A-optimal designs for generalized linear model with two parameters

Min Yang *
University of Missouri - Columbia

Abstract

An algebraic method for constructing A-optimal designs for two parameter generalized linear models is presented. It gives sufficient conditions to identify the A-optimal design. When the conditions are satisfied, the A-optimal has exactly two points, which is symmetric but not weight symmetric. The methodology is illustrated by means of selected examples. This result proves the conjecture of Mathew and Sinha (2001), which is for logistic model, and shows that the conjecture is also true for probit models and some cases of double exponential and double reciprocal models.

KEY WORDS: A-optimality; Binary response; Generalized linear model; Logistic regression; Probit regression.

1 Introduction

Generalized linear models are widely applied in the experiments where the responses are categorical. These models provide the experimenters with a rich and rewarding modelling environment. Methods of analysis and inference for these models are well established (McCullagh and Nelder, 1989; Agresti, 2002). The efficiency of resulting estimates depends largely on the method of data collection, i.e., the design of experiments.

There are extensive results on the theory of optimal design for classical linear model with normal errors. The problem is relative easy because the information matrix is independent of the unknown parameters. For the generalized linear and nonlinear models, the information matrix is dependent on the unknown parameters. The challenge in designing experiments for such model is that: one is looking for the best design with the aim of estimating the unknown parameters, and yet one has to know the parameters to find the best design. One way to solve this problem is to use the locally optimal design based on

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the best guess of the parameters. In spite of this unpleasant feature, it is important to construct the optimal designs in this context; see the arguments in Ford, Torsney, and Wu (1992).

In this article, we will restrict to the binary response, where the response Y follows a binomial distribution with the probability $P(\alpha + \beta x)$ that Y takes value 1. Here x is the explanatory variable and (α, β) are the unknown parameters. For the joint estimation of (α, β), the optimal design minimizes suitable scalar-valued functions of the information matrix of the two parameters. The A-, D-, and E-optimality criteria are well known examples. Some of the relevant references on this specific optimal problem include Abdelbasit and Plackett (1983), Minkin (1987), Wu (1988), Ford, Torsney, and Wu (1992), Sitter and Wu (1993a, 1993b), Dette and Haines (1994), Hedayat, Yan, and Pezzuto (1997), and Mathew and Sinha (2001). The A-optimal design for (α, β) has an appealing property. It minimizes the sum of the two variances of the two estimated parameters. While most of these results are about D- and E-optimal design, especially D-optimal design, A-optimal design has also been considered. Sitter and Wu (1993a) obtained A-optimal design for transformed parameters $(\alpha/\beta, 1/\beta)$ by using the geometric approach provided by Elfving's Theorem (Elfving, 1952). For $(\alpha/\beta, 1/\beta)$, they show that the optimal design is among the symmetric designs. This greatly simplifies the information matrix. By using an algebraic approach, Mathew and Sinha (2001) obtained a series of optimality results for logistic model. They also found that A-optimal design for (α, β) cannot be symmetric design in general. By restricting consideration to two points designs, they obtained some A-optimal designs through numerical search. The result shows that the best symmetric design under A-criterion could be up to 36% less efficient compared with the best design they found (it could be even worse). What is the A-optimal design for (α, β)? Mathew and Sinha (2001)'s numerical result shows that the A-optimal designs are point symmetric, but not weight symmetric. They conjecture that (i) the A-optimal design in the entire class has exact two points and (ii) the A-optimal design is point symmetric, but not weight symmetric. Is the conjecture true? How about other models such as probit, double exponential, and double reciprocal?

The aim of the present paper is to present an algebraic approach for constructing A-optimal design under generalized linear model. The approach is based on the min-max idea from Kunert and Stufken (2002). Mathew and Sinha (2001)'s conjecture can be proved by applying this approach to logistic model. Section 2 introduces the necessary notations. Section 3 develops the algebraic approach. Some illustrative examples including logistic, probit, double exponential, and double reciprocal models are studied in Section 4. Discussion is in Section 5.

