NONSTANDARD ANALYSIS AND TOPOLOGY

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Abstract. In this survey we present the machinery of nonstandard analysis. We introduce the axiomatic approach and develop the classical construction of the nonstandard extension through ultrafilters. The main content of the survey is the nonstandard characterisation of topological concepts, culminating with a treatment of compactifications.

1. A little history

Calculus, as originally conceived by Gottfried Wilhelm Leibniz, involved infinitesimals. For reasons of mathematical rigour, however, this approach to analysis has been substituted for one using limits that is due largely to Cauchy and Weierstrass. Almost 300 years after the invention of calculus a rigourous way to deal with infinitesimals emerges in Abraham Robinson's nonstandard analysis. Although it was initially developed through a model theoretic approach in the early 60s [5], others, such as Wilhelmus Luxemburg [4], showed that the same results could be achieved using ultrafilters. This made Robinson's work more accessible to mathematicians who lacked training in formal logic. (The ultrafilter construction that is presented here in 2.2 is a particular case of this idea.)

One of the most popular among researchers applying the nonstandard methods is the axiomatic approach. The three axioms – **Extension**, **Transfer**, and **Saturation** – arise from the close historical connection to model theory and mathematical logic. The ultrapower construction can be viewed as either a proof of the consistency of the axioms or as an independent approach to building the theory. We will present the axioms with the basic concepts in 2.1, then we give an overview of the construction of a nonstandard extension of a mathematical universe V. We finish the section with an extended look at a concrete construction of one of nonstandard analysis' flagship systems – that of the hyperreals.

In 3 we consider a nonstandard extensions of topological spaces and the properties of these spaces that translate neatly. We present the classic results by Robinson [5] on openness, closedness, and compactness before reworking the separation axioms for topological spaces into claims about nonstandard extensions. The section and the survey are capped by a look at the nonstandard treatment of compactifications of a space.

2. Introduction to nonstandard analysis

We start with a mathematical universe V containing all relevant mathematical objects - $\mathbb{N}$, $\mathbb{R}$, various groups G, various topological spaces X, cartesian products of the above, their powersets, etc.

We then want to extend to a nonstandard universe V^*.

2.1. The axiomatic approach.

To state the three axioms we need some machinery. The first piece is a formal language in which to evaluate statements. Taking a lengthy detour into formal logic is unnecessary – we are only concerned with the following class of well-formed sentences:

Definition 2.1. *(The Language $\mathcal{L}(V)$) The formal statements we will be concerned with are the **bounded quantifier formulae** of the first order language $\mathcal{L}$. These are the sentences of $\mathcal{L}$ with*

- *logical symbols among: $=, \in, \neg, \wedge, \vee, \forall, \exists, \rightarrow, \leftrightarrow, (,)$,*
- *countably many variables: $x, y, x_1, y_1, \ldots$,*
- *countably many constants from V, and*
- *no free variables (i.e. unbounded quantifiers).*

We denote a bounded quantifier formula Φ of $\mathcal{L}$, containing the constants $A_1, A_2, \ldots \in V$, by $\Phi(A_1, A_2, \ldots)$.

We can now state the first two axioms:

Axiom 1. *(Extension) For any $A \in V$, we have its nonstandard extension $A^* \supseteq A$.*

We thus have a mapping $* : V \to V^*$ such that $A^* = *(A) \supseteq A$, where we identify a copy of V inside V^*. There is a much more robust correspondence between the universe and its nonstandard extension:

Axiom 2. *(Transfer) A bounded quantifier formula $\Phi(A_1, A_2, \ldots)$ is true in $\mathcal{L}(V)$ if and only if $\Phi(A_1^*, A_2^*, \ldots)$ is true in $\mathcal{L}(V^*)$, where $\Phi(A_1^*, A_2^*, \ldots)$ is obtained from $\Phi(A_1, A_2, \ldots)$ by replacing every occurrence of A_i with A_i^*, its nonstandard counterpart.*

For the last axiom we need to define some key properties of a set $\mathcal{A} \in V^*$:

Definition 2.2.

- *The objects in the image of the $*$-mapping are called **standard**. In other words, $\mathcal{A} \in V^*$ is standard if $\mathcal{A} = A^*$ for some $A \in V$.*
- *If $A \in V$, then A^* is called the **nonstandard extension** of A.*
- *If $A \in V$, then the set $A^\sigma = \{a^* \mid a \in A\}$ is called the **standard copy** of A.*
- *An object in V^* is called **internal** if it is an element of a standard set of V^*; otherwise it is called **external**. We denote the set of internal objects of V^* by V^*_{int}.*

Now let κ be an infinite cardinal. The last axiom depends on a choice of κ.

