HYPERBOLIC STRUCTURE OF KNOT COMPLEMENTS

MIHAIL HURMUZOV

Abstract. In this survey we demonstrate the construction of a hyperbolic structure on several knot/link complements. We mainly follow a manuscript edition of William P. Thurston's famous unpublished textbook on the geometry and topology of three-manifolds. For each knot/link complement, a cell-complex is found, giving the space as ideal polyhedra with face identifications.

1. A little history

A graduate student at the University of Southampton in England, Robert Riley in 1974 showed that the complement of the figure-eight knot has a hyperbolic structure [3]. He did this "indirectly" – he proved that the fundamental group of the figure-eight knot complement is isomorphic to a subgroup of $PSL_2\mathbb{C}$ (the projective special linear group) and then used the theory of Haken manifolds to show that the figure-eight knot complement is homeomorphic to $\mathbb{H}^3$, the hyperbolic 3-space, mod a discrete group of isometries [5]. Riley later showed that several other knot complements admit a hyperbolic structure and conjectured that indeed all knot complements apart from torus knots and satellite knots admit a hyperbolic structure.

Meeting with Riley in Princeton in 1976, William P. Thurston became interested in the subject of knot complements and hyperbolic geometry. In Thurston's grand scheme, it is the topology of a manifold that limits and frequently determines its possible geometries. He came up with an explicit way of constructing hyperbolic structures on knot complements; his ideas lead the discussion in the rest of this survey.

Relying in part on his experience with knot complements, in 1978 Thurston completed his "hyperbolisation theorem" for Haken manifolds, for which he won the Fields medal in 1982. The full geometrisation conjecture was famously proved by Grigori Perelman in 2003 using Ricci flow with surgery; Perelman subsequently declined both a Fields Medal and the Millennium Prize for his contribution.

In what follows, we construct a cell complex M such that for a particular point $v \in M$, *1)* $M\setminus\{v\}$ is homeomorphic to $S^3\setminus E$ (where E is the figure-eight knot) and *2)* $M\setminus\{v\}$ is a hyperbolic manifold, i.e. every point has a hyperbolic neighbourhood. Throughout the survey, the images used are either taken from George K. Francis' stunning book [1], or drawn by me.

2. Preliminaries

2.1. **Hyperbolic Geometry.** Proving the parallel postulate – that through every given point exists a unique line parallel to a given line – from the other axioms of Euclid was a millennia-long endeavour that ended with the discovery of non-Euclidean geometry and sparked interest in independency proofs throughout mathematics. The two possible alternatives to the parallel postulate give rise to the two non-Euclidean geometries – elliptic geometry, where there is *no line* passing through the point and parallel to a given line; and hyperbolic geometry, where there are *many* such lines. The underlying spaces for these geometries are naturally Riemannian manifolds of constant sectional curvature $+1$ for elliptic, 0 for Euclidean, and -1 for hyperbolic geometry.

Certain surfaces of revolution in $\mathbb{R}^3$ have constant curvature -1 and so give an idea of the local picture of the hyperbolic plane. The simplest of these is the pseudosphere (Figure 1), the surface of revolution generated by a tractrix. A tractrix is the track of a point, which starts at $(0, 1)$ and is dragged by a point walking along the x-axis from $(0, 0)$ via a chain of unit length. The pseudosphere is not complete, howerer – it has an edge, beyond which it cannot be extended. Hilbert proved the remarkable theorem that *no* complete surface with curvature -1 can exist in $\mathbb{R}^3$.

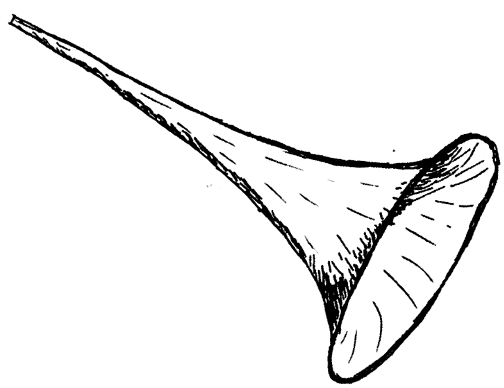


Figure 1. The pseudosphere

There are several useful models of hyperbolic geometry:

Definition 2.1. *The* Poincaré disk model *(Figure 2) takes the interior of the n-disk D^n as a map of hyperbolic space. A hyperbolic line is any Euclidean circle (or line) orthogonal to ∂D^n; a hyperbolic plane is a Euclidean sphere or plane orthogonal to ∂D^n; etc. Hyperbolic arc length $\sqrt{ds^2}$ is given by*

$$ds^2 = \left(\frac{1}{1-r^2}\right)^2 dx^2,$$

where $\sqrt{dx^2}$ *is Euclidean arc length and* r *is distance from the origin. The boundary* S^{n-1}_∞ *of the disk is called* the sphere at infinity.