2 Notations

Consider the broad class of models for which the response, Y, follows a binomial distribution with the expectation $P(\alpha + \beta x)$, where P is a cumulative distribution function. For the estimation of α and β, the exact optimal design problem is to choose k distinct x_1,...,x_k and n_i observations on each of x_i respect to some optimality criterion for fixed n. Here $\sum_{i=1}^k n_i = n$. Since this is a difficult and often intractable optimization problem, the corresponding approximate design, in which n_i/n is replaced by ξ_i, is considered. Thus a design can be denoted by $d = \{(x_i, \xi_i), i = 1, ..., k\}$, where $\xi_i > 0$ and $\sum_{i=1}^k \xi_i = 1$. We shall denote the entire class of all such designs by $\mathcal{D}$.

It is well known that the information matrix for a given design d is

$$I_d(\alpha, \beta) = \begin{pmatrix} \sum_{i=1}^k \xi_i \Psi(c_i) & \sum_{i=1}^k \xi_i x_i \Psi(c_i) \\ \sum_{i=1}^k \xi_i x_i \Psi(c_i) & \sum_{i=1}^k \xi_i x_i^2 \Psi(c_i) \end{pmatrix}.$$

Here $c_i = \alpha + \beta x_i$ and $\Psi(c_i) = \{P'(c_i)\}^2/[P(c_i)\{1 - P(c_i)\}]$. We shall assume that P satisfies following condition.

Conditions (i): the density function P' is symmetric about zero; $\Psi(0) > 0$ and $\lim_{c\to\infty} \Psi(c) = 0$; when $c > 0$, $\Psi(c) > 0$, $\left(\Psi^{-1/2}(c)\right)' > 0$, and $\left(\Psi^{-1/2}(c)\right)'' > 0$.

Condition (i) is not demanding. In fact, commonly used generalized linear models for binary response, such as logistic, probit, double exponential, and double reciprocal models, satisfy condition (i). We shall verify this in Section 4.

By the symmetric property of P', we can easily see that $P(c) + P(-c) = 1$. Thus, by the definition of $\Psi(c)$, we have $\Psi(c) = \Psi(-c)$. An A-optimal design for α and β minimizes the sum of the variances of the maximum likelihood estimators. This is equivalent to minimizing $tr[I_d(\alpha, \beta)^{-1}]$. It is clear that

$$tr[I_d(\alpha, \beta)^{-1}] = \frac{\sum_{i=1}^k \xi_i \Psi(c_i) + \sum_{i=1}^k \xi_i x_i^2 \Psi(c_i)}{(\sum_{i=1}^k \xi_i \Psi(c_i))(\sum_{i=1}^k \xi_i x_i^2 \Psi(c_i)) - (\sum_{i=1}^k \xi_i x_i \Psi(c_i))^2}. \tag{2.1}$$

By the facts $\Psi(c) = \Psi(-c)$ and (2.1), for any design $d = \{(x_i, \xi_i), i = 1, ..., k\}$, we can easily verify $tr[I_d(\alpha, \beta)^{-1}] = tr[I_d(-\alpha, -\beta)^{-1}]$ and $tr[I_d(\alpha, \beta)^{-1}] = tr[I_{d'}(\alpha, -\beta)^{-1}]$, where $d' = \{(-x_i, \xi_i), i = 1, ..., k\}$. Thus the A-optimality problem when either α or β is negative or both are negative can be transformed to the same problem when both α and β are positive. Without loss of generation, we assume α and β are both positive in this paper.

3 The approach

From the expression of (2.1), we can see that directly minimizing $tr[I_d(\alpha, \beta)^{-1}]$ is not feasible. The strategy to identify the A-optimal design here consists three steps: (i) identify the A-optimal design, say d^*, among a subclass of designs $\mathcal{D}_1$. In this subclass $\mathcal{D}_1$, each design d only has two symmetric design points, i.e., $d = \{(x_1, \xi_1), (x_2, \xi_2)\}$, where $\alpha + x_1\beta = -\alpha - x_2\beta$. (ii) derive the sufficient conditions such that $tr[I_d(\alpha, \beta)^{-1}] \geq tr[I_{d^*}(\alpha, \beta)^{-1}]$ for any arbitrary design d. (iii) verify whether the model satisfies the sufficient conditions. If yes, this establishes that d^* is an A-optimal design among the entire class. In this section, we will focus on step (i) and (ii). Step (iii) will be illustrated in next section by a few selected models.