Axiom 3. *(Saturation) V^* is κ-saturated in the sense that*

$$\bigcap_{\gamma\in\Gamma} A_\gamma \neq \emptyset$$

for any family of internal sets $\{A_\gamma\}_{\gamma\in\Gamma}$ in V^* with the finite intersection property and index set Γ with $\operatorname{card}\Gamma \leq \kappa$.

The last axiom makes one fact clear – there are actually many nonstandard extensions of a universe V, although they can be shown to be isomorphic under some extra set-theoretical assumptions at least in the case when they have the same degree of saturation κ. The choice of κ, however, is open and depends on the standard theory and the specific goals. In particular, if (X, T) is a topological space, we apply a κ-saturated nonstandard model to a universe V containing X and the reals $\mathbb{R}$, and a degree of saturation $\kappa \geq \operatorname{card}\mathcal{B}$, where $\mathcal{B}$ is a basis for T.

At this point the emergence of a nonstandard extension of any system does not seem necessary – we have mentioned no concrete structure until now, only what properties such a structure would have to satisfy. The next subsection provides such a concrete structure.

2.2. **The ultrafilter construction.** The idea is the following. Given a set X and a set I, we want to define a notion of 'closeness' on X^I, and then take the quotient over that relation. We will say that $f \in X^I$ **agrees with** $g \in X^I$ **on a set** $A \subseteq I$ if $f(i) = g(i)$ for all $i \in A$. Two elements of X^I will be considered 'close' if they agree on a 'big enough' subset of I. Several intuitive consequences of that idea become immediate:

(1) If a and b do not agree on any subset of I they shouldn't be considered 'close'.
(2) If a is 'close' to b, because they agree on a set $\rho \subseteq I$, then if c agrees with b on a larger set $\rho \subseteq \tau \subseteq I$, then c should also be considered 'close' to b.
(3) If a is 'close' to b and b is 'close' to c, then a should be 'close' to c (after all we want 'closeness' to be an equivalence relation so we would be able to quotient it out later). This means that if a and b agree on $\rho \subseteq I$ and b and c agree on $\tau \subseteq I$, then it is sufficient that a and c agree on $\rho \cap \tau \subseteq I$ for them to be considered 'close'.
(4) For every a and b in X^I they should either be considered 'close' or not.

We have a topological tool that gives us exactly that.

Definition 2.3. *Given a set I, a **filter** on I is a set $\mathcal{F} \subseteq \mathcal{P}(I)$ such that:*

(1) $\emptyset \notin \mathcal{F}$.
(2) $A \in \mathcal{F}, A \subseteq B \implies B \in \mathcal{F}$.
(3) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$.

*An **ultrafilter** is a filter that cannot be enlarged, i.e. a filter that satisfies the further condition:*

(4) $\forall A \subseteq \mathcal{P}(I)$ *either* $A \in \mathcal{F}$ *or* $I \backslash A \in \mathcal{F}$.

Now we can make precise the ultrapower construction.

Definition 2.4. *Given sets X and I and an ultrafilter $\mathcal{F}$ of I, the* ***ultrapower*** *of X with respect to $\mathcal{F}$ is $X^{\mathcal{F}} = X^I/\sim_{\mathcal{F}}$, where for $f, g \in X^I$*

$$f \sim_{\mathcal{F}} g \text{ if and only if } \{i \in I | f(i) = g(i)\} \in \mathcal{F}.$$

We can now identify X as a subset of $X^{\mathcal{F}}$ by identifying X with

$$\{f \in X^I | f(i) = f(j) \forall i, j \in I\},$$

where $x \in X$ is identified with $f_x \in X^I$ if $f_x(i) = x$ for all $i \in I$. And since with this identification $f_x \sim_{\mathcal{F}} f_y$ if and only if $x = y$, then we can consider $X \subseteq X^{\mathcal{F}}$. Moreover, a function $f : X \to Y$ naturally extends to a function $f : X^{\mathcal{F}} \to Y^{\mathcal{F}}$ by $f([(i)]_{\mathcal{F}}) := [f(i)]_{\mathcal{F}}$. Therefore $X^{\mathcal{F}}$serves as a nonstandard extension of X in the sense defined above.

The transfer principle in this setting is known as **Łoś's Theorem** and is due to Jerzy Łoś. It states that any first-order formula is true in the ultraproduct if and only if the set of indices i, such that the formula is true in the copy of X in the ultraproduct corresponding to the index i, is in $\mathcal{F}$. Its proof is a classic induction by complexity of formulae.

What the largest cardinal κ such that X^* is κ-saturated depends both on the cardinality of I and on the particular ultrafilter $\mathcal{F}$ we choose in the ultrapower construction. However, since we can always choose a larger I and a 'finer' ultrafilter, it is usually the case that we do not belabour the point and we assume that our nonstandard extension is saturated for a large enough cardinal κ as is required by our arguments.