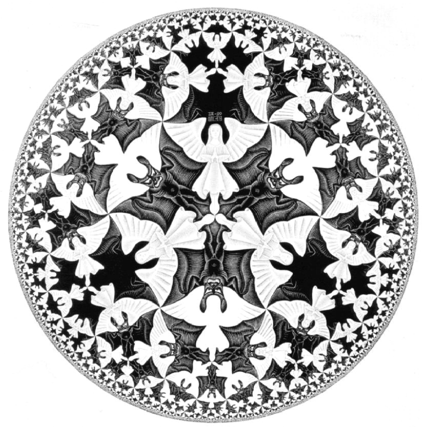


FIGURE 2. "Angels and Devils" – one of M.C. Escher's famous drawings using the Poincaré disk model

Definition 2.2. *The* Poincaré half-space model *(Figure 3) takes the upper half Euclidean space* $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ *as a map of hyperbolic space. Here a line is any vertical line (line in the* x_n *direction) or arc of circle that intersects the plane* $x_n = 0$ *at right angles. The hyperbolic metric is*

$$ds^2 = \left(\frac{1}{x_n}\right)^2 dx^2.$$

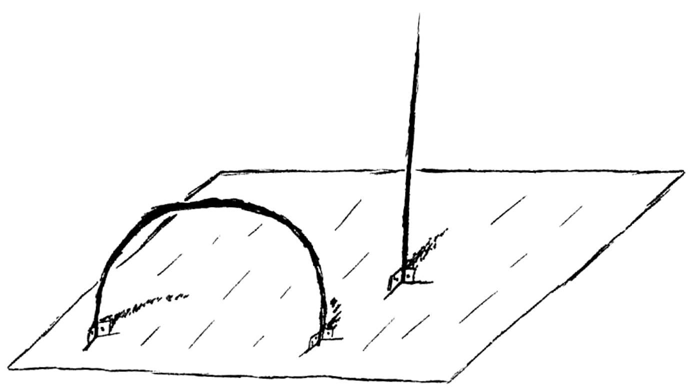


FIGURE 3. The half-space model for $n = 3$

A well-known fact about the above two models is that they are *conformal* – they preserve angles. That means that the angles we measure in the Euclidean space representation are the same as the actual angles in hyperbolic space.

As both the above are maps of hyperbolic space, we can formally require a hyperbolic manifold to have Riemannian neighbourhoods isometric to one of the above models.

Definition 2.3. Hyperbolic n-manifolds *are the Riemannian manifolds with isometric charts to either the Poincaré n-disk or the Poincare n-half-space.*

2.2. **Knot Theory.**

Definition 2.4. *A* knot *is an embedding of S^1 into S^3. Two knots are considered the same if one can distort one knot into the other. More precisely, if K and L are two knots with embedding maps f and g, respectively, then $K \sim L$ if and only if there is a continuous map*

$$F : S^3 \times [0,1] \to S^3$$

such that F_0 is the identity map, F_t is a homeomorphism for each t, and $F_1 \circ f = g$.

Definition 2.5. *A* torus knot *(Figure 4) is a knot that can be represented as a closed simple curve on the surface of a 2-torus.*

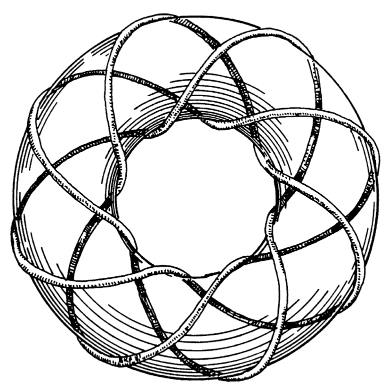


FIGURE 4. The $(3, 8)$ torus knot

Definition 2.6. *A* satellite knot *(Figure 5) is a knot that contains an incompressible, non boundary-parallel torus in its complement.*

Equivalently, K is a satellite knot if there is a nontrivial knot K' lying inside the solid torus V (nontrivial here is in the sense that K' is not allowed to sit inside of a 3-ball in V and is not allowed to be isotopic to the core curve of V) and a nontrivial embedding $f : V \to S^3$ such that $f(V) = K$.

