vector space. At the end we obtain the result that locally compact NSS local groups are locally euclidean.

Definition 7.1. *A* ***subbasis*** *for a topological space X with topology $\mathcal{T}$ is a subcollection B of $\mathcal{T}$, which generates $\mathcal{T}$. That is, every element of $\mathcal{T}$ is an arbitrary union of finite intersections of elements of B.*

So by specifying a subbasis B we fix a topology $\mathcal{T}$ on the space, there $\mathcal{T}$ is the smallest(coarsest) topology such that B is a subbasis.

We specify a subbasis B of the space $L(G)$:

$$B := \{B_{C,U} \,|\, C \subseteq (-2,2) \text{ is compact, } U \subseteq G \text{ is open}\}, \text{ where}$$
$$B_{C,U} := \{\mathbb{X} \in L(G) \,|\, C \subseteq domain(\mathbb{X}),\, \mathbb{X}(C) \subseteq U\}.$$

This is usually called *the compact-open topology* on a functional space. We now investigate the monad structure of $L(G)$ with this topology.

Lemma 7.2. *Let $\mathbb{X} \in L(G)$, $\mathbb{Y} \in L(G)^*$. Then $\mathbb{Y} \in \mu(\mathbb{X})$ if and only if*

(1) $domain(\mathbb{X}) \cap (-2,2) \subseteq domain(\mathbb{Y})$, and

(2) for every $t \in domain(\mathbb{X}) \cap (-2,2)$ and for every $t' \in \mu(t)$, $\mathbb{Y}(t') \in \mu(\mathbb{X}(t))$.

Proof.

(=>): Suppose $\mathbb{Y} \in \mu(\mathbb{X})$. Then $\mathbb{Y}$ is in the extension of any open set of $L(G)$ containing $\mathbb{X}$.

Take the sets $B_{C,G}$ for $C \subseteq domain(\mathbb{X}) \cap (-2,2)$ compact. Then $\mathbb{X}$ is in any such set. So $\mathbb{Y} \in B^*_{C,G}$ for $C \subseteq domain(\mathbb{X}) \cap (-2,2)$ compact. This means that $C^* \subseteq domain(\mathbb{Y})$ for all $C \subseteq domain(\mathbb{X}) \cap (-2,2)$ compact. Therefore

$$domain(\mathbb{X}) \cap (-2,2) \subseteq domain(\mathbb{X})^* \cap (-2,2)^* \subseteq domain(\mathbb{Y}).$$

For the second part suppose $t \in domain(\mathbb{X}) \cap (-2,2)$, $t' \in \mu(t)$. We will show that $\mathbb{Y}(t') \in U^*$ for every U-open in G with $X(t) \in U$. This will imply $\mathbb{Y}(t') \in \mu(\mathbb{X}(t))$.

Let U be an arbitrary open neighbourhood of $\mathbb{X}(t)$ in G. Choose C_U to be a compact neighbourhood of t with $C_U \subseteq domain(\mathbb{X}) \cap (-2,2)$ and $\mathbb{X}(C_U) \subseteq U$. Then $C_U^* \subseteq domain(\mathbb{Y}) \cap (-2,2)^*$ by the above, and since $\mathbb{X} \in B_{C_U,U}$, then $\mathbb{Y} \in B^*_{C_U,U}$. Therefore $\mathbb{Y}(C_U^*) \subseteq U^*$ and in particular $\mathbb{Y}(t') \in U^*$. Hence $\mathbb{Y}(t') \in \mu(\mathbb{X}(t))$ and we are done.

(<=): We now suppose that $domain(\mathbb{X}) \cap (-2,2) \subseteq domain(\mathbb{Y})$ and for every $t \in domain(\mathbb{X}) \cap (-2,2)$ and for every $t' \in \mu(t)$, $\mathbb{Y}(t') \in \mu(\mathbb{X}(t))$. Note that the sets $B_{C,U}$ as defined above with $\mathbb{X} \in B_{C,U}$ form a neighbourhood basis for $\mathbb{X}$. So if we show that $\mathbb{X} \in B_{C,U}$ implies $\mathbb{Y} \in B^*_{C,U}$, then $\mathbb{Y} \in \mu(\mathbb{X})$.

Suppose $\mathbb{X} \in B_{C,U}$. Then for all $t \in C^*$ we have $\mathbb{Y}(t) \in \mu(\mathbb{X}(st(t))) \subseteq U^*$, since C is compact. Then $\mathbb{Y} \in B^*_{C,U}$. So $\mathbb{Y} \in \mu(\mathbb{X})$.

□

From now on we assume our special neighbourhood of choice $\mathcal{U}$ of an NSS local group is chosen such that $\mathcal{U} \subseteq \mathcal{U}_6$.

Lemma 7.3. *$\mathcal{K}$ is a compact neighbourhood of $\mathbb{O}$ in $L(G)$.*

Proof. We first show that $\mathcal{K}$ is a neighbourhood of $\mathbb{O}$. Choose $W \subseteq \mathcal{U}$ such that W is open. Then

$$B_{I,W} = \{\mathbb{X} \in L(G) \,|\, I \subseteq domain(\mathbb{X}),\, \mathbb{X}(I) \subseteq W\} \subseteq \mathcal{K}.$$

Since $\mathbb{O} \in B_{I,W}$ and $B_{I,W}$ is open by definition, so $\mathcal{K}$ is a neighbourhood of $\mathbb{O}$.

Now we show that $\mathcal{K}$ is compact. It is enough to show that for an arbitrary $\mathbb{Y} \in \mathcal{K}^*$, there is an $\mathbb{X} \in \mathcal{K}$ with $\mathbb{Y} \in \mu(\mathbb{X})$, by Remark 2.14.

Let $\mathbb{Y} \in \mathcal{K}^*$. Pick $Y \in \mathbb{Y}$ with $I^* \subseteq domain(Y)$. We now put $X : I \to G$ to be defined by $X : t \mapsto st(Y(t))$. By Lemma 3.12, we know that there is an LPS $\overline{X} : (-1-\epsilon, 1+\epsilon) \to G$ of G, for some $\epsilon \in \mathbb{R}^+$, with $\overline{X}|_I = X$. We claim that if $\mathbb{X} = [\overline{X}]$, then $\mathbb{Y} \in \mu(\mathbb{X})$. We will prove the two conditions from the previous lemma, equivalent to $\mathbb{Y} \in \mu(\mathbb{X})$.

For $t \in (1,2)^*$ we have $(Y(1), Y(t-1)) \in \mathcal{U}^* \times \mathcal{U}^* \subseteq \Omega^*$ and for $t \in (-2,-1)^*$ we have $(Y(t+1), Y(-1)) \in \mathcal{U}^* \times \mathcal{U}^* \subseteq \Omega^*$. Therefore we can define

$$\overline{Y} : t \mapsto \begin{cases} Y(t) & \text{for } t \in I^* \\ Y(1) \cdot Y(t-1) & \text{for } t \in (1,2)^* \\ Y(t+1) \cdot Y(-1) & \text{for } t \in (-2,-1)^* \end{cases}$$

Now, analogously to the proof of Lemma 3.12, we can prove that $\overline{Y}$ is an internal LPS of G^*. It is also clear that $[\overline{Y}] = [Y]$. Since $domain(\overline{Y}) = (-2,2)^*$, then $domain(\mathbb{Y}) \supseteq (-2,2) \supseteq (-2,2) \cap domain(\mathbb{X})$.