Theorem 2.5. *(Boolean properties) The extension mapping $* : V \to V^*$ is injective and its restriction on sets satisfies*

$$(A \cup B)^* = A^* \cup B^*$$
$$(A \cap B)^* = A^* \cap b^*$$
$$(A \backslash B)^* = A^* \backslash B^*,$$

for any sets $A, B \in V$.

Proof. To show injectivity, note that if $A^* = B^*$, then the first order formula $\Phi(A^*, B^*) = [A^* = B^*]$ is true, so by Transfer we have that the formula $\Phi(A, B) = [A = B]$ is true. The Boolean properties follow directly from the fact that they are all expressible as first order sentences in $\mathcal{L}(V)$. $\square$

2.3. **The hyperreals $\mathbb{R}^*$.** We can now explicitly construct the hyperreals, making precise the intuition that both Liebniz and Newton had about the properties of infinitesimals. The key fact that allows infinitesimal quantities in the hyperreal extension $\mathbb{R}^*$ is the fact that $\mathbb{R}^*$ is taken to be at least $\aleph_0$-saturated. Therefore, when we take the family of nested internal intervals $\{(0, \frac{1}{n})^* \in \mathbb{R}^* | n \in \mathbb{N}\}$: saturation tells us that (since clearly every finite intersection of these sets in nonempty) there is an element (which we sometimes will call ϵ) that is in the intersection of all of them, and is hence smaller than any representative from $\mathbb{R}$.

But we can construct this extension explicitly. There are many ways to do this, of course. We show the (arguably) simplest one:

Take $I = \omega$. We now want to pick an ultrafilter $\mathcal{F}$ on ω such that $\mathbb{R}^{\mathcal{F}}$ is $\aleph_0$-saturated.

Proposition 2.6. *If there is a finite set $\{a_1, a_2, \ldots, a_n\} \in \mathcal{F}$, then $\mathcal{F}$ is a principal ultrafilter, i.e. $\mathcal{F} = \{S \subseteq \omega | a \in S\}$ for some $a \in \omega$, abbreviated $\mathcal{F} = \uparrow a$.*

Proof. We proceed by induction on n.

Base case. If $\{a_1\} \in \mathcal{F}$, then $\mathcal{F}$ is clearly principal.

Inductive hypothesis. If $\{a_1, a_2, \ldots, a_n\} \in \mathcal{F}$, then $\mathcal{F}$ is a principal ultrafilter, for all $n < k$.

Inductive step. Suppose $\{a_1, a_2, \ldots, a_k\} \in \mathcal{F}$. Consider $\{a_1, a_2, \ldots, a_{k-1}\}$. Either $\{a_1, a_2, \ldots, a_{k-1}\} \in \mathcal{F}$ or $\{a_1, a_2, \ldots, a_{k-1}\} \notin \mathcal{F}$. The first case implies that $\mathcal{F}$ is a principal ultrafilter by IH. The second case implies (by property (4) of ultrafilters) that $\omega \backslash \{a_1, \ldots, a_{k-1}\} \in \mathcal{F}$. But then, by property (3) of ultrafilters $\{a_k\} = (\omega \backslash \{a_1, \ldots, a_{k-1}\}) \cap (\{a_1, a_2, \ldots, a_k\}) \in \mathcal{F}$ and so $\mathcal{F}$ is a principal filter. $\square$

We want to avoid $\mathcal{F}$ being a principal ultrafilter. This is because in the case where $\mathcal{F} = \uparrow a$ for some $a \in \omega$, when we construct the ultraproduct $\mathbb{R}^{\mathcal{F}}$ as above the equivalence classes of the relation $\sim_{\mathcal{F}}$ would correspond to the values of $f(a)$ in $\mathbb{R}$, and so $\mathbb{R}^{\mathcal{F}}$ will be isomorphic to $\mathbb{R}$. $\mathbb{R}$, however, is as we know *not* $\aleph_0$-saturated, which is what we are aiming for.

So we want for all finite sets S, S not to be in $\mathcal{F}$. This means (by property (4) of ultrafilters) that our ultrafilter $\mathcal{F}$ has to be an extension of the Fréchet filter

$$\mathcal{C} := \{A \subseteq \omega \mid \omega \backslash A \text{ is finite}\},$$

the set of all cofinite sets in ω. We have to show that such an ultrafilter exists. We do that by showing that $\mathcal{C}$ is a filter and that any filter can be extended to an ultrafilter.