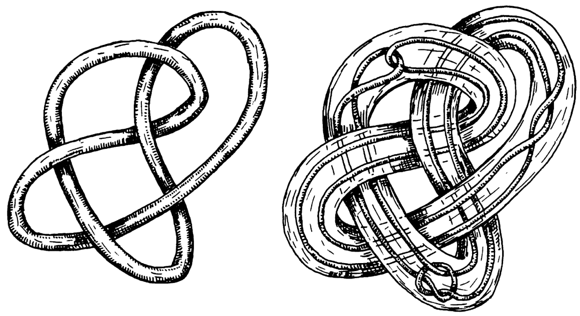


Figure 5. A satellite knot on the right with its companion knot (the embedding f) on the left

Definition 2.7. *A* hyperbolic knot *is a knot that has a complement that admits a hyperbolic structure.*

Following from the hyperbolisation theorem of Thurston's mentioned in Chapter 1, we have the confirmation of Riley's conjecture that almost all knots are hyperbolic:

Theorem (Classification of Knots)**.** *All knots are either torus knots, satellite knots, or hyperbolic knots, and these categories are disjoint.*

3. From Gluing Polyhedra to Hyperbolic Structures

For the remainder of the survey, unless otherwise noted, we will identify $\mathbb{H}^n$ with the Poincaré n-disk D^n. Note then that the natural $\mathbb{R}^n$ inclusions $D^1 \subseteq D^2 \subseteq \cdots \subseteq D^n$ induce the inclusions $\mathbb{H}^1 \subseteq \mathbb{H}^2 \subseteq \cdots \subseteq \mathbb{H}^n$ in $\mathbb{H}^n$.

Definition 3.1. *We build ideal polyhedra in $\mathbb{H}^n$ using some basic geometry in hyperbolic space.*

(1) A k-dimensional hyperplane *in $\mathbb{H}^n$ is the image of $D^k \subseteq D^n$ after an isometry $D^n \to D^n$.*
(2) A half-space *is the closure in D^n of one component of the complement of an $(n-1)$-dimensional hyperplane.*
(3) A polyhedron *in $\mathbb{H}^n$ is a compact subset of $\mathbb{H}^n$ that is the intersection of finitely many half-spaces. The dimension of a polyhedron is the smallest dimension of a hyperplane containing it. We will call a polyhedron* standard*, if its dimension is n in $\mathbb{H}^n$.*
(4) A face *of a standard polyhedron P is the intersection $P \cap \pi$, where π is an $(n-1)$-dimensional hyperplane, such that P is disjoint from one component of $\mathbb{H}^n \backslash \pi$ and $P \cap \pi \neq \emptyset$.*
(5) A facet *of a standard polyhedron P is a face of dimension $n-1$. The* vertices *are the faces of dimension zero.*
(6) An ideal polyhedron *is the intersection of finitely many half-spaces in $\mathbb{H}^n$ whose closure in $\mathbb{H}^n \cap S^{n-1}_\infty$ intersects S^{n-1}_∞ in a finite number of points, and which has no vertices in $\mathbb{H}^n$.*

Now suppose M is obtained by identifying the facets of hyperbolic polyhedra $P_1, \dots, P_m$. Let $P = \bigsqcup P_i$ and $q : P \to M$ be the quotient map. Note that $q|_{P \setminus \partial P}$ is a homeomorphism, so $P \setminus \partial P$ inherits a hyperbolic structure. We want M to be a hyperbolic manifold.

Theorem 3.2. *[[4]] Suppose that for each $x \in M$ there is a neighbourhood U_x of x such that*

(1) there exists an open mapping $\varphi_x : U_x \to B_{\epsilon_x}(0) \subseteq D^n$ for some ϵ_x;
(2) φ_x is a homeomorphism onto its image;
(3) $\varphi_x(x) = 0$; and
(4) φ_x restricts to an isometry on each component of $U_x \cap q(P \setminus \partial P)$.

Then M inherits a hyperbolic structure.