For the second condition suppose $t \in domain(\mathbb{X}) \cap (-2,2)$ and $t' \in \mu(t)$. For $t \in [-1,1]$, $\mathbb{Y}(t') \in \mu(\mathbb{X}(t))$ is clear by how we defined $\mathbb{X}$. If $t \in (1,2)$, then

$$\mathbb{Y}(t') = \mathbb{Y}(1) \cdot \mathbb{Y}(t'-1) \in \mu(X(1) \cdot X(t-1)) = \mu(\overline{X}(t)).$$

And finally, if $t \in (-2,-1)$, then

$$\mathbb{Y}(t') = \mathbb{Y}(t'+1) \cdot \mathbb{Y}(-1) \in \mu(X(t+1) \cdot X(-1)) = \mu(\overline{X}(t)).$$

□

We prove an easy topological lemma that we need in order to prove that $E : \mathcal{K} \to K$ is a homeomorphism.

Lemma 7.4. *If T_1 is a compact space, T_2 is a Hausdorff space and $f : T_1 \to T_2$ is a continuous bijection, then f is a homeomorphism.*

Proof. We need to show that $g = f^{-1} : T_2 \to T_1$ is continuous. We will show that if V is closed in T_1, then $g^{-1}(V) = f(V)$ is closed in T_2.

Let V be closed in T_1. Then, since T_1 is compact, we have that V is compact. Since the continuous image of a compact set is compact, we have that $f(V)$ is compact. Moreover, a compact set in a Hausdorff space is always closed. So $g^{-1}(V) = f(V)$ is closed and we are done. □

Corollary 7.5. *$E : \mathcal{K} \to K$ is a homeomorphism.*

Proof. Lemma 7.2 implies that $\mathbb{Y} \in \mu(\mathbb{X})$ only if $E(\mathbb{Y}) \in \mu(E(\mathbb{X}))$, in other words $\mathbb{Y}(1) \in \mu(\mathbb{X}(1))$. So E is continuous. We know from Lemma 4.8 that E is also bijective. By Lemma 7.3 we know that $\mathcal{K}$ is compact.

Then, by our previous lemma, E is a homeomorphism, since G is Hausdorff. □

Now suppose $\mathbb{X} \in L(G)$ and $[-s,s] \subseteq domain(\mathbb{X})$ for some $s \in (0,2)$, and $\mathbb{X}([-s,s]) \subseteq \mathcal{U}_2$. Let $U \subseteq \mathcal{U}_2$ be a neighbourhood of 1. Previously we defined $xH := \{xh \,|\, h \in H \text{ and } (x,h) \in \Omega\}$. We investigate the set $N_{\mathbb{X}}(s,U)$ of equivalence classes of LPSs that have images of $[-s,s]$ 'U-close' to those of $\mathbb{X}$. Formally,

$$N_{\mathbb{X}}(s,U) := \{\mathbb{Y} \in L(G) \,|\, \mathbb{Y}(t) \in \mathbb{X}(t)U \text{ for all } t \in [-s,s]\}.$$

Lemma 7.6. *For each neighbourhood U of 1 in G with $U \subseteq \mathcal{U}_2$, $N_{\mathbb{X}}(s,U)$ is a neighbourhood of $\mathbb{X}$ in $L(G)$ and the collection*

$$\{N_{\mathbb{X}}(s,U) \,|\, U \text{ is a neighbourhood of } 1 \text{ in } G \text{ and } U \subseteq \mathcal{U}_2\}$$

is a neighbourhood basis of $\mathbb{X}$ in $L(G)$.

Proof. We first show that $N_{\mathbb{X}}(s,U)$ is a neighbourhood of $\mathbb{X}$ in $L(G)$. So we have to show that $\mu(\mathbb{X}) \subseteq N_{\mathbb{X}}(s,U)$.

Let $\mathbb{Y} \in \mu(\mathbb{X})$. By Lemma 7.2, this means that

$$[-s,s] \subseteq domain(\mathbb{X}) \cap (-2,2) \subseteq domain(\mathbb{Y}),$$

so $[-s,s]^* \subseteq domain(\mathbb{Y})$. Now fix $t \in [-s,s]^*$ and $U \subseteq \mathcal{U}_2$ a neighbourhood of 1 in G. Then, again by Lemma 7.2, we have $\mathbb{Y}(t), \mathbb{X}(t) \in \mu(\mathbb{X}(st(t)))$. This implies $\mathbb{X}(t)^{-1}\mathbb{Y}(t) \in \mu \subseteq U^*$. Therefore $\mathbb{Y}(t) = \mathbb{X}(t)\mathbb{X}(t)^{-1}\mathbb{Y}(t) \in \mathbb{X}(t)U^*$. So $\mathbb{Y} \in N^*_{\mathbb{X}}(s,U)$. Thus $\mu(\mathbb{X}) \subseteq N_{\mathbb{X}}(s,U)^*$.

Next we show that the collection in the lemma is a neighbourhood basis. We do this by showing that $\bigcap N_{\mathbb{X}}(s,U) \subseteq \mu(\mathbb{X})$, where the intersection runs through all neighbourhoods $U \subseteq \mathcal{U}_2$ of 1 in G.

Let $\mathbb{Y} \in \bigcap N_{\mathbb{X}}(s,U)$. Suppose $domain(\mathbb{X}) \cap (-2,2) = (-r,r)$. Choose $n > 0$ such that $\frac{1}{n}(-r,r) \subseteq [-s,s]$. Then, since $\mathbb{Y} \in N_{\mathbb{X}}(s,\mathcal{U}_{2n})$, we have $domain(\mathbb{Y}) \supseteq (-r,r)$, because $\mathbb{Y}(t) = \mathbb{Y}(\frac{t}{n})^n$ for $t \in (-r,r)$ defines an internal LPS of G^*. This means that $domain(\mathbb{X} \cap (-2,2) \subseteq domain(\mathbb{Y})$. (1)

Now fix $t \in (-r,r)$ and $t' \in \mu(t)$. Let U be a neighbourhood of $\mathbb{X}(t)$. Recall that the partial map $p_n : G \rightharpoonup G$ is defined by $p_n : a \mapsto a^n$, if a^n is defined, and that the domain of this map is open. Since $\mathbb{X}(\frac{t}{n}) \in domain(p_n)$, then we can choose an open neighbourhood V of $\mathbb{X}(\frac{t}{n})$ such that $V \subseteq domain(p_n)$ and $p_n(V) \subseteq U$. By local compactness, we can choose a compact neighbourhood W of 1 in G with $W \subseteq \mathcal{U}_2$, $\{\mathbb{X}(\frac{t}{n})\} \times W \subseteq \Omega$ and $\mathbb{X}(\frac{t}{n})W \subseteq V$. By supposition $\mathbb{Y} \in N_{\mathbb{X}}(s,W)$, which implies that $\mathbb{Y}(\frac{t'}{n}) \in \mathbb{X}(\frac{t'}{n})W^*$. Now, suppose $a \in W^*$. Then $\mathbb{X}(\frac{t'}{n}) \cdot a = \mathbb{X}(\frac{t}{n}) \cdot (\mathbb{X}(\frac{t'-t}{n}) \cdot a)$, and we know that $\mathbb{X}(\frac{t'-t}{n}) \cdot a \sim a$. So

$$\mathbb{X}\left(\frac{t'}{n}\right) \cdot a \sim \mathbb{X}\left(\frac{t}{n}\right) \cdot a \in \mathbb{X}\left(\frac{t}{n}\right) W \subseteq V.$$

Since V is open, we see that $\mathbb{X}(\frac{t'}{n}) \cdot a \in V^*$. This implies $\mathbb{Y}(\frac{t'}{n}) \in \mathbb{X}(\frac{t'}{n})W^* \subseteq V^*$. By our choice of V, we have $\mathbb{Y}(\frac{t'}{n}) \in domain(p_n)^*$. Furthermore, we see that $\mathbb{Y}(t') = \mathbb{Y}(\frac{t'}{n})^n \in U^*$. Therefore, $\mathbb{Y}(t') \in \mu(\mathbb{X}(t))$. (2)

By Lemma 7.2, (1) and (2) imply $\mathbb{Y} \in \mu(\mathbb{X})$. □

For the proof of the next theorem we need to borrow a lemma of Reisz.