Proposition 2.7. *$\mathcal{C}$ is a filter.*

Proof. We check the three requirements in the filter definition:

(1) $\emptyset$ is a finite set, so $\emptyset \notin \mathcal{C}$.
(2) Suppose $A \in \mathcal{C}$. So $A = \omega \backslash S$ for some finite S. If $B \supseteq A$, we have $B = (\omega \backslash S) \cup T = \omega \backslash (S \cap T)$ for some $T \subseteq \omega$. Since S is finite, then $S \cap T$ is also finite, so $B \in \mathcal{C}$.
(3) Suppose $A = \omega \backslash S \in \mathcal{C}$ and $B = \omega \backslash T \in \mathcal{C}$. Then

$$A \cap B = (\omega \backslash S) \cap (\omega \backslash T) = \omega \backslash (S \cup T).$$

Since both S and T are finite, then $S \cup T$ is finite. So $A \cap B \in \mathcal{C}$. $\square$

Lemma 2.8. *(The Ultrafilter Lemma) If $\mathcal{K}$ is a filter, then there exists $\mathcal{F} \supseteq \mathcal{K}$ – an ultrafilter.*

Proof. Let $\mathcal{K}$ be a filter on a set I.

Let $\Omega = \{\mathcal{W} \subseteq \mathcal{P}(I) | \mathcal{W} \text{ is a filter on } I\}$.

Now, considering the subset relation as an ordering, we can see that that gives us Ω as a partially ordered set.

If $N \subseteq \Omega$ is a non-empty chain, then we check that $\bigcup N$ is a filter as well (and so is in Ω):

(1) $\emptyset \notin \mathcal{W}$ for all $\mathcal{W} \in N$. So $\emptyset \notin \bigcup N$.
(2) Suppose $A \in \bigcup N$. Then $A \in \mathcal{W}_k$ for some $\mathcal{W}_k \in N$. If $B \supseteq A$, then $B \in \mathcal{W}_k$ (by $\mathcal{W}_k$'s being a filter). So $B \in \bigcup N$.
(3) Suppose $A, B \in \bigcup N$. Then, by N's being a chain, $A, B \in \mathcal{W}_k$ for some $\mathcal{W}_k \in N$. Then $A \cap B \in \mathcal{W}_k \subseteq \bigcup N$.

So by Zorn's Lemma, there is a maximal element in Ω for every $\mathcal{W} \in \Omega$. So there is a maximal element $\mathcal{F}$ of Ωsuch that $\mathcal{K} \subseteq \mathcal{F}$. In this context this means that $\mathcal{F}$ is an ultrafilter extending $\mathcal{K}$. □

The previous two results show that there is an ultrafilter $\mathcal{F}$ that contains all cofinite sets. We can now construct the ultraproduct $\mathbb{R}^{\mathcal{F}}$. Choosing $\mathcal{F}$ to be nonprincipal is enough to ensure that we have at least $\aleph_0$-saturation. We shall thus not endeavour to make the choice of ultrafilter any more precise, for this would be enough for the present purposes. We denote the set $\mathbb{R}^{\mathcal{F}}$ by $\mathbb{R}^*$ from now on and we will be concerned with the properties shared by all ultrapowers of $\mathbb{R}$ with respect to a nonprincipal ultrafilter.

We can give a concrete example of saturation. Take $[f] \in \mathbb{R}^*$ for $f \in \mathbb{R}^\omega$ such that $f(n) = \frac{1}{n}$. It is clear that $-x < [f] < x$ for all $x \in \mathbb{R}$, because for any such x we have $x \geq f(n)$ for only finitely many $n \in \omega$. So we have an infinitesimal element of $\mathbb{R}^*$. Call this element ϵ.

Definition 2.9. *We call numbers e in $\mathbb{R}^*$, such that $-x < e < x$ for all $x \in \mathbb{R}$, **infinitesimal.** We call numbers f in $\mathbb{R}^*$ with infinitesimal multiplicative inverse **infinite.** We further call all non-infinite numbers **finite**.*

Consider the ring $\tilde{\mathbb{R}}^*$ of all finite numbers in $\mathbb{R}^*$. It is clear that $e.x$ is infinitesimal for all e – infinitesimal and x – finite. Therefore the set $(\epsilon) = \{e \text{ -infinitesimal}\}$ is an ideal. Moreover, it is a maximal ideal. So we can now take the quotient $\tilde{\mathbb{R}}^*/(\epsilon)$. It is obvious that $\tilde{\mathbb{R}}^*/(\epsilon) \cong \mathbb{R}$.

Definition 2.10. *We denote by $st(.) : \tilde{\mathbb{R}}^* \to \mathbb{R}$ the **standard part function** – the composition of the natural map $\tilde{\mathbb{R}}^* \to \tilde{\mathbb{R}}^*/(\epsilon)$ with the isomorphism alluded to above $\tilde{\mathbb{R}}^*/(\epsilon) \to \mathbb{R}$. The $st(.)$ map is thus an additive and multiplicative homomorphism.*

We can now do some simple analysis in $\mathbb{R}$ by simply doing algebra in $\tilde{\mathbb{R}}^*$.