3.1 A-optimal design in $\mathcal{D}_1$

It is relative easy to identify an A-optimal design in this subclass. Let $c = \alpha + x_1\beta$. Then $x_1 = (c - \alpha)/\beta$ and $x_2 = (-c - \alpha)/\beta$. So for any design $d \in \mathcal{D}_1$, by (2.1) and the fact $\xi_1 + \xi_2 = 1$, we have

$$\begin{aligned} tr[I_d(\alpha, \beta)^{-1}] &= \frac{1 + \xi_1(\frac{c-\alpha}{\beta})^2 + \xi_2(\frac{-c-\alpha}{\beta})^2}{\left[\xi_1(\frac{c-\alpha}{\beta})^2 + \xi_2(\frac{-c-\alpha}{\beta})^2 - \left(\xi_1(\frac{c-\alpha}{\beta}) + \xi_2(\frac{-c-\alpha}{\beta})\right)^2\right]\Psi(c)} \\ &= \frac{[\beta^2 + (c+\alpha)^2]/\xi_1 + [\beta^2 + (c-\alpha)^2]/\xi_2}{4c^2\Psi(c)} \\ &\geq T^2(c, \alpha, \beta). \end{aligned} \tag{3.1}$$

Here,

$$T(c, \alpha, \beta) = \frac{\sqrt{\beta^2 + (c+\alpha)^2} + \sqrt{\beta^2 + (c-\alpha)^2}}{2c(\Psi(c))^{\frac{1}{2}}}. \tag{3.2}$$

The equality holds in the last inequality of (3.1) when $\xi_1 = \xi_{c,\alpha,\beta}$ and $\xi_2 = 1 - \xi_1$, where

$$\xi_{c,\alpha,\beta} = \frac{\sqrt{\beta^2 + (c+\alpha)^2}}{\sqrt{\beta^2 + (c+\alpha)^2} + \sqrt{\beta^2 + (c-\alpha)^2}}. \tag{3.3}$$

We are ready to present our first result.

Theorem 1. *Suppose that P satisfies condition (i) in Section 2. Then $d^* = \{(x_1^*, \xi_1^*), (x_2^*, \xi_2^*)\}$ is the A-optimal in $\mathcal{D}_1$. Here $x_1^* = (c^* - \alpha)/\beta$, $x_2^* = (-c^* - \alpha)/\beta$, $\xi_1^* = \xi_{c^*,\alpha,\beta}$, and $\xi_2^* = 1 - \xi_1^*$, where $\xi_{c^*,\alpha,\beta}$ is defined in (3.3) and $c^* > 0$ is the only positive solution of the following equation*

$$\frac{c^2 - \alpha^2 - \beta^2}{\sqrt{\beta^2 + (c+\alpha)^2}\sqrt{\beta^2 + (c-\alpha)^2}} = 1 + \frac{c\Psi'(c)}{\Psi(c)}. \tag{3.4}$$

Proof. By (3.1) and (3.3), the conclusion is sufficient if we can show that c^* minimizes $T^2(c, \alpha, \beta)$. Since $T^2(c, \alpha, \beta)$ is symmetric about zero, we could restrict our consideration for $c > 0$. Since $T(c, \alpha, \beta) > 0$ when $c > 0$, minimizing $T^2(c, \alpha, \beta)$ is equivalent to minimizing $T(c,\alpha,\beta)$. By simple computation, we have

$$T'(c,\alpha,\beta) = \frac{T_1(c,\alpha,\beta)}{2c^2},$$

where

$$\begin{aligned} T_1(c,\alpha,\beta) = &\left(c\left(\Psi^{-1/2}(c)\right)' - \Psi^{-1/2}(c)\right)\left(\sqrt{\beta^2+(c+\alpha)^2}+\sqrt{\beta^2+(c-\alpha)^2}\right) \\ &+ \Psi^{-1/2}(c)\left(\frac{c^2+\alpha c}{\sqrt{\beta^2+(c+\alpha)^2}}+\frac{c^2-\alpha c}{\sqrt{\beta^2+(c-\alpha)^2}}\right). \end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned} T_1'(c,\alpha,\beta) =& c\left(\Psi^{-1/2}(c)\right)''\left(\sqrt{\beta^2+(c+\alpha)^2}+\sqrt{\beta^2+(c-\alpha)^2}\right) \\ &+ 2c\left(\Psi^{-1/2}(c)\right)'\left(\frac{c+\alpha}{\sqrt{\beta^2+(c+\alpha)^2}}+\frac{c-\alpha}{\sqrt{\beta^2+(c-\alpha)^2}}\right) \\ &+ c\Psi^{-1/2}(c)\left(\frac{\beta^2}{(\sqrt{\beta^2+(c+\alpha)^2})^3}+\frac{\beta^2}{(\sqrt{\beta^2+(c-\alpha)^2})^3}\right). \end{aligned} \tag{3.6}$$