Lemma 7.7. *(Reisz* [6, p. 47]*) Let X be a normed linear space, Y be a closed proper subspace of X and α be a real number with $0 < \alpha < 1$. Then there exists an $x \in X$ with $|x| = 1$ such that $|x - y| > \alpha$ for all $y \in Y$.*

Corollary 7.8. *If the closed ball is compact, then X is finite dimensional.*

Theorem 7.9. *$L(G)$ is locally compact, finite dimensional real topological vector space.*

Proof. We know by Lemma 7.3 that $L(G)$ is locally compact. Then, by the above corollary, we only need to show that $L(G)$ is a topological vector space. We first prove that scalar multiplication and addition are continuous in the topology we previously defined. After that we prove that $L(G)$ is a vector space.

(Scalar multiplication is continuous): We show that if $r \in \mathbb{R}\backslash\{0\}$, $\mathbb{X} \in L(G)$, $r' \in \mu(r)$ and $\mathbb{X}' \in \mu(\mathbb{X})$, then $r'\mathbb{X}' \in \mu(r\mathbb{X})$. By Lemma 7.2 we have that $\mathbb{X}' \in \mu(\mathbb{X})$ implies $domain(\mathbb{X}) \cap (-2,2) \subseteq domain(\mathbb{X}')$. We can then easily see that $domain(r\mathbb{X}) \cap (-2,2) \subseteq domain(r'\mathbb{X}')$. Also, suppose $t \in domain(r\mathbb{X}) \cap (-2,2)$ and $t' \in \mu(t)$. Then

$$(r'\mathbb{X}')(t') = \mathbb{X}'(r't') \in \mu(\mathbb{X}(rt)) = \mu((r\mathbb{X})(t)).$$

Therefore, by Lemma 7.2, we have $r'\mathbb{X}' \in \mu(r\mathbb{X})$.

(Addition is continuous): We show that $+$ is continuous at $(\mathbb{O}, \mathbb{O})$, that for any subbasic set $B_{C,U}$ of $L(G)$ containing $\mathbb{O}$ (and thus U containing 1), there is a subbasic set $B_{C,W}$ of $L(G)$ containing $\mathbb{O}$ such that $B_{C,W} + B_{C,W} \subseteq B_{C,U}$. Assume $V \subseteq U$ is a compact neighbourhood of 1. Let $Z \subseteq \mu$ be an internally open set. Then, by Lemma 5.2, we know that for every $a, b \in Z$ and for every σ such that for all $i \in \{1, \ldots, \sigma\}$, a^i and b^i are defined and $a^i, b^i \in Z$, we have $(ab)^\sigma$ is defined and $(ab)^\sigma \in V^*$. Hence, by transfer, there is an open set W such that for all $a, b \in W$ and all n with a^i, b^i defined and $a^i, b^i \in W$ for all $i \in \{1, \ldots, n\}$, we have $9ab)^n$ defined and $(ab)^n \in V$. Now suppose $\mathbb{X}, \mathbb{Y} \in B_{C,W}$. Let $t \in C$ and r be such that $\frac{r}{\sigma} \sim t$. Then

$$(\mathbb{X} + \mathbb{Y})(t) \sim \left[\mathbb{X}\left(\frac{1}{\sigma}\right)\mathbb{Y}\left(\frac{1}{\sigma}\right)\right]^r.$$

Let $a := \mathbb{X}(\frac{1}{\sigma}), b := \mathbb{Y}(\frac{1}{\sigma})$. By assumption, for all $i \in \{1, \ldots, r\}$, a^i, b^i are defined and $a^i, b^i \in W^*$. Thus, $(ab)^r \in V^*$, from which we infer that $(\mathbb{X} + \mathbb{Y})(t) \in V \subseteq U$. Therefore $\mathbb{X} + \mathbb{Y} \in B_{C,U}$. So $B_{C,W} + B_{C,W} \subseteq B_{C,U}$.

To prove that $\mathbb{X} \mapsto \mathbb{X} + \mathbb{Y}$ is continuous at $\mathbb{O}$, we need to prove that if $W \subseteq L(G)$ is a neighbourhood of $\mathbb{Y}$, then there is a neighbourhood V of $\mathbb{O}$ in $L(G)$ with image under the aforementioned map inside W. We now fix $s \in domain(\mathbb{Y}) \cap (-2,2)$ such that $\mathbb{Y}([-s,s]) \subseteq \mathcal{U}_2$. Fix a compact neighbourhood U of 1 in G with $U \subseteq \mathcal{U}_2$ and $N_{\mathbb{Y}}(s, U) \subseteq W$. Now suppose $\mathbb{X} \in \mu(\mathbb{O}) \subseteq L(G)^*$. By Lemma 5.2, for all $i \leq r$ (where r is such that $\frac{r}{\sigma} \sim s$) we have $(\mathbb{X}(\frac{1}{\sigma})\mathbb{Y}(\frac{1}{\sigma}))^i$ is defined and infinitely close to $\mathbb{Y}(\frac{i}{\sigma})$. Therefore, the internal set of all members $\mathbb{X}$ of $L(G)^*$ such that whenever $i \leq r$, we have $(\mathbb{X}(\frac{1}{\sigma})\mathbb{Y}(\frac{1}{\sigma}))^i$ is defined and

$$\left(\mathbb{X}\left(\frac{1}{\sigma}\right)\mathbb{Y}\left(\frac{1}{\sigma}\right)\right)^i \in \mathbb{Y}\left(\frac{i}{\sigma}\right)U^*,$$

contains all $\mathbb{X} \in \mu(\mathbb{O})$. We know that $\mu(\mathbb{O})$ is not an internal set, so there is a neighbourhood V of $\mathbb{O}$ in $L(G)$ such that for all $\mathbb{X} \in V$ we have $(\mathbb{X}(\frac{1}{\sigma})\mathbb{Y}(\frac{1}{\sigma}))^i$ is defined and

$$\left(\mathbb{X}\left(\frac{1}{\sigma}\right)\mathbb{Y}\left(\frac{1}{\sigma}\right)\right)^i \in \mathbb{Y}\left(\frac{i}{\sigma}\right)U^*,$$

whenever $i \leq r$. This implies that for $\mathbb{X} \in V$, we have

$$t \in domain(\mathbb{X} + \mathbb{Y}) \text{ and } (\mathbb{X} + \mathbb{Y})(t) \in \mathbb{Y}(t)U,$$

for all $t \in [-s, s]$. Therefore $\mathbb{X} + \mathbb{Y} \in N_{\mathbb{Y}}(s, U) \subseteq W$ for all $\mathbb{X} \in V$.

We can prove that $\mathbb{X} \mapsto \mathbb{X} - \mathbb{Y}$ is continuous at $\mathbb{Y}$ in exactly the same way.

($L(G)$ is a vector space): Only one of the vector space axioms is not trivial to verify, namely that $r \cdot (\mathbb{X} + \mathbb{Y}) = r \cdot \mathbb{X} + r \cdot \mathbb{Y}$. We show that it holds first fot $r \in \mathbb{Z}$, then for $r \in \mathbb{Q}$ at which point we conclude that it holds for all $r \in \mathbb{R}$, because of the continuity of the operations of scalar multiplication and addition.