Example 2.11. (The derivative of x^2.) We can calculate the derivative of the function $x \mapsto x^2$ by considering the infinitesimal change of x^2 when x

changes infinitesimally and taking the standard part:

$$\begin{aligned} st\left(\frac{(x+e)^2-x^2}{e}\right) &= st\left(\frac{x^2+2xe+e^2-x^2}{e}\right) \\ &= st\left(\frac{2xe+e^2}{e}\right) \\ &= st(2x+e) = 2x \end{aligned}$$

Another useful idea is to identify $\mathbb{Z}^*$ as a subset of $\mathbb{R}^*$. We take the floor function

$$(\lfloor\cdot\rfloor) : \mathbb{R} \to \mathbb{Z} \text{ defined by } (\lfloor\cdot\rfloor) : x \mapsto \max\{n \in \mathbb{Z} \mid n \leq x\}$$

We know this map is onto $\mathbb{Z}$. We can take its nonstandard extension in $\mathbb{R}^*$ and identify $\operatorname{im}(\lfloor\cdot\rfloor^*) = \mathbb{Z}^*$. This way we actually have some information as to what kind of set $\mathbb{Z}^*$ actually is.

Since $\mathbb{R}^*$ inherits its order from $\mathbb{R}$, then it is totally ordered. It is clear that $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\}$. Therefore $\mathbb{N}^* = \{n \in \mathbb{Z}^* | n \geq 0\}$.

If we apply the transfer principle to the process of induction on $\mathbb{N}$ we get the following analogue for $\mathbb{N}^*$:

Theorem 2.12. *(Internal Induction) For any internal subset A of $\mathbb{N}^*$ if*

(1) 1 is an element of A, and

(2) for every $n \in A$, $n+1 \in A$,

then $A = \mathbb{N}^*$.

Theorem 2.13. *(Overflow Principle) Assume $A \subseteq \mathbb{N}^*$ is internal and that $\mathbb{N} \subseteq A$. Then there is a $\nu \in \mathbb{N}^* \backslash \mathbb{N}$ such that $\nu \in A$.*

Proof. Assume for a contradiction that $A = \mathbb{N} \subseteq \mathbb{N}^*$ is internal. Then, clearly, for all $n \in A$, $n+1 \in A$. But by Theorem 2.12 this implies that $\mathbb{N} = A = \mathbb{N}^*$. However, we are assuming sufficient saturation for this to not be true (even $\aleph_0$-saturation is enough here). $\square$

3. Nonstandard analysis in topology

Let now (X, T) be a topological space. To apply nonstandard methods, we require the universe V to contain both the space X and the reals $\mathbb{R}$, and we take V^* – a κ-saturated nonstandard model with $\kappa > \operatorname{card} T$. In the cases where we consider two topological spaces $(X, T), (X', T')$, we assume $X \cup X' \cup \mathbb{R} \subseteq V$ and

$$\kappa > \max(\operatorname{card} T, \operatorname{card} T').$$

3.1. **Classical results.**

Definition 3.1. *(Monads) Let X^* be the nonstandard extension of X. Then:*

- *For any $\alpha \in X^*$ define the monad $\mu(\alpha)$ of α by*
$$\mu(\alpha) = \bigcap\{G^* \mid \alpha \in G^*, G \in T\}.$$
- *For any $A \subseteq X^*$ define*
$$\mu(A) = \bigcup\{G^* \mid A \subseteq G, G \in T\}.$$

The following properties of the monad follow directly from the definition.

Lemma 3.2. *If $A, B \subseteq X^*, \alpha, \beta \in X^*$, then:*

(1) $A \subseteq \mu(A)$.
(2) $A \subseteq B$ *implies* $\mu(A) \subseteq \mu(B)$.
(3) $\mu(\mu(A)) = \mu(A)$.
(4) $\alpha \in A$ *implies* $\mu(\alpha) \subseteq \mu(A)$.
(5) $\alpha \in \mu(\beta)$ *if and only if* $\mu(\alpha) \subseteq \mu(\beta)$.
(6) $\alpha \in \mu(\beta)$ *and* $\beta \in \mu(\alpha)$ *if and only if* $\mu(\alpha) = \mu(\beta)$.

Theorem 3.3. *(Balloon and Nuclei Principles) Let $x \in X$ and $\mu(x)$ be the monad of x in (X, T).*

- Balloon Principle: *If $\mu(x) \subset \mathcal{B}$ for some internal set $\mathcal{B} \subseteq X^*$, then there exists $G \in T$ such that $\mu(x) \subset G^* \subseteq \mathcal{B}$ (ballooning of $\mu(x)$ into G^*).*
- Nuclei Principle: *There exists an internal set $\mathcal{A} \subseteq X^*$ such that $x \in \mathcal{A} \subset \mu(x)$. The set $\mathcal{A}$ is called a nuclei of $\mu(x)$.*

Proof. For the Balloon Principle, suppose towards a contradiction that we have $G^* \backslash \mathcal{B} \neq \emptyset$ for all $G \in T, x \in G$. Observe that the family of sets $\{G^* \backslash \mathcal{B}\}_{G \in T, x \in G}$ has the finite intersection property, since
$$(G_1^* \backslash \mathcal{B}) \cap (G_2^* \backslash \mathcal{B}) = (G_1 \cap G_2)^* \backslash \mathcal{B}.$$
It follows from Saturation then that
$$\mu(x) \backslash \mathcal{B} = \bigcap_{x \in G \in T} (G^* \backslash \mathcal{B}) \neq \emptyset,$$
which contradicts out assumption.