Notice that $c>0$ and $\alpha>0$, it can be verified that

$$\frac{(c+\alpha)^2}{\beta^2+(c+\alpha)^2} \geq \frac{(c-\alpha)^2}{\beta^2+(c-\alpha)^2},$$

which implies

$$\frac{c+\alpha}{\sqrt{\beta^2+(c+\alpha)^2}}+\frac{c-\alpha}{\sqrt{\beta^2+(c-\alpha)^2}} > 0.$$

By the assumptions that $\Psi^{-1/2}(c)$ has positive first and second derivatives, we have $T_1'(c,\alpha,\beta)>0$. So $T_1(c,\alpha,\beta)=0$ has at most one solution. Notice that $\lim_{c\to 0} T(c,\alpha,\beta) = \infty$, $\lim_{c\to\infty} T(c,\alpha,\beta)=\infty$, and $T(c,\alpha,\beta)<\infty$, there exist a point $c^*>0$ such that $T'(c^*,\alpha,\beta)=0$. This implies $T_1(c^*,\alpha,\beta)=0$. By applying the fact that $T_1'(c,\alpha,\beta)>0$ again, $T_1(c,\alpha,\beta)<0$ when $0<c<c^*$ and $T_1(c,\alpha,\beta)>0$ when $c>c^*$. This implies that $T'(c,\alpha,\beta)$ has the same pattern as $T_1(c,\alpha,\beta)$. Thus we can conclude that c^* is the point which minimizes $T(c,\alpha,\beta)$. On the other hand, from (3.5), it is clear that $T_1(c,\alpha,\beta)=0$ has the same solutions as the following equation

$$\frac{\frac{c^2+\alpha c}{\sqrt{\beta^2+(c+\alpha)^2}}+\frac{c^2-\alpha c}{\sqrt{\beta^2+(c-\alpha)^2}}}{\sqrt{\beta^2+(c+\alpha)^2}+\sqrt{\beta^2+(c-\alpha)^2}} = \frac{2\Psi(c)+c\Psi'(c)}{2\Psi(c)}. \tag{3.7}$$

By (5.1) of Proposition 1 in appendix, (3.7) is equivalent to (3.4). Thus c^* is the unique solution of (3.4). □

Remark: There is no explicit expression of c^* in general. However $T_1(c,\alpha,\beta)$ is a increasing function of c. This is very helpful for us to find the solution c^* numerically.

3.2 A-optimal design in $\mathcal{D}$

In this section, we shall show that under some conditions, d^*, introduced in Theorem 1, is the A-optimal design in the entire class $\mathcal{D}$, i.e., $tr[I_d(\alpha,\beta)^{-1}]$ is minimized only at design d^*. The structure of $tr[I_d(\alpha,\beta)^{-1}]$ is complicated. To circumvent this problem, we will the following strategy: (i) for any design $d \in \mathcal{D}$, find a manageable lower bound for $tr[I_d(\alpha,\beta)^{-1}]$ (which may depend on d); and (ii) identify that d^* minimizes the lower bound.

In pursuing this strategy, we start with a lemma that provides an lower bound for $tr[I_d(\alpha,\beta)^{-1}]$. In this lemma, the min-max idea from Kunert and Stufken (2002) is used. Before we state the lemma, we need define some notations. Hereafter, c^* and $d^* = \{(x_1^*, \xi_1^*), (x_2^*, \xi_2^*)\}$ are defined in Theorem 1. Let

$$\begin{aligned} Z_1^* &= \frac{1}{\beta}\left(\alpha - (\xi_1^* - \xi_2^*)c^*\right), \\ Z_2^* &= \frac{\beta\left(\alpha - (\xi_1^* - \xi_2^*)c^*\right)}{\xi_1^*(\alpha - c^*)^2 + \xi_2^*(\alpha + c^*)^2}, \\ Y_1^* &= -\frac{\beta^2}{4(c^*)^2\xi_1^*\xi_2^*\Psi(c^*)}, \end{aligned} \tag{3.8}$$

and

$$Y_2^* = -\frac{\xi_1^*(\alpha - c^*)^2 + \xi_2^*(\alpha + c^*)^2}{4(c^*)^2\xi_1^*\xi_2^*\Psi(c^*)}.$$