Assume $r \in \mathbb{Z}$. Let $a := \mathbb{X}(\frac{1}{\sigma})$, $b := \mathbb{Y}(\frac{1}{\sigma})$. Then we have that

$$S(r \cdot \mathbb{X} + r \cdot \mathbb{Y}) = (a^r b^r)G^o(\sigma) \text{ and } S(r \cdot (\mathbb{X} + \mathbb{Y})) = ((ab)^r)G^o(\sigma).$$

But $G(\sigma)/G^o(\sigma)$ is abelian, so $(a^r b^r)G^o(\sigma) = ((ab)^r)G^o(\sigma)$, and hence $S(r \cdot \mathbb{X} + r \cdot \mathbb{Y}) = S(r \cdot (\mathbb{X} + \mathbb{Y}))$. Since S is an injection, we conclude that $r \cdot \mathbb{X} + r \cdot \mathbb{Y} = r \cdot (\mathbb{X} + \mathbb{Y})$.

Let $r \in \mathbb{Z}\backslash\{0\}$. By the above we know that $r \cdot (\frac{1}{r}\mathbb{X} + \frac{1}{r}\mathbb{Y}) = \mathbb{X} + \mathbb{Y}$. This is equivalent to $\frac{1}{r}\mathbb{X} + \frac{1}{r}\mathbb{Y} = \frac{1}{r}(\mathbb{X} + \mathbb{Y})$. So the axiom holds for $r \in \mathbb{Z}$ and $r \in \frac{1}{\mathbb{Z}\backslash\{0\}}$. Therefore it holds for $r \in \mathbb{Q}$.

□

Corollary 7.10. *G is locally euclidean.*

Proof. K is a neighbourhood of 1 by Theorem 6.12, so we can choose $W \subseteq K$ – an open neighbourhood of 1 in G. By the above, $L(G)$ is a finite dimensional topological vector space over $\mathbb{R}$, so we have an isomorphism $L(G) \cong \mathbb{R}^n$ of real vector spaces for some n, and this isomorphism is also a homeomorphism. Hence, by Lemma 7.3, we can choose an open neighbourhood U of $\mathbb{O}$ with $U \subseteq \mathcal{K}$ and $E(U) \subseteq W$ such that U is homeomorphic to an open subset of $\mathbb{R}^n$. Then, since E is a homeomorphism by Corollary 7.5, $E(U)$ is an open neighbourhood of 1 in G homeomorphic to an open subset of $\mathbb{R}^n$. So G is locally euclidean. □

8. LH5 for NSS local groups

We are now almost ready to prove the LH5 for NSS local groups. To finish this part of the proof, we will need the local version of the Adjoint Representation Theorem.

In this section we, again, assume G is NSS and that our special neighbourhood of choice $\mathcal{U}$ is such that $\mathcal{U} \subseteq \mathcal{U}_6$.

Definition 8.1. *We define the* ***adjoint representation map for*** $g \in \mathcal{U}_6$ *to be* $Ad_g : L(G) \to L(G)$*, defined by*

$$Ad_g : \mathbb{X} \mapsto g\mathbb{X}g^{-1}.$$

Here if $X \in \mathbb{X}$ *and* $r \in domain(\mathbb{X}) \cap \mathbb{R}^+$ *is such that* $X((-r,r)) \subseteq \mathcal{U}_6$*, then* $gXg^{-1} : (-r,r) \to G$ *is defined by* $gXg^{-1} : t \mapsto gX(t)g^{-1}$.

Lemma 8.2. *The above definition produces a well-defined map. That is, for every* $X \in \mathbb{X}$*,* gXg^{-1} *is an LPS of* G*, and for every* $X_1, X_2 \in \mathbb{X}$*, we have that* $[gX_1g^{-1}] = [gX_2g^{-1}]$.

Proof. Let $X \in \mathbb{X}$ and gXg^{-1} be as defined above. It is clear that gXg^{-1} is continuous, as X, ρ, and ι are continuous. Now let $domain(gXg^{-1}) = (-r,r)$ and $s, t, s+t \in (-r,r)$. Then, since $g, g^{-1}, X(t), X(s) \in \mathcal{U}_6$, we have

$$\begin{aligned}(gXg^{-1}(t+s) &= g(X(t+s))g^{-1}\\ &= g(X(t)X(s))g^{-1}\\ &= (gX(t)g^{-1})(gX(s)g^{-1})\\ &= (gXg^{-1})(t)(gXg^{-1})(s)\end{aligned}$$

Therefore, gXg^{-1} is an LPS of G.

Now suppose $X_1, X_2, \in \mathbb{X}$. We also put $domain(gX_1g^{-1}) = (-r_1, r_1)$ and $domain(gX_2g^{-1}) = (-r_2, r_2)$. Since $[X_1] = [X_2]$, there exists r_3 such that we get $X_1|_{(-r_3,r_3)} = X_2|_{(-r_3,r_3)}$. Let $r = min\{r_1, r_2, r_3\}$. By the definition of gX_1g^{-1} and gX_2g^{-1}, it is clear that $gX_1g^{-1}|_{(-r,r)} = gX_2g^{-1}|_{(-r,r)}$. This means that $[gX_1g^{-1}] = [gX_2g^{-1}]$. □

Lemma 8.3. *Suppose* $\sigma > \mathbb{N}$*,* $a \in G(\sigma)$ *and* $g \in \mathcal{U}_6$*. Then*

(1) $gag^{-1} \in G(\sigma)$*;*
(2) $Ad_g([X_a]) = [X_{gag^{-1}}]$.

Proof. Let $\tau := ord(a)$. Then, $\sigma \in O(\tau)$.

To show (1) we use the usual trick - we show by internal induction that for $i \in \{1, \dots, \tau\}$ (gag^{-1}) is defined and equal to ga^ig^{-1}. It is clearly true for $i = 1$, so suppose it is true for $i < \tau$. By Lemma 2.36, to prove that $(gag^{-1})^{i+1}$ is defined it is enough to show that $((gag^{-1})^k, (gag^{-1})^l) \in \Omega^*$ for all $k, l \in \{1, \dots, i\}$ with $k + l = i + 1$. But by induction $(gag^{-1})^k = ga^kg^{-1}$ and $(gag^{-1})^l = ga^lg^{-1}$ and since $k, l < \tau$, then $g, a^k, a^l \in \mathcal{U}_6^*$. So we have $(ga^kg^{-1}, ga^lg^{-1}) \in \Omega^*$. Therefore $(gag^{-1})^{i+1}$ is defined. Furthermore,

$$\begin{aligned}(gag^{-1})^{i+1} &= (gag^{-1})^i(gag^{-1})\\ &= (ga^ig^{-1})(gag^{-1}) && \text{(by IH)}\\ &= ga^{i+1}g^{-1} && (\text{since } g, a^i, a, g^{-1} \in \mathcal{U}_6^*).\end{aligned}$$

Therefore for $i \in \{1, \ldots, \tau\}$ (gag^{-1}) is defined and equal to $ga^i g^{-1}$. This implies that if $i \in o(\sigma)$, then $(gag^{-1})^i$ is defined and $(gag^{-1})^i = ga^i g^{-1} \in \mu$ since $a \in G(\sigma)$. This proves (1).