For the Nuclei Principle, define the family $\{S_G\}_{x \in G \in T}$. where
$$S_G = \{H \in T \mid x \in H \in G\},$$
and observe that it has the finite intersection property since $G \in S_G$, thus, $S_G \neq \emptyset$, and, on the other hand, $S_{G_1} \cap S_{G_2} = S_{G_1 \cap G_2}$. It follows by Saturation that there exists $\mathcal{A}$ in the intersection
$$\bigcap_{x \in G \in T} S_G^*.$$

But note that

$$S_G^* = \{H \in T^* \mid x \in H \subseteq G^*\}.$$

Thus $\mathcal{A}$ is internal (as an element of T^*) and $\mathcal{A} \subset \mu(x)$, as required. $\square$

Theorem 3.4. *(A. Robinson [5])*

(1) *Let $x \in H \subseteq X$. Then x is an interior point of H in (X, T) if and only if $\mu(x) \subset H^*$. Consequently, H is open in (X, T) if and only if $\mu(x) \subset H^*$ for all $x \in H$.*

(2) *A set $F \subseteq X$ is closed in (X, T) if and only if $F^* \cap \mu(x) \neq \emptyset$ implies $x \in F$ for any $x \in X$.*

(3) *Let $A \subseteq X$ and $\mathrm{cl}_X(A)$ be the closure of A in (X, T). Then*

$$\mathrm{cl}_X(A) = \{x \in X \mid A^* \cap \mu(x) \neq \emptyset.$$

Proof.

(1) ($\Rightarrow$): If x is an interior point of H, then $\mu(x) \subset H^*$ by the definition of the monad.

($\Leftarrow$): Suppose that x is not an interior point of H, i.e. that $G \backslash H \neq \emptyset$ for all G such that $x \in G \in T$. Observe that the family of sets $\{G \backslash H\}_{x \in G \in T}$ has the finite intersection property. Therefore the family of internal (even standard) sets $\{G^* \backslash H^*\}_{x \in G \in T}$ has the finite intersection property, since $(G \backslash H)^* = G^* \backslash H^*$. If follows that the intersection $\mu(x) \backslash H^*$ is non-empty by Saturation $\lightning$.

(2) Suppose that $x \in X \backslash F$. We have $\mu(x) \in X^* \backslash F^*$, by the above part, since $X \backslash F$ is open by assumption, and $X^* \backslash F^* = (X \backslash F)^*$. Therefore $\mu(x) \cap F^* = \emptyset$ $\lightning$.

(3) ($\subseteq$): Let $x \in \mathrm{cl}_X(A)$, i.e. $x \in F$ for all F such that $A \subset F \subseteq X$, $X \backslash F \in T$. Suppose that $A^* \cap \mu(x) = \emptyset$. Then, by the Balloon Principle (applied for $\mathcal{B} = X^* \backslash A^*$), there exists $G \in T, x \in G$, such that $A^* \cap G^* = \emptyset$. Thus we have $A^* \subseteq (X \backslash G)^*$, implying $A \subseteq X \backslash G$, by the Boolean Properties. Hence it follows that $x \in X \backslash G$, by our assumption (since $X \backslash G$ is a closed set) $\lightning$.

($\supseteq$): Let $x \in X$ and $A^* \cap \mu(x) \neq \emptyset$. We have to show that $x \in F$ for all F such that $A \subset F \subseteq X$ and $X \backslash F \in T$. Suppose that $x \notin F$ for some F as above. Then $x \in X \backslash F$. On the other hand, $A \subset F$ implies $A^* \subset F^*$, by the Boolean Properties. Hence, $A^* \cap (X^* \backslash F^*) = \emptyset$, which implies $A^* \cap \mu(x) \neq \emptyset$ (because $A^* \cap \mu(x) \subseteq A^* \cap (X^* \backslash F^*)$) $\lightning$.