Lemma 1. *For any design* $d = \{(x_i, \xi_i), i = 1, ..., k\}$,

$$tr[I_d(\alpha,\beta)^{-1}] \geq -\sum_{i=1}^{k} \xi_i f(c_i).$$

Here, $c_i = \alpha + \beta x_i$ *and*

$$f(c) = \Psi(c)\left((Z_1^* + \frac{c-\alpha}{\beta})^2 (Y_1^*)^2 + (\frac{c-\alpha}{\beta} Z_2^* + 1)^2 (Y_2^*)^2\right) + 2Y_1^* + 2Y_2^*. \tag{3.9}$$

Proof. For any design $d = \{(x_i, \xi_i), i = 1, ..., k\}$, let

$$G_d(z_1, z_2, y_1, y_2) = -g_1(z_1)y_1^2 - 2y_1 - g_2(z_2)y_2^2 - 2y_2,$$

where

$$g_1(z_1) = \left(\sum_{i=1}^{k} \xi_i \Psi(c_i)\right) z_1^2 + 2\left(\sum_{i=1}^{k} \xi_i x_i \Psi(c_i)\right) z_1 + \sum_{i=1}^{k} \xi_i x_i^2 \Psi(c_i)$$

and

$$g_2(z_2) = \left(\sum_{i=1}^{k} \xi_i x_i^2 \Psi(c_i)\right) z_2^2 + 2\left(\sum_{i=1}^{k} \xi_i x_i \Psi(c_i)\right) z_2 + \sum_{i=1}^{k} \xi_i \Psi(c_i)$$

By the property of quadric function, we have

$$g_1(z_1) \geq \frac{(\sum_{i=1}^k \xi_i \Psi(c_i))(\sum_{i=1}^k \xi_i x_i^2 \Psi(c_i)) - (\sum_{i=1}^k \xi_i x_i \Psi(c_i))^2}{\sum_{i=1}^k \xi_i \Psi(c_i)}$$

and (3.10)

$$g_2(z_2) \geq \frac{(\sum_{i=1}^k \xi_i \Psi(c_i))(\sum_{i=1}^k \xi_i x_i^2 \Psi(c_i)) - (\sum_{i=1}^k \xi_i x_i \Psi(c_i))^2}{\sum_{i=1}^k \xi_i x_i^2 \Psi(c_i)}.$$

The preceding inequalities imply that $g_1(z_1) > 0$ and $g_2(z_2) > 0$. By this fact and the quadric property of $G_d(z_1, z_2, y_1, y_2)$ about y_1 and y_2, for any z_1 and z_2, we have

$$\max_{y_1, y_2} G_d(z_1, z_2, y_1, y_2) = \frac{1}{g_1(z_1)} + \frac{1}{g_2(z_2)}. \tag{3.11}$$

Applying (3.10) into (3.11) and by (2.1), we have

$$\max_{z_1, z_2, y_1, y_2} G_d(z_1, z_2, y_1, y_2) = tr[I_d(\alpha, \beta)^{-1}],$$

which implies that

$$tr[I_d(\alpha, \beta)^{-1}] \geq G_d(Z_1^*, Z_2^*, Y_1^*, Y_2^*).$$

On the other hand, by the definitions of $g_1(z_1)$ and $g_2(z_2)$, we have

$$\begin{aligned} G_d(Z_1^*, Z_2^*, Y_1^*, Y_2^*) &= -\sum_{i=1}^k \left(\xi_i \Psi(c_i)[(Z_1^* + x_i)^2 (Y_1^*)^2 + (x_i Z_2^* + 1)^2 (Y_2^*)^2] + 2\xi_i Y_1^* + 2\xi_i Y_2^*\right) \\ &= -\sum_{i=1}^k \xi_i f(c_i). \end{aligned}$$

Thus the conclusion follows. □

From Lemma 1