The evolution of the classical viscous incompressible fluid flow is governed by the Navier-Stokes equation (NSE). The theory regarding the qualitative properties of solutions to the NSE is far from being complete. For instance, the global regularity for the 3D NSE has been a longstanding open problem. My research has been mainly concerned with regularity problems, well/ill-posedness problems, the long time dynamics of solutions to the NSE and other related fluid equations, and finite dimensionality of flows.

Regularity and blow-up criteria. It is known that solutions of the 3D NSE are regular if they belong to some scaling invariant (critical) space (e.g. Prodi-Serrin type), or if the initial data is small in some critical space, such as BMO^{-1}. The classical Prodi-Serrin regularity criterion states that: if a regular solution u on $(0,T)$ satisfies $u \in L^p(0,T;L^q)$ with $\frac{2}{p}+\frac{3}{q}=1$, then it can be extended beyond time T. Another classical criterion is due to Beale-Kato-Majda (BKM): if $\int_0^T \|\nabla\times u\|_\infty \, dt<\infty$, u does not blow up at time T. Extensions and improvements of these criteria have been obtained by many authors.

Recently in [16], we established a new regularity criterion for the 3D NSE:

Theorem 1. *A solution u does not blow up at $t=T$ provided* $\limsup_{q\to\infty} \displaystyle\int_{\mathcal{T}_q}^T \|\Delta_q(\nabla\times u)\|_\infty \, dt$ *is small enough, where Δ_q is the Littlewood-Paley projection, and $\mathcal{T}_q$ is a certain sequence such that $\mathcal{T}_q \to T$ as $q\to\infty$.*

This criterion improves many existing ones, specifically the classical Prodi-Serrin, BKM, and their extensions, such as [21], as we take advantage of a dissipation wavenumber $\Lambda^{\mathrm{dis}}(t)$ similar to the one introduced in [21] by Cheskidov and Shvydkoy. The dissipation wavenumber is defined so that some critical norm of u is small above Λ^{dis}. The wavenumber separates the inertial range from the dissipation range where the viscous term Δu dominates.

In [16] we also show that the same criterion as in Theorem 1 (with a condition only on the velocity) holds for the magneto-hydrodynamic (MHD) system, which improves many existing regularity criteria in the literature. This result is also important from another perspective. It is consistent with various indications, both numerical and theoretical, that suggest the velocity field u plays a dominant role in interactions between u and b.

I have applied this method of wavenumber splitting to the supercritical surface quasi-geostrophic (SQG) equation [26] and Hall-MHD system [27] as well. Regularity criteria obtained in these papers are weaker than all the corresponding Prodi-Serrin type criteria.

The technique of wavenumber splitting is also utilized to study the finite dimensionality of flows, particularly the determining modes in [11, 12, 13, 15, 19]. The definition of a determining wavenumber Λ^{det} is much more involved in these cases as there are less cancelations and more terms to control in estimates.

Finite dimensionality of flows: determining modes. The first result for the finite dimensionality of the 2D NSE appeared in Foias and Prodi's work [38] where they showed that high modes of a solution are controlled by low modes asymptotically as time goes to infinity. The number of these low modes, called determining modes, was estimated by Foias, Manley, Temam and Treve [37] and improved by Jones and Titi [39].

The situation is drastically different for the 3D NSE as it has thus far eluded a proof for the existence of classical solutions. The existence of a finite number of determining modes is not known either. However, one might ask whether it can be estimated in some average sense; or an alternative question: what is the number of determining modes for a time discretization of the 3D NSE?

Without making any assumptions on regularity properties of solutions, we prove the existence of a time-dependent determining wavenumber $\Lambda_u^{\det}(t)$ defined for each individual solution u in [19]. We show that any weak solutions on the global attractor u and v that coincide below $\max\{\Lambda_u^{\det}, \Lambda_v^{\det}\}$ have to be identical. The dominance of the dissipation term above Λ_u^{dis} (see Section of Regularity) is already reflected in improved BKM and Prodi-Serrin criteria. The determining wavenumber $\Lambda_u^{\det}$ imposes tougher condition on high modes, as well as requires a control on low modes via the low frequency Reynolds number. The wavenumber $\Lambda_u^{\det}(t)$ blows up if and only if the solution $u(t)$ blows up. Indeed, we establish the pointwise bound:

$$\Lambda_u^{\det}(t) \lesssim \frac{\|\nabla u(t)\|_{L^2}^2}{\nu^2} \tag{0.1}$$

with ν being the viscosity, which holds in general for arbitrary forces. It implies that the time average $\langle \Lambda^{\det} \rangle$ is uniformly bounded for all Leray-Hopf solutions on the global attractor. On the other hand, (0.1) provides a finite number of determining modes and recovers the results by Constantin, Foias, Manley, and Temam in [23] where $\|\nabla u(t)\|_{L^2}^2$ is bounded on the global attractor.