$\square$

Definition 3.5. *(Nearstandard Points and Standard Part) Let $\mu(x)$ be the monad of $x \in (X, T)$.*

(1) *If $A \subseteq X$, then the points in the union*

$$\tilde{A} = \bigcup_{x \in A} \mu(x)$$

are called the ***nearstandard points of*** A^**. In particular, the points in* $\tilde{X}$ *are called the nearstandard points of* X^**.*

(2) Assume, in addition, that (X, T) *is a regular Hausdorff space. Then the mapping* $\mathrm{st}_X : \tilde{X} \to X$*, defined by* $\mathrm{st}_X(\chi) = x, \chi \in \mu(x)$*, is called the* ***standard part mapping****.*

Lemma 3.6.

(1) Let $A, B \subseteq X^*$*. Then* $\mu(A) \cap \mu(B) = \emptyset$ *if and only if there exist open disjoint sets* G *and* H *such that* $A \subseteq G^*$ *and* $B \subseteq H^*$*.*

(2) Let $\alpha, \beta \subseteq X^*$*. Then* $\mu(\alpha) \cap \mu(\beta) = \emptyset$ *if and only if there exist open disjoint sets* G *and* H *such that* $\alpha \in G^*$ *and* $\beta \in H^*$*.*

Proof. It is enough to prove the first part, since setting $A = \{\alpha\}, B = \{\beta\}$ implies the second part.

Let $\mu(A) \cap \mu(B) = \emptyset$ and suppose that $G \cap H \neq \emptyset$ for all open G and H such that $A \subseteq G^*$ and $B \subseteq H^*$. By Saturation

$$\mu(A) \cap \mu(B) = \bigcap\{(G \cap H)^* \mid G, H \in T, A \subseteq G^*, B \subseteq H^*\} \neq \emptyset.$$

☇ . The converse is immediate. $\square$

Theorem 3.7. *(Characterisation) Let* $A \subseteq X$*. Then the following conditions are equivalent:*

(1) A *is compact in* (X, T)*.*
(2) $A^* \subseteq \bigcup_{x \in A} \mu(x)$*.*
(3) $\bigcup_{x \in A} \mu(x) = \bigcup_{\alpha \in A^*} \mu(\alpha)$*.*
(4) $\bigcup_{x \in A} \mu(x) = \mu(A)$*.*

3.2. **Separation properties.** We can translate the separation properties of topological spaces into the language of NSA. For clarity we state their definitions as well:

Let (X, T) be a topological space. Then:

(1) A space (X, T) is T_0 if $x \neq y \Rightarrow$ there is $U \in T$ with either $x \in U, y \notin U$ or $y \in U, x \notin U$.
(2) A space (X, T) is T_1 if $\{x\}$ is closed for all $x \in X$.
(3) A space (X, T) is T_2 (or Hausdorff) if $x \neq y \Rightarrow$ there are $U_x, U_y \in T$ with $x \in U_x, x \notin U_y$ and $y \in U_y$, $y \notin U_x$.
(4) A space (X, T) is regular if for any closed $F \subseteq X$ and $x \in X$ such that $x \notin F$, there are $U_x, U_F \in T$ with $x \in U_x, x \notin U_F$ and $F \subseteq U_F, F \cap U_x = \emptyset$.
(5) A space (X, T) is normal if for any two disjoint closed $E, F \subseteq X$ there are $U_E, U_F \in T$ with $E \subseteq U_E$, $E \cap U_F = \emptyset$ and $F \subseteq U_F, F \cap U_E = \emptyset$.

Theorem 3.8. *The topological space* (X, T) *is:*

(1) T_0 *if and only if* $x = y \Leftrightarrow \mu(x) = \mu(y)$ *for all* $x, y \in X$*.*
(2) T_1 *if and only if* $x = y \Leftrightarrow$ *either* $\mu(x) \subseteq \mu(y)$ *or* $\mu(y) \subseteq \mu(x)$ *for all* $x, y \in X$*.*

(3) T_2 *or Hausdorff if and only if* $x \neq y \Rightarrow \mu(x) \cap \mu(y) = \emptyset$ *for all* $x, y \in X$.
(4) *regular if and only if* $\alpha \notin \mu(x) \Rightarrow \mu(\alpha) \cap \mu(x) = \emptyset$ *for any* $\alpha \in X^*$ *and* $x \in X$.
(5) *normal if and only if* $E \cap F = \emptyset \Rightarrow \mu(E) \cap \mu(F) = \emptyset$ *for any two closed sets* $E, F \subseteq X$.

Proof.

(1) ($\Rightarrow$): $x = y \Rightarrow \mu(x) = \mu(y)$ is clear. Suppose $x \neq y$ and $U \in T$ with $x \in U, y \notin U$. Then $y \in \bigcap_{U \in T, x \in U} U$, so $\mu(y) \neq \mu(x)$.
($\Leftarrow$): Suppose $x \neq y$ and $\alpha \in \mu(x) \backslash \mu(y)$. So $\exists U \in T$ with $x \in U$ and $\alpha \notin U^*$. But then $x \notin U$, since otherwise $\alpha \in \mu(y) \subseteq U^*$.