To prove (2) we fix $r \in \mathbb{R}^+$ such that there is $i \leq \tau$ with $\frac{i}{\sigma} \sim r$, $X_a((-r,r)) \subseteq \mathcal{U}_6$, and $gX_a((-r,r))g^{-1} \subseteq \mathcal{U}$. Then for $t \in (-r,r)$ and i such that $\frac{i}{\sigma} \sim t$ we have

$$\begin{aligned}(Ad_g([X_a])(t)) &= gX_a(t)g^{-1} \\ &= g(st(a^i))g^{-1} \\ &= st(ga^i g^{-1}) \\ &= st((gag^{-1})^i) \\ &= X_{gag^{-1}}(t).\end{aligned}$$

Hence $Ad_g([X_a])|_{(-r,r)} = [X_{gag^{-1}}]|_{(-r,r)}$ and so $Ad_g([X_a]) = [X_{gag^{-1}}]$.

□

Corollary 8.4. *For $g \in \mathcal{U}_6$, $Ad_g : L(G) \to L(G)$ is a vector space automorphism with inverse $Ad_{g^{-1}}$.*

Remark 8.5. We denote by $Aut(L(G))$ the group of isomorphisms of $L(G)$. We now know that $dim_{\mathbb{R}}(L(G)) = n$ for some $n > 0$, so we can choose an $\mathbb{R}$-linear isomorphism $L(G) \cong \mathbb{R}^n$. It induces a group isomorphism

$$Aut(L(G)) \cong GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2},$$

and we take the topology on $Aut(L(G))$ that makes this group isomorphism a homeomorphism. (Note that this topology does not depend on our choice of $\mathbb{R}$-linear isomorphism $L(G) \cong \mathbb{R}^n$.)

Now we can characterise the structure of $Aut(L(G))^*$. For $T \in Aut(L(G))^*$, $T \in Aut(L(G))^*_{ns}$ if $T(\mathbb{X}) \in L(G)^*_{ns}$ for all $\mathbb{X} \in L(G)$. For $T, T' \in Aut(L(G))^*_{ns}$, we see that $T \sim T'$ if and only if $T(\mathbb{X}) \sim T'(\mathbb{X})$ for all $\mathbb{X} \in L(G)$.

Theorem 8.6. *(Local Adjoint Representation Theorem) We have a morphism $Ad : (\mathcal{U}_6)_G \to Aut(L(G))$ of local groups given by $Ad : g \mapsto Ad_g$.*

Proof. First note taht if $g, h, gh \in \mathcal{U}_6$ and $[X_a] \in L(G)$, then

$$\begin{aligned}Ad_{gh}([X_a]) &= [X_{(gh)a(gh)^{-1}}] \\ &= [X_{g(hah^{-1})g^{-1}}] \\ &= Ad_g(Ad_h([X_a])).\end{aligned}$$

This implies that $Ad(gh) = Ad(g) \circ Ad(h)$. Therefore, taking also into account Corollary 8.4, the map Ad satisfies the three conditions in Definition 2.18. Remains to show that Ad is continuous.

We first show that Ad is continuous at 1. Let $a \in \mu$, $\mathbb{X} \in L(G)$. We will show that $Ad_a(\mathbb{X}) \sim \mathbb{X}$ by showing the two equivalent conditions in Lemma 7.2.

Suppose $(-r,r) \subseteq domain(\mathbb{X})$. We show that $(-r,r) \subseteq domain(Ad_a(\mathbb{X}))$. To do this we need that if $s, t, s+t \in (-r,r)$, then $(a\mathbb{X}(s)a^{-1}, a\mathbb{X}(t)a^{-1}) \in \Omega^*$ and $(a\mathbb{X}(s)a^{-1})(a\mathbb{X}(t)a^{-1}) = a\mathbb{X}(s+t)a^{-1}$. But we know that $(\mathbb{X}(s), \mathbb{X}(t)) \in \Omega$, so $(a\mathbb{X}(s)a^{-1}, a\mathbb{X}(t)a^{-1}) \in \Omega^*$, because $a \in \mu$. Furthermore, we can see that

$(a\mathbb{X}(s)a^{-1})(a\mathbb{X}(t)a^{-1}) = a\mathbb{X}(s+t)a^{-1}$ from the usual calculations involving infinitesimals and nearstandard elements, since $(\mathbb{X}(s), \mathbb{X}(t)) \in \Omega$ and $a \in \mu$. So $(-r, r) \subseteq domain(Ad_a(\mathbb{X}))$ and hence

$$domain(\mathbb{X}) \cap (-2, 2) \subseteq domain(\mathbb{X}) \subseteq domain(Ad_a(\mathbb{X})).$$

Now suppose $t \in domain(\mathbb{X}) \cap (-2, 2)$ and $t' \in \mu(t)$. Then

$$Ad_a(\mathbb{X})(t') = a\mathbb{X}(t')a^{-1} \sim \mathbb{X}(t') \sim \mathbb{X}(t).$$

Therefore we have $Ad_a(\mathbb{X}) \sim \mathbb{X}$ and since $\mathbb{X}$ was arbitrary, by Remark 8.5, $Ad(a) \sim id_{L(G)}$. So Ad is continuous at 1.

To show that Ad is continuous at $g \in \mathcal{U}_6$, take $g' \in \mu(g)$. Then we already know that $Ad_{g^{-1}g'} \sim id_{L(G)}$, which implies, by the first part of the proof, that

$$Ad_{g'} = Ad_g \circ Ad_{g^{-1}g'} \sim Ad_g \circ id_{L(G)} = Ad_g.$$

So Ad is continuous everywhere. □

Next, we develop several ways of obtaining local Lie groups.

Definition 8.7. *A local group G is **abelian** if there is a neighbourhood U of 1 in G such that $U \subseteq \mathcal{U}_2$ and $ab = ba$ for all $a, b \in U$.*

Theorem 8.8. *Suppose G is abelian. Then G is locally isomorphic to a Lie group.*

Proof. Since G is abelian, we can choose out special neighbourhood $\mathcal{U}$ to be such that $\mathcal{U} \subseteq \mathcal{U}_6$ and its elements commute with each other. We now want to show that if a^n, b^n, and $(ab)^n$ are defined with $a^i, b^i \in \mathcal{U}$ for all $i \in \{1, \ldots, n\}$, then $(ab)^n = a^n b^n$. We do this by induction, the case $n = 1$ being trivial.

Suppose a^n, b^n and $(ab)^n$ are defined, $a^i, b^i \in \mathcal{U}$ for all $i \in \{1, \ldots, n\}$ and also $(ab)^i = a^i b^i$ for $i \in \{1, \ldots, n-1\}$. Then

$$(ab)^n = (ab)^{n-1}ab = (a^{n-1}b^{n-1})ab = a^{n-1}b^{n-1}ab = a^n b^n,$$

and we are done. By transfer, we have the nonstandard variant of our claim: if a^η, b^η and $(ab)^\eta$ are defined with $a^i, b^i \in \mathcal{U}^*$ for all $i \in \{1, \ldots, \eta\}$, then $(ab)^\eta = a^\eta b^\eta$.

Now we choose a symmetric open neighbourhood $\mathcal{V}$ of $\mathbb{O}$ in $L(G)$ with $\mathcal{V} \subseteq \mathcal{K}$, $\mathbb{X} + \mathbb{Y} \in \mathcal{K}$ for all $\mathbb{X}, \mathbb{Y} \in \mathcal{V}$, and $E(\mathcal{V}) \subseteq \mathcal{U}$ (the last part of the choice is possible, since E is a homeomorphism). Then, if $\mathbb{X}, \mathbb{Y} \in \mathcal{V} \subseteq \mathcal{K}$ and $a := \mathbb{X}(\frac{1}{\sigma}), b := \mathbb{Y}(\frac{1}{\sigma})$, we have $a^\sigma, b^\sigma, (ab)^\sigma$ defined with $a^i, b^i \in \mathcal{U}^*$ for all $i \in \{1, \ldots, \sigma\}$. Hence we have

$$\begin{aligned}(\mathbb{X} + \mathbb{Y})(1) &= st((ab)^\sigma) \\ &= st(a^\sigma b^\sigma) \\ &= \mathbb{X}(1)\mathbb{Y}(1).\end{aligned}$$

We know that $E(\mathcal{V})$ is a symmetric open neighbourhood of 1 in G. Therefore, by the above, the equivalence class of $E|_\mathcal{V}$ is a local isomorphism from $L(G)$ to G. Since $L(G)$ is a Lie group, we are done. □

For the next lemma we will need the following theorem of John von Neumann.