In a similar way, a time-dependent wavenumber based on the scaling of the SQG equation is introduced in [11]. It is shown to be a determining wavenumber. In the subcritical and critical cases this wavenumber, and hence the number of determining modes, has a uniform upper bound, while it may blow up when the equation is supercritical.

The above framework of time-dependent determining wavenumber is general and can be applied to different fluid equations since $\Lambda^{\det}$ is defined solely based on the structures (scaling) of equations. We have further investigated this method by introducing a wavenumber based on known estimates of the solutions as well as the scaling. For the subcritical SQG equation, it is known that solutions are bounded in L^∞ (see [18, 8]). Utilizing this bound, in [15] we find a new determining wavenumber which is smaller than the one obtained in [11].

For the 2D NSE, the global regularity is known. In [13], we use the fact that $\int_t^{t+T} \|\Delta u\|_2^2 \, d\tau$ is uniformly bounded on the global attractor and prove the existence of uniformly bounded determining wavenumber and hence finite number of determining modes. Remarkably, this number is of the order of Kraichnan's dissipation wavenumber, which is consistent with predictions of physicists. On the other hand, it is only known that the average of enstrophy $\int_t^{t+T} \|\nabla u\|_2^2 \, d\tau$ is uniformly bounded for the 3D NSE. In [12], by applying this bound we establish the existence of a determining wavenumber whose time average is uniformly bounded on the global attractor and is of the order of Kolmogorov's dissipation wavenumber. The results are significant, in the sense that they are consistent with predictions of physicists.

Indeed, Kraichnan's and Kolmogorov's dissipation wavenumbers are frequently used to set resolutions for direct numerical simulations.

As a continuation of my previous work, I plan to keep developing the wavenumber splitting technique which may result in improved regularity criteria for the 3D NSE and other fluid equations, and solving some open problems in the area of determining modes.

Well/ill-posedness. Since there is no hope in breaking the scaling with current techniques, behavior of solutions in the largest critical space attracted a lot of attention recently. Below are some of the projects I have completed in this area.

NSE and MHD. Koch and Tataru [40] established the global well-posedness of the NSE with small initial data in BMO^{-1}. Then the question whether this result can be extended to the largest critical space $\dot{B}^{-1}_{\infty,\infty}$ had become of great interest. The first indication that such an extension might not be possible came in the work by Bourgain and Pavlović [7] who showed the norm inflation in $\dot{B}^{-1}_{\infty,\infty}$. In [33] we extended this result to the 3D MHD system and showed that some solutions to the Cauchy problem develop various norm inflation in $\dot{B}^{-1}_{\infty,\infty}$. In other words, starting with arbitrarily small data the solutions become arbitrarily large almost instantaneously. The method involves a construction of initial data with a sequence of plane waves. It utilizes the fact that the energy transferred to low modes increases norms of the Besov spaces with negative smoothness indexes.

Though the method used in [7] breaks down for the NSE with fractional Laplacian $(-\Delta)^\alpha$ with $\alpha > 1$, in [17] we prove the existence of a smooth solution with arbitrarily small data in $\dot{B}^{-\alpha}_{\infty,p}$ $(\alpha \geq 1, 2 < p \leq \infty)$ that becomes arbitrarily large in $\dot{B}^{-s}_{\infty,\infty}$ for all $s > 0$ in arbitrarily small time. It is remarkable that the space $\dot{B}^{-\alpha}_{\infty,\infty}$ is supercritical for $\alpha > 1$. Moreover, the norm inflation occurs even in the case $\alpha > 5/4$ where the global regularity is known. We also study the generalized MHD system with fractional Laplacians in [14], and establish norm inflation in a wide range of spaces which include critical, supercritical and subcritical spaces. This also shows a clear obstacle in extending the small data result to the classical NSE in $\dot{B}^{-1}_{\infty,\infty}$, which we hope to overcome in the future.

Another construction to prove ill-posedness of the NSE in $\dot{B}^{-1}_{\infty,\infty}$ was introduced by Cheskidov and Shvydkoy [22] who proved that there are weak solutions discontinuous in this space. These solutions move the energy forward to high modes, and local interactions of high modes produce the discontinuity at the initial time. In [10] we improve this method and find that, in fact, this type of ill-posedness can be obtained in a very wide range of spaces (critical, supercritical and subcritical), which include Besov spaces $\dot{B}^{s}_{r,\infty}$ for $s > 0$ and r depending on s. An analogous ill-posedness result is also obtained for the MHD system. It is worth to mention that the space $\dot{B}^{0}_{2,\infty}$, which is close to L^2, is at the boarder line of being an ill-posdness space.