Proof. We can choose the neighbourhoods U_x so that the closure of each component of $U_x \cap q(P \setminus \partial P)$ contains x. Then φ_x will be the charts for M. We have to check that if X is a component of $U_x \cap U_y$, then the transition map

$$\varphi_y \varphi_x^{-1} : \varphi_x(X) \to \varphi_y(X)$$

is the restriction of a hyperbolic isometry. By assumption this is true for each component of $\varphi_x(X \cap q(P \setminus \partial P))$. But if x is in the image of an interior of a polyhedron, then $\varphi_x(X \cap q(P \setminus \partial P))$ is the whole $\varphi_x(X)$. So what is left is to show that these isometries agree over all of $\varphi_x(X)$ when x and y are on the image of faces of the polyhedra.

Since any two points of $\varphi_x(X \cap q(P \setminus \partial P))$ are joined by a path in $\varphi_x(X)$ which avoids the image of the polyhedron faces of dimension less than $n-1$, it is enough to show that $\varphi_y \varphi_x^{-1}$ is an isometry of all points z that lie in $\varphi_x(X \cap q(\partial P))$ but not in a face of dimension less than $n-1$.

Suppose $z \in \varphi_x(X \cap q(\partial P))$ and $z_1, z_2 \in q^{-1} \varphi_x^{-1}(z)$. Let x_1, x_2 be the unique points of $q^{-1}(x)$ lying in the same component of $q^{-1}(U_x)$ as z_1, z_2, respectively. Let y_1, y_2 be the unique points of $q^{-1}(y)$ lying in the same component of $q^{-1}(U_y)$ as z_1, z_2, respectively. Let F_1, F_2 be the facets containing z_1, z_2, respectively, and let $k : F_1 \to F_2$ be the identification isometry between them. Note that x_1, y_1 and x_2, y_2 lie on (possibly the boundary of) F_1 and F_2, respectively.

The chart φ_x determines for each $x_i \in q^{-1}(x)$ an isometry

$$h_{x_i} : B_{\epsilon_x}(x_i)[\subseteq P_i] \to B_{\epsilon_x}(0)[\subseteq D^n],$$

such that $h_{x_i}|_{P \setminus \partial P} = \varphi_x \circ q$.

Now, since F_i are facets, we have two ways of extending k to an isometry of $\mathbb{H}^n$. Choose the one that makes the following diagram commute:

$$
\begin{tikzcd}
\mathbb{H}^n \arrow[rr, "k"] \arrow[dr, "h_{x_1}"'] & & \mathbb{H}^n \arrow[dl, "h_{x_2}"] \\
& D^n &
\end{tikzcd}
$$

Hence, running down the left-hand side of the following diagram is the same as running down the right-hand side (where the maps are defined):

$$
\begin{tikzcd}
& D^n \arrow[dl, "h_{x_1}^{-1}"'] \arrow[dr, "h_{x_2}^{-1}"] & \\
\mathbb{H}^n \arrow[rr, "k"] \arrow[dr, "h_{y_1}"'] & & \mathbb{H}^n \arrow[dl, "h_{y_2}"] \\
& D^n &
\end{tikzcd}
$$

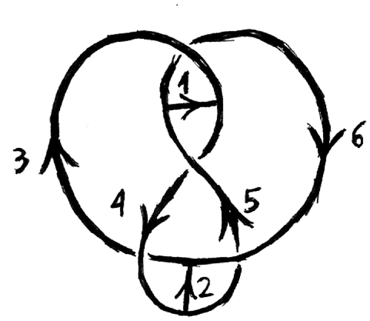
This ensures that $\varphi_y \varphi_x^{-1}$ is a well-defined isometry in a neighbourhood of z.

Finally, we check the requirements for M to be a topological manifold. We can refine $\{U_x \mid x \in M\}$ to a countable cover, so M is second countable. Hausdorffness is clear from the construction. $\square$