Theorem 8.9. *(von Neumann, [7, p. 82]) If H is a hausdorff topological group which admits an injective continuous homomorphism into $GL_n(\mathbb{R})$ for some n, then H is a Lie group.*

Lemma 8.10. *Suppose $f : G \to GL_n(\mathbb{R})$ is an injective morphism of local groups. Then G is a local group.*

Proof. Let $G' := f(G) \subseteq GL_n(\mathbb{R})$. Let H be the subgroup of $GL_n(\mathbb{R})$ generated by G'. Let $\mathcal{F}'$ be the filter of neighbourhoods of 1 in G and $\mathcal{F}''$ be the image of $\mathcal{F}'$ under f. Finally, let $\mathcal{F}$ be the filter in H, generated by $\mathcal{F}''$.

It is easy to verify that $\mathcal{F}$ becomes a neighbourhood filter in any topology on H that makes H a topological group. Thus, in any such topology, the inclusion map $H \hookrightarrow GL_n(\mathbb{R})$ is continuous. By von Neumann's theorem then, H is a Lie group. However, f becomes a homeomorphism when $f(G)$ is given the induced topology from H. So G is homeomorphic to $f(G)$ and thus G is a local Lie group. $\square$

We finally need the following theorem of Kuranishi [8].

Theorem 8.11. *Let G be a locally compact local group and let H be a normal sublocal group of G. Consider the local coset space $(G/H)_W$ as in Lemma 2.25. Suppose*

(1) H is an abelian local Lie group;
(2) $(G/H)_W$ is a local Lie group;
(3) there is a set $M \subseteq W$ containing 1 and $W' \subseteq W$, an open neighbourhood of 1 in G, such that for every $(zH) \cap W \in \pi(W')$, there is exactly one $a \in M$ such that $a \in (zH) \cap W$; moreover, the map $\pi(W') \to M$ that assigns to each element of $\pi(W')$ the corresponding a in M is continuous.

Then W_G is a local Lie group.

Proof of the LH5 for NSS local groups.

We are now ready to provide a proof of the LH5 for NSS local groups. We use the following notation

- $G' := \mathcal{U}_{6G}$, then the adjoint representation map is $Ad : G' \to Aut(L(G))$;
- $H := ker(Ad)$;
- $G'' := (G'/H)_W$ is some local coset space.

We will prove that

(1) H is abelian with a restriction that is a local Lie group. After this point we abuse notation slightly and write H for the Lie restriction, which is an equivalent sublocal group of G';
(2) G'' is a local Lie group;
(3) there is a set $M \subseteq W$ satisfying condition (3) of Kuranishi's theorem.

These three results then, by Kuranishi's theorem, imply that W_G is a Lie group, which is exactly what LH5 requires.

(1) Since $Ad : G' \to Aut(L(G))$ is a morphism of local groups, then H is a normal sublocal group of G'. Then H is a locally compact NSS local group and, by Lemma 6.12, we know that H must have an open (in H) neighbourhood V of 1 ruled by LPSs of H, which are also LPSs of G. To show that H is abelian it is enough to prove that $gh = hg$ for all $g, h \in V$.

Let $\mathcal{U}$ be a special neighbourhood for H and $g, h \in V$. By our choice of V, we have $h = \mathbb{X}(1)$ for some $\mathbb{X} \in L(H)$ with $I \subseteq domain(\mathbb{X})$ and $\mathbb{X}(I) \subseteq \mathcal{U}$. Then it is clear that $domain(g\mathbb{X}g^{-1}) \supseteq I$. But since $g \in H = ker(Ad)$, we have

$$g\mathbb{X}g^{-1} = Ad_g(\mathbb{X}) = (Ad(g))(\mathbb{X}) = (id_{L(H)})(\mathbb{X}) = \mathbb{X}.$$

Therefore

$$ghg^{-1} = g\mathbb{X}(1)g^{-1} = (g\mathbb{X}g^{-1})(1) = \mathbb{X}(1) = h,$$

and we are done.

Finally, by Theorem 8.8, we have that a restriction of H is a local Lie group.

(2) For this part we simply note that the adjoint representation map induces an injective morphism $G'' \to Aut(L(G))$. Then G'' is a local Lie group by Lemma 8.10.

(3) Since G'' is a local Lie group, we can introduce *canonical coordinates of the second kind*. More precisely, we can find an open neighbourhood $\mathcal{U}'$ of 1 in G'' and a basis $\mathbb{X}'_1, \dots, \mathbb{X}'_r$ of $L(G'')$ such that the closure of $\mathcal{U}'$ is a special neighbourhood of G'', and every element of $\mathcal{U}'$ is of the form $\mathbb{X}'_1(s_1) \cdots \mathbb{X}'_r(s_r)$ for a *unique* tuple $(s_1, \dots, s_r) \in [-\beta', \beta']^r$. Now we choose a special neighbourhood $\mathcal{U}$ of G' such that $\pi(\mathcal{U}) \subseteq \mathcal{U}'$. Let $Z \subseteq \mathcal{U}$ be an open neighbourhood of 1 ruled by LPSs of G'. Fix a small $s_0 \in (0, \beta')$ such that $\mathbb{X}'_1(s_0), \dots, \mathbb{X}'_r(s_0) \in \pi(Z)$. Choose $x_i \in Z$ such that $\pi(x_i) = \mathbb{X}'_i(s_0)$. Let $\mathbb{X}_i \in L(G')$ be such that $\mathbb{X}_i(s_0) = x_i$.

Let $\beta < s_0$ be so that $\mathbb{X}_i(s) \in \mathcal{U}_{2r}$ if $|s| \leq \beta$ and $\mathbb{X}_1(s_1) \cdots \mathbb{X}_r(s_r) \in W$ if $|s_i| \leq \beta$ for all $i \in \{1, \dots, r\}$. A uniqueness of root argument yields that $\pi(\mathbb{X}_i(s)) = \mathbb{X}'_i(s)$ for $|s| \leq \beta$. Set

$$M := \{\mathbb{X}_1(s_1) \cdots \mathbb{X}_r(s_r) \,|\, |s_i| \leq \beta \text{ for all } i = 1, \dots, r\}$$

and let $W' \subseteq W$ be an open neighbourhood of 1 in G contained in the image of the map $[-\beta, \beta]^r \to G$ defined by

$$(s_1, \dots, s_r) \mapsto \mathbb{X}_1(s_1) \cdots \mathbb{X}_r(s_r).$$

These choices for M and W' satisfy condition (3) of Kuranishi's theorem.

□

9. Locally Euclidean local groups are NSS

In order to finish the proof of the LH5 we need to show that all locally euclidean local groups are NSS. We do this in two steps - from locally euclidean to NSCS, and from NSCS to NSS. We start by borrowing a theorem from Montgomery and Zippin [7, p. 105], the proof of which does not benefit from our use of nonstandard analysis.

Theorem 9.1. *Every compact connected nontrivial hausdorff topological group has a nontrivial 1-parameter subgroup (OPS).*

Lemma 9.2. *Let G be a topological group and $X : \mathbb{R} \to G$ an OPS of G.*

(1) If H is a closed subgroup of G, then either $X(\mathbb{R}) \subseteq H$ or there is a neighbourhood D of 0 in $\mathbb{R}$ such that $X(D) \cap H = \{1\}$.
(2) If X is nontrivial, then there is a neighbourhood D of 0 in $\mathbb{R}$ on which X is injective.