BBM equation. The norm inflation argument can be applied to dispersive equations, for instance, the Benjamin-Bona-Mahony (BBM) equation. The BBM equation [3] describes (approximately) the evolution of small-amplitude long waves in nonlinear dispersive media. Bona and Tzvetkov [6] have shown that the Cauchy problem of the BBM equation is globally well-posed in Sobolev spaces H^s with $s \geq 0$, while the solution map is not of C^2 class in H^{-s}

($s > 0$). So it is conjectured that certain ill-posedness behavior may happen in the Sobolev spaces with negative indexes. Actually, in [5] we are able to prove that the solutions of the BBM equation develop norm inflation phenomena in H^{-s}. Due to the essential difference between the dispersive and dissipative structures, the analysis to produce norm inflation is different from the ones described above but interesting.

LCD Systems. The liquid crystal (LCD) system is a coupling of the NSE and the equation of the local configuration of molecules in the complex fluid of nematic liquid crystals. The two most often used local configurations are a unit director field [36] and an Q-tensor [2].

For a simplified LCD system with the director field configuration and non constant density in the whole space $\mathbb{R}^3$, with the possibility of vacuum, in [32], we establish the existence of weak solutions. On bounded domains in two and three dimensions, in [34], global regularity is obtained in 3D with small initial data, and in 2D for general data. The main ingredients include the higher order (Ladyzhenskaya) energy method, the frozen coefficient argument, the standard L^p theory and a bootstrapping argument. For a full LCD model with non-constant density where rotating and stretching effects of the director field are considered, in [25], we establish global regularity for the 3D system with small initial data and for the 2D system with large data. The result is particularly interesting due to the much more complicated structure of the full model.

Tumor growth model. Recently, we studied a diffuse interface model for tumor growth which is a continuum thermodynamically consistent model in [29]. The system couples four different types of equations: a Cahn-Hilliard type equation for the tumor cells, a Darcy law for the tissue velocity field, a transport equation, and a quasi-static reaction diffusion equation. The different nature of the four equations as well as their nonlinear coupling make the analysis of the problem particularly challenging. However, we are able to establish the existence of weak solutions for the system coupled with suitable initial and boundary conditions.

Long time dynamics and stability. An important feature of dissipative systems is that trajectories forget about the initial condition and the long-time behavior is described by a global attractor. When the force is large, the attractor is usually very complicated, and even its existence can be a challenging problem. When the force is small or zero, the convergence to a unique steady state (the only element of the attractor) is expected in an autonomous case, and we are interested in the decay rate of the convergence.

The global attractor problem of the SQG equation with a time independent large force f was investigated in [18]. Combining the De Giorgi method, the Littlewood-Paley decomposition approach, and the evolutionary system framework [9, 20] for global attractor theory, we proved that the SQG equation possesses a strong global attractor in L^2, providing the force $f \in L^p$ for some $p > 2$. This result extends the work of Constantin, Tarfulea and Vicol [24] on the existence of global attractor of SQG in H^1 with the assumption that f is in $L^\infty \cap H^1$ and the initial data is in H^1.

For a domain where Poincaré's inequality holds, it is known that with an autonomous small force, the strong (in L^2) global attractor for the 3D NSE is the unique steady state. Without Poincaré's inequality the situation is more delicate. Remarkably, Bjorland and Schonbek [4] proved that, in the whole space $\mathbb{R}^3$, every solution of the NSE converges to a unique steady state in L^2. Inspired by their work, in [28] I considered the stationary SQG equation with time independent force f. With certain smallness conditions on f, the existence of a unique steady state is obtained. With even smaller f, using the Fourier splitting method, we show that every evolutionary solution of the SQG with initial data in L^2 converges to the steady state.

The Fourier splitting method, introduced by Schonbek in [41], is a powerful technique to obtain decay rate for solutions in situations where Poincaré's inequality is not available. In [31, 35] we studied a simplified LCD system in $\mathbb{R}^3$ and obtained optimal decay rates for regular solutions in the Sobolev spaces $H^m(\mathbb{R}^3)$ for all $m \geq 0$, by applying the the higher order energy estimates and the Fourier splitting approach. In [30] we study a nematic LCD model with local configuration of the Q-tensor. Providing that the potential energy functional of the Q-tensor is strictly convex in a neighborhood of the isotropic state $\mathbb{Q} = 0$, we show that any weak solution converge to the equilibrium with optimal decay rate $(1+t)^{-3/4}$ in energy space. Without of the convexity condition, the convergence rate is $(1+t)^{-\beta}$ with $\beta > 1/2$. The result is optimal since our hypotheses are optimal for unconditional convergence to an equilibrium. Notice that the convergence rate is obtained for weak solutions satisfying the basic energy inequality.

The Fourier splitting method can be also used to obtain pointwise decay and spacial decay for solutions, see [1, 42]. Some of my projects are in progress in this direction as well.

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