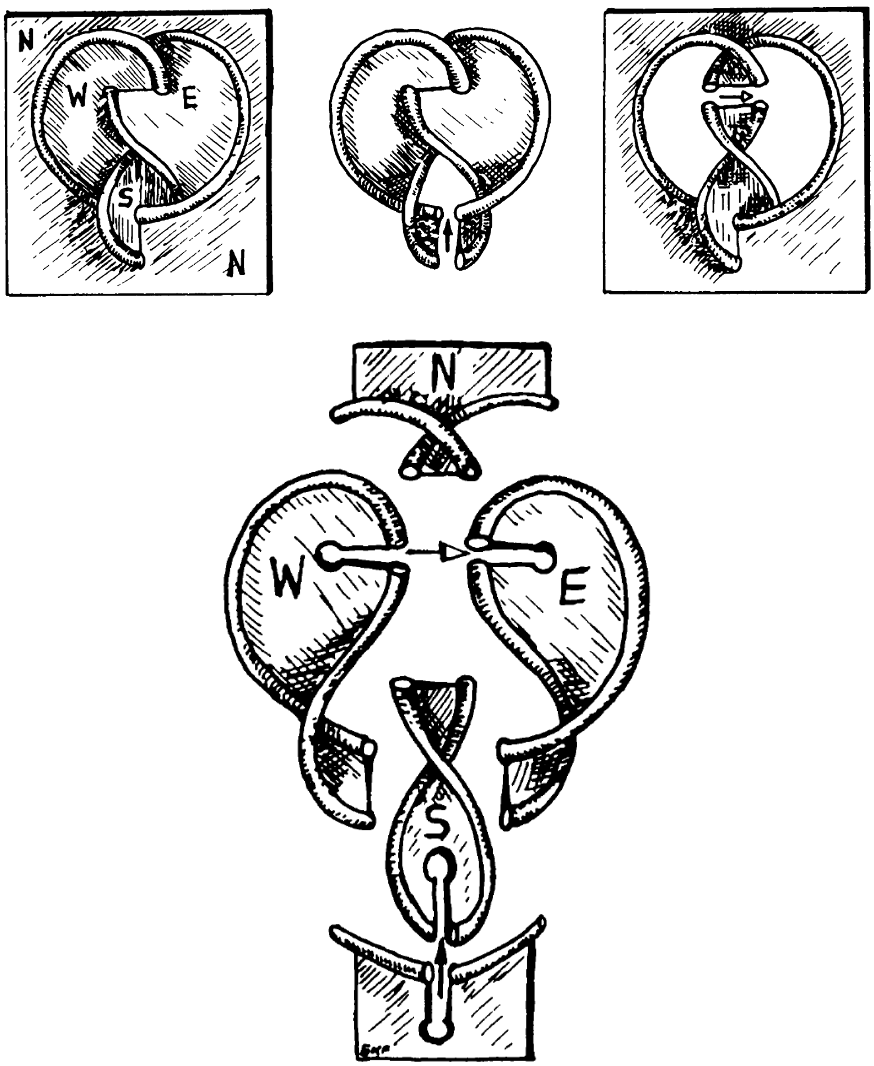
We can now build hyperbolic structures of spaces by just finding ideal polyhedra decompositions, where each point has a full hyperbolic neighbourhood. We do this for the figure-eight knot and Whitehead link in turn.

4. The Complement of the Figure-eight Knot

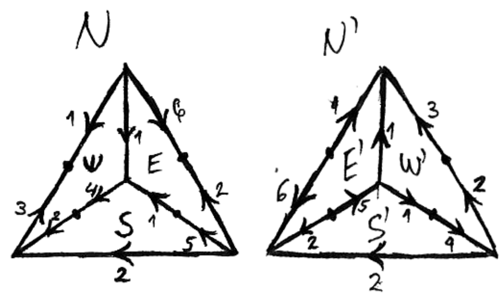
We begin by constructing a cell complex K, homeomorphic to the complement of E – the figure-eight knot – in S^3. Take K^1 as the following 1-complex (embedded in S^3)



Now attach four 2-cells – cell N along $(3 \cdot 1^{-1} \cdot 6 \cdot 2^{-1} \cdot 2)$, cell W along $(3{\cdot}1_{-1}{\cdot}1{\cdot}5^{-1}{\cdot}2)$, cell E along $(6{\cdot}2^{-1}{\cdot}5{\cdot}1{\cdot}1_{-1})$, and cell S along $(1{\cdot}4{\cdot}2{\cdot}2^{-1}{\cdot}5)$ – to get K^2.