Proof.

(1) Suppose there is no such neighbourhood D. Then we can find a sequence (a_n) with $lim_{n\to\infty} a_n = 0$ and $X(a_n) \in H$ for all $n \in \mathbb{N}$. Then, since H is a subgroup and X is an OPS, we have that if
$$A := \{m.a_n \in \mathbb{R} \,|\, m \in \mathbb{Z}, n \in \mathbb{N}\},$$
then $X(A) \subseteq H$. But A is dense in $\mathbb{R}$ and X is continuous. Therefore $X(\mathbb{R}) \subseteq H$.
(2) If there is no such neighbourhood D, then the zeros of X get arbitrarily close as one gets closer to $X(0)$. Since $X(\mathbb{R})$ is a subgroup in G, then X is the trivial OPS. □

Lemma 9.3. *Suppose V is a neighbourhood of 1 in G. Then V contains a compact subgroup H of G and a neighbourhood W of 1 in G such that every subgroup of G contained in W is contained in H.*

Proof. Let $W \subseteq \mu$ be an infinitesimal internal neighbourhood of 1 in G^*. We let
$$S := \{a_1 \cdots a_\nu \,|\, \text{For all } i \in \{1,\dots,\nu\},\ a_i \in E_i \text{ for some internal subgroup } E_i \text{ of } G^*\}.$$
By Lemma 5.1, all the products in the definition of S are defined. Also, by the same lemma, S is an internal subgroup of G^* with $S \subseteq \mu$. So every internal subgroup of G^* contained in W is contained in S. Furthermore, if H is the internal closure of S in G^*, then H is an internally compact internal subgroup of G^* containing all of the subgroups of W and $H \subseteq V^*$. The desired result follows by transfer. □

Definition 9.4. *We call a topological space **feebly finite-dimensional** if, for some n, it does not contain a homeomorphic copy of $[0,1]^n$. Clearly locally euclidean local groups are feebly finite-dimensional.*

Lemma 9.5. *If G is feebly finite-dimensional, then G is NSCS.*

Proof. Suppose, for a contradiction, that G is feebly finite-dimensional, but not NSCS. We show that for every compact symmetric neighbourhood U of 1 in G contained in $\mathcal{U}_2$ and for arbitrarily large n, there is a compact subgroup of G

contained in U which contains a homeomorphic copy of $[0,1]^n$. Assume this holds for a given n and U.

Now Lemma 9.3 implies that there is a compact symmetric neighbourhood V of 1 in G, containing 1, and a compact subgroup $H \subseteq U$ that contains every subgroup of G contained in V. Since G is not NSCS, then V contains a nontrivial connected compact subgroup of G, and so we have a nontrivial OPS X of H. By shrinking V if necessary, we can suppose $X(\mathbb{R}) \nsubseteq V$. By our assumption we have a compact subgroup $G(V) \subseteq V$ of G and a homeomorphism $Y : [0,1]^n \to Y([0,1]^n) \subseteq G(V)$. We can assume, because we can replace X with $r \cdot X$ for a suitable $r \in \mathbb{R}$, that $X([0,1]) \subseteq V$, X is injective on $[0,1]$, and $X([0,1]) \cap G(V) = \{1\}$. Since H is a group, we can define $Z : [0,1] \times [0,1]^n \to H$ by

$$Z : (s,t) \mapsto X(s) \cdot Y(t),$$

which is a continuous map, as the continuous product of two other continuous maps.

Suppose $Z(s,t) = Z(s',t')$ with $s \geq s'$. Then we have

$$X(s - s') = Y(t')Y(t)^{-1} \in X([0,1]) \cap G(V) = \{1\}.$$

Since X is injective on $[0,1]$, we have $s = s'$, and since Y is injective, we have $t = t'$. It follows that Z is injective, and thus a homeomorphism onto its image. $\square$

We now need to show that NSCS groups are NSS. We do this using two lemmas.

Lemma 9.6. *Suppose H is a normal sublocal group of G which is totally disconnected, in the sense that there are no nontrivial connected subsets of H. Let $\pi : G \to G/H$ be the canonical projection. Then the map*

$$L(\pi) : L(G) \to L(G/H) \text{ defined by } L(\pi) : \mathbb{X} \mapsto \pi \circ \mathbb{X},$$

is surjective.

Proof. We put $G' := (G/H)_W$, where W is as in Lemma 2.25, and we let $\mathbb{Y} \in L(G')$. We want to find $\mathbb{X} \in L(G)$ such that $\pi \circ \mathbb{X} = \mathbb{Y}$.

If $\mathbb{Y}$ is trivial, then we let $\mathbb{X}$ be trivial and we are done. So assume $1 \in domain(\mathbb{Y})$ and $\mathbb{Y}(1) \neq 1_{G'}$. Fix $\nu > \mathbb{N}$ and let $h := \mathbb{Y}(\frac{1}{\nu}) \in \mu(1_{G'})$. We pick a compact symmetric neighbourhood of $1'_G$ with $\mathbb{Y}(1) \notin V$ and then pick a compact symmetric neighbourhood of 1_G with $U \subseteq W$ and $\pi(U) \subseteq V$.

We know that π is an open map, so we have $\mu(1_{G'}) \subseteq \pi(\mu(1_G))$ and we can choose $a \in \mu(1_G)$ with $\pi(a) = h$. Now let $\sigma := ord_U(a)$.

If $\nu \leq \sigma$, then $\pi(a^\nu) \in \pi(U)^* \subseteq V^*$, contradicting our choice, because we have $\pi(a^\nu) = h^\nu = 1_{G'}$. Therefore $\nu > \sigma$.

If $i \in o(\sigma)$, then $\pi(st(a^i)) = st(h^i) = 1_{G'}$. Let $G_U(a) = \{st(a^i) \mid i \in o(\sigma)\}$. Now, $\pi(st(a^i)) = 1_{G'}$ implies that $st(a^i) \in H$, and so by Lemma 3.5 $G_U(a)$ is a connected subgroup of G contained in H. Since H is totally disconnected, this implies that $G_U(a) = \{1_G\}$, i.e. $a^i \in \mu(1_G)$ for all $i \in o(\sigma)$, thus $a \in G(\sigma)$. Moreover, $a \notin G^o(\sigma)$, because $a \notin G^o(ord_U(a))$. So $[X_a] \neq \mathbb{O}$.

Now suppose $\sigma \in o(\nu)$ and let $t \in domain(\pi \circ [X_a])$. Then if $\frac{i}{\nu} \sim t$, then

$$\pi([X_a](t)) = \pi(st(a^i)) = st(h^i) = 1_{G'},$$

since $i \in o(\sigma)$. Therefore $[X_a] \in L(H)$. But H is totally disconnected, so $L(H)$ is trivial and hence $[X_a] = \mathbb{O}$, a contradiction.

Thus we have $\sigma = (r+\epsilon)\nu$ for some $r \in \mathbb{R}^+$ and $\epsilon \in \mu(1) \subseteq \mathbb{R}^*$. Thus $\pi \circ [X_a] = r\mathbb{Y}$ and $\mathbb{X} := \frac{1}{r}[X_a]$ is the desired lift. $\square$

Lemma 9.7. *Suppose G is a pure topological group such that there are no nontrivial OPSs $X : \mathbb{R} \to G$. Then G has a neighbourhood base at 1 of open subgroups of G. In particular, G is totally disconnected.*

Proof. We will show that for any neighbourhood V of 1 in G, there is a subgroup H of G in V that is also an open neighbourhood of 1 in G. Thus, we can always take a neighbourhood smaller than H and do the same, ultimately arriving at a neighbourhood basis for 1 consisting of open subgroups of G.