(2) ($\Rightarrow$): $x = y \Rightarrow \mu(x) \subseteq \mu(y)$ or $\mu(y) \subseteq \mu(x)$ – clear.
Suppose T_1 and $\mu(x) \subseteq \mu(y)$. If $x \neq y$, then $G = X \backslash \{x\}$ is open and $x \notin G$, $y \in G$. This means $x \notin \mu(y)$. ☿
($\Leftarrow$): Suppose not T_1 – $\exists x \neq y$ with $y \in \mathrm{cl}\{x\}$. But then $x \in \mu(y)$, so $\mu(x) \subseteq \mu(y)$ and so $x = y$. ☿

(3) ($\Rightarrow$): Suppose T_2 and $x \neq y$. Then clearly $\mu(x) \cap \mu(y) = \emptyset$.
($\Leftarrow$): Suppose $x \neq y$. Then $\mu(x) \cap \mu(y) = \emptyset$. Then

$$x \notin \bigcap_{G \in T, y \in G} G^*, \text{ so } x \notin G_1 \ni y, G \in T$$

$$y \notin \bigcap_{G \in T, y \in G} G^*, \text{ so } y \notin G_1 \ni x, G \in T.$$

(4) ($\Rightarrow$): Suppose regular, $\alpha \in X^*, x \in X$ with $\alpha \notin \mu(x)$. Then $\exists G \in T$ with $x \in G$, $\alpha \notin G^*$. By regularity, $\exists U \in T$ with $x \in U$ and $\mathrm{cl}_X U \subseteq G$. So $\mu(x) \subseteq U^*$ and $\mu(\alpha) \subseteq (X - \mathrm{cl}_X U)^*$ since

$$\alpha \in X^* \backslash G^* = (X \backslash G)^* \subseteq (X - \mathrm{cl}_X U)^*.$$

So $\mu(\alpha) \cap \mu(x) = \emptyset$.
($\Leftarrow$): Suppose (X, T) is not regular. We find $\alpha \in X^*, x \in X$ with $\alpha \notin \mu(x)$ but $\mu(\alpha) \cap \mu(x) \neq \emptyset$.
By non-regularity, there are $x \in X$ and $G \in T$ with $x \in G$ and $\mathrm{cl}_X H \cap G^c \neq \emptyset$ for all $H \in T$ with $x \in H$. But then let

$$\alpha \in \bigcap \{(\mathrm{cl}_X H)^* \mid H \in T, x \in H\} \backslash G^*.$$

Since $\alpha \in (\mathrm{cl}_X H)^* = \mathrm{cl}_{X^*} H^*$, then $\forall\, O, H \in T$ with $\alpha \in O^*$ and $x \in H$, we have $O^* \cap H^* \neq \emptyset$. Also $\alpha \notin \mu(x)$, since $\alpha \notin G^*$. But then

$$\mu(\alpha) \cap \mu(x) = \bigcap \{O^* \cap H^* \mid O, H \in T, \alpha \in O^*, x \in H\} \neq \emptyset.$$

(5) ($\Rightarrow$): If (X, T) is normal, then the statement is clear.
($\Leftarrow$): Suppose (X, T) is not normal, Then $\exists E, F$ – closed, such that $\forall\, U_1, U_2 \in T$ with $E \subseteq U_1, F \subseteq U_2$, we have $U_1 \cap U_2 \neq \emptyset$. So

$$\mu(E) \cap \mu(F) = \bigcap \{U_1^* \cap U_2^* \mid U_1, U_2 \in T, E \subseteq U_1, F \subseteq U_2\} \neq \emptyset.$$

□

3.3. **Compactifications – a taster.** Let (X,T) be a topological space. Then the collection of sets $T^\sigma := \{G^* \mid G \in T\}$ forms a base for a topology on X^*. We denote this topology by T^s and the corresponding topological space by (X^*, T^s). We call T^s **the standard topology** on X^* and the space (X^*, T^s) – **the nonstandard compactification** of (X,T).

Theorem 3.9. *Let (X,T) be a topological space and (X^*,T^s) its nonstandard compactification (as defined above). Then:*

(1) Every internal subset A of X^* is compact in (X^*,T^s).
(2) (X^*,T^s) is a compact topological space and (X,T) is a dense subspace of it.

We have a number of satisfying results connecting properties of the nonstandard compactification of a space and the space itself:

(1) (X^*,T^s) is normal iff (X,T) is normal.
(2) (X^*,T^s) is regular iff every open set in (X,T) is closed.
(3) The previous statement implies that if D is the discrete topology on $\mathbb{N}$, then $(\mathbb{N}^*, D^s)$ is not a T_0 space.
(4) Therefore, (X^*,T^s) is a T_0 space iff X is finite.

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