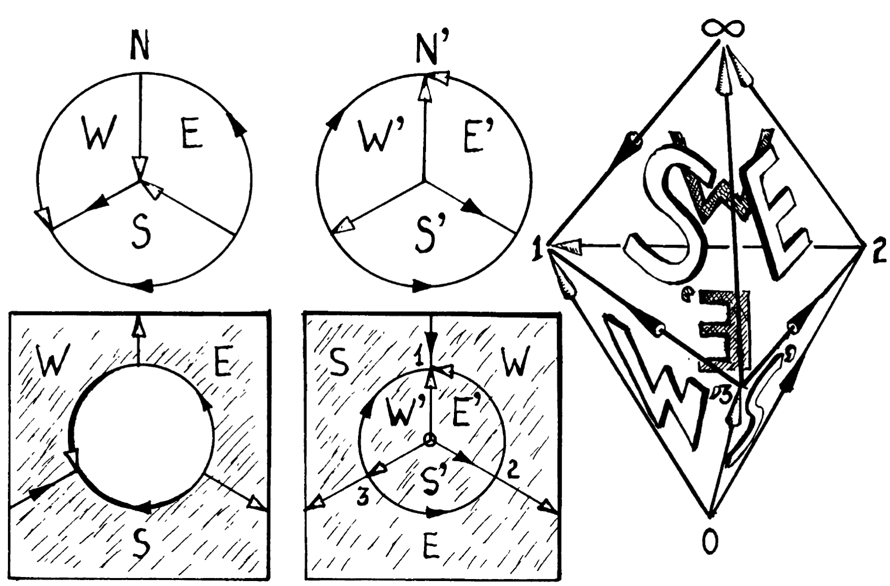


Now, embedding K^2 in S^3 separates the space. This can be seen on the above images – only one side of each 2-cell is reachable from the viewing point without crossing any cell. Thus $S^3 \backslash K^2$ is a union of two disjoint 3-balls. One 3-cell will be attached to the 'front' faces of the four 2-cells (the faces with the letters visible on the picture), one 3-cell will be attached to the 'back' faces of the 2-cells (represented by the primed letters). Below are the attaching maps with the corresponding 2- and 1-cells.



Now note that with the above attachments, each 3-cell can be realised as a tetrahedron with faces N, W, E, S and N', W', E', S', respectively. The identification arising from the attaching maps identifies the faces of the two tetrahedra in the natural way (N is identified with N', etc.), reversing orientation. Thus, $S^3\backslash\{\infty\}$ is realised as two tetrahedra with face identifications.

Deleting the figure-eight knot from this space is just removing the $3, 4, 5$ and 6 1-cells. This does not conflate any edge of the two tetrahedra – notice that on the identifications above, every edge of the tetrahedra contains exactly one of the 1 and 2 1-cells. If we denote 1 by a white arrow and 2 by a black arrow, we get the following resulting structure.



Thus deleting the knot from $S^3\backslash\{\infty\}$ we end up with the two tetrahedra with identifications and deleted vertices. We can identify the two tetrahedra with two ideal tetrahedra in $\mathbb{H}^3$ via homeomorphisms. Now, to use Theorem 3.2 to get a hyperbolic manifold, we need each point to have a hyperbolic neighbourhood when the facets of the tetrahedra are identified. This is clear for the interiors and the facets. We only need to show that points on the 1-dimensional faces of the tetrahedra have hyperbolic neighbourhoods.

Proposition 4.1 ([2]). *The sum of the dihedral angles between any three facets of an ideal tetrahedron in $\mathbb{H}^3$ is π.*

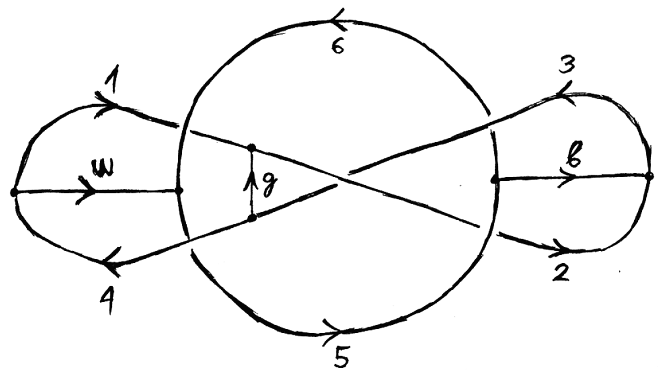
Proof. Working in the Poincaré half-space model, there is an isometry taking any ideal tetrahedron to the tetrahedron having three facets as Euclidean planes. But since the half-space model is conformal, it is clear that the sum of the dihedral angles in π. □

Definition 4.2. *An* ideal n-simplex *is the ideal polyhedron determined by* $n+1$ *points on* S^{n-1}_∞. *An ideal* n*-simplex is* regular *if, for any permutation of its vertices, there is a hyperbolic isometry which realises this permutation.*