Let V be a neighbourhood of 1 in G. By Lemma 9.3 we have a neighbourhood $W \subseteq V$ of 1 in G and a compact subgroup $H \subseteq V$ of G containing all subgroups contained in W.

By assumption, we have that all nondegenerate infinitesimal elements of G^* are pure. But since G has no OPSs, then there are no pure elements. So all $a \in \mu$ are degenerate. So all $a \in \mu$ (internally) generate internal subgroups of G^* entirely contained in μ. However, W is a neighbourhood, so $\mu \subseteq W^*$. This means that $\mu \subseteq H^*$. Moreover, since we can multiply any element of G^* by an element of μ, this means that $\mu(b) \subseteq H^*$ for any $h \in H$. Thus H is an open neighbourhood of 1 in G. □

We are now ready to complete the proof of the LH5 with the last needed result. Note that all locally euclidean local groups are clearly locally connected. Then the following theorem completes the second step of this chapter.

Theorem 9.8. *If G is locally connected and NSCS, then G is NSS.*

Proof. We begin with several nested choices. Since G is NSCS we can choose a compact symmetric neighbourhood V of 1 in G such that V contains no nontrivial connected subgroups. By Lemma 9.3 we can choose an open neighbourhood $W \subseteq V$ of 1 in G and a compact subgroup $H_1 \subseteq V$ such that every subgroup of G contained in W is contained in H_1. By shrinking the above sets we can always make it so that $W \subseteq \mathcal{U}_6$.

Our previous proof shows that all degenerate $a \in \mu$ are such that $a \in H_1^*$. By Corollary 3.10, we have that G is pure, and so if $a \in \mu$ is nondegenerate, then it is pure. For such an a, if $a \in H_1^*$, then $a^\nu \in H_1^* \subseteq V^*$ for all ν, contradicting our choice of a neighbourhood V with no connected subgroups. Therefore, for nondegenerate $a \in \mu$, $a \notin H_1$.

Since $H_1 \subseteq V$, H_1 admits no nontrivial OPSs, and hence by Lemma 9.7 H_1 is totally disconnected. Choose an open (in H_1) subgroup H of H_1 contained in $H_1 \cap W$. Write $H = H_1 \cap W_1$ with W_1 an open neighbourhood of 1 in G and $W_1 \subseteq W$. Since H is an open neighbourhood of H_1, it is also closed in H_1 and thus H is a compact subset of G. This implies that the set

$$U_1 := \{a \in W_1 \mid aHa^{-1} \subseteq W_1\}$$

is open. Fix $a \in U_1$. Since aHa^{-1} is a subgroup of G contained in $W_1 \subseteq W$, then we have $aHa^{-1} \subseteq H_1$. Thus $aHa^{-1} \subseteq H_1 \cap W_1 = H$, implying that H, considered as a sublocal group of G, is normal. Putting $U = U_1 \cap U_1^{-1}$ we can take U as the associated normalising neighbourhood for H.

Note that $H \subseteq \mathcal{U}_6$ and $H^6 \subseteq U$. The facts that G is a regular topological space and H is compact allow is to choose symmetric open neighbourhoods W_2, W_3 of 1 in G such that $H \subseteq W_3 \subseteq \overline{W}_3 \subseteq W_2 \subseteq \mathcal{U}_6$ with $W_2^6 \subseteq U$. Let $G' := (G/H)_{W_2}$. If $a \in \mu$ is degenerate, then we have that $a \in H_1^* \cap W_1^* = H^*$, so $\pi(a) = 1_{G'}$. If $a \in \mu$ is nondegenerate, let $\tau := ord_{\overline{W}_3}(a)$. Then $a^\tau \in (\overline{(W)}_3)^*$, but $a^{\tau+1} \in W_2^* \backslash (\overline{W}_3)^*$. Choose an open neighbourhood U' of 1 in G such that $U'H \subseteq W_3$. Suppose $\pi(a^{\tau+1}) = \pi(x)$ for some $x \in (U')^*$. Then $a^{\tau+1} = xh$ for some $h \in H^*$, and thus $a^{\tau+1} \in W_3^*$, a contradiction. So $\pi(a)^{\tau+1} \notin \pi((U')^*)$. But also, for $i \in o(\tau)$, we have $\pi(a)^i \in \mu(1_{G'})$. Thus $\pi(a)$ is pure in G' and we have shown that G/H has no degenerate infinitesimals other than $1_{G'}$. That is, we have shown that G' is NSS.

Now, since G and G' are pure, $L(\pi) : L(G) \to L(G')$ is a continuous $\mathbb{R}$-linear map. Since H is totally disconnected, $L(H)$ is trivial, and so $\mathbb{X} = \mathbb{O}$. Since $L(G')$ is finite-dimensional, we can conclude that the map $L(\pi)$ is an isomorphism of real topological vector spaces.

We now choose a special neighbourhood $\mathcal{U}'$ of G' and let $E' : \mathcal{K}' \to K'$ be the bijective local exponential map for G'. Take a connected neighbourhood $\mathcal{U} \subseteq W_2$ of 1 in G with $\pi(\mathcal{U}) \subseteq K'$. If $x \in \mathcal{U}$, then there is a unique $\mathbb{Y} \in \mathcal{K}'$ with $\pi(x) = E(\mathbb{Y})$. Since $L(\pi)$ is a bijection, there is a unique $\mathbb{X} \in L(G)$ with $\pi \circ \mathbb{X} = \mathbb{Y}$. Thus we can write $x = \mathbb{X}(1) \cdot x_H$, where $x_H \in H$. It is easy to see that the map assigning a $\mathbb{Y} \in L(G')$ to each $x \in \mathcal{U}$ is continuous. From this and the fact that $L(\pi)$ is a homeomorphism, we can see that the map

$$\delta : \mathcal{U} \to H \text{ defined by } \delta : x \mapsto x_H$$

is continuous. Since $\mathcal{U}$ is connected, H is totally disconnected and $\delta(1) = 1$, it follows that $\delta(\mathcal{U}) = 1$. Now by the injectivity of $L(\pi)$ and E', we have that $\pi|_{\mathcal{U}}$ is injective, implying that G cannot hace any subgroups other than $\{1\}$ contained in $\mathcal{U}$. We thus conclude that G is NSS. □

References

[1] A.M. Gleason, *Groups without small subgroups*, Ann. of Math. **56** (1952), 193-212.

[2] D. Montgomery and L. Zippin, *Small subgroups of finite dimensional groups*, Ann. of Math. **56** (1952), 213-241.

[3] A. Robinson, *Nonstandard analysis*, North-Holland, Amsterdam, 1966.

[4] J. Hirschfeld, *The nonstandard treatment of Hilbert's fifth problem*, Transactions of the AMS, Vol. 321, No. 1 (Sep. 1990), 379-400.

[5] I. Goldbring, *Hilbert's fifth problem for local groups*, Ann. of Math., Vol. 172 (2010), No. 2, 1269-1314.

[6] B. Rynne and M. Youngson, *Linear functional analysis*, London: Springer (2nd ed.), 2008.

[7] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.

[8] M. Kuranishi, *On local euclidean groups satisfying certain conditions*, Proc. Amer. Math. Soc. **1** (1950), 372-380.

St. Hilda's College, University of Oxford, Oxford, United Kingdom OX4 1DY

E-mail address: mihail.hurmuzov@st-hildas.ox.ac.uk