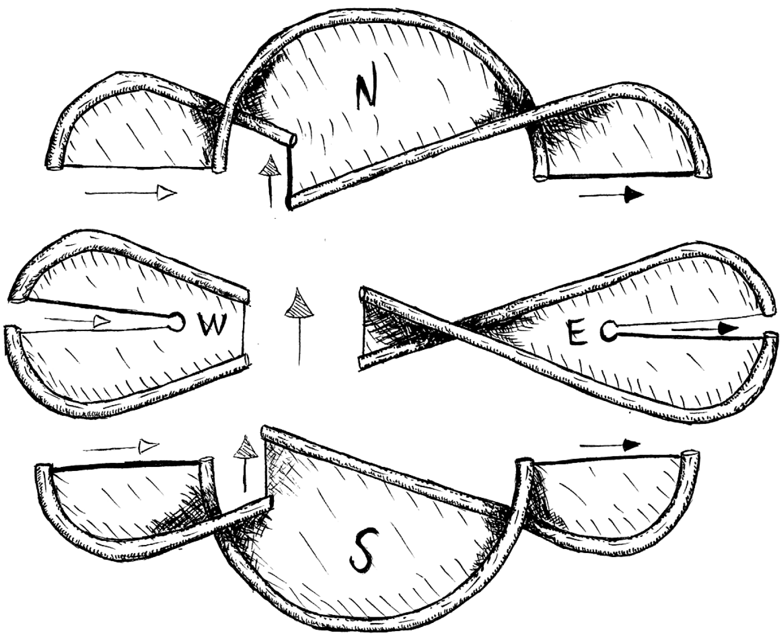
Choosing the tetrahedra regular, the above proposition says that all dihedral angles are $\pi/3$. Finally, the 'black arrow' edge occurs a total of 6 times in the disjoint tetrahedra – summing the dihedral angles gives the needed 2π. Similarly for the 'white arrow' edge. Therefore all points in our construction have hyperbolic neighbourhoods and hence the two tetrahedra with identifications as above realise $S^3\backslash E$ as a hyperbolic manifold.

5. The Complement of the Whitehead link

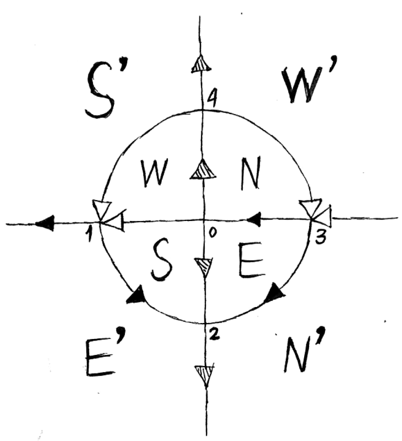
As before, we aim to construct a cell complex K, homeomorphic to the complement of W – the Whitehead link – in S^3. Take K^1 as the following 1-complex:



We attach four 2-cells as shown:



Embedding K^2 in S^3 does not separate the space, so we only need one 3-cell. Deleting the 1-cells corresponding to the strands of the link, we have the following attachment map for the 3-cell, giving an octahedron with face identifications:



Deleting the vertices, we can identify the octahedron with a regular ideal octahedron in $\mathbb{H}^3$. Analogously to the case of the tetrahedron, the dihedral angles between the facets of the octahedron are all $\pi/2$. Each of the coloured arrows occurs four times in the octahedron, hence all points on the edges of the figure have hyperbolic neighbourhoods. By Theorem 3.2, this gives a hyperbolic structure for $S^3\backslash W$.

References

[1] George K. Francis. *A Topological Picturebook*. New York: Springer-Verlag, 1987.

[2] Alexander J. Gutierrez. "Hyperbolic Geometry on the Figure-Eight Knot Complement". Arizona State University, 2012. URL: https://math.la.asu.edu/~paupert/Gutierrezproject.pdf.

[3] I.M. James. *History of topology*. North Holland, 2006.

[4] Marc Lackenby. "Notes for graduate hyperbolic manifolds course, Hilary term". University of Oxford, 2000. URL: http://people.maths.ox.ac.uk/lackenby/.

[5] J. Milnor. "Hyperbolic geometry: the first 150 years". In: *Bull. Amer. Math. Soc* 6.1 (1982).

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, USA

E-mail address: mhurmu2@uic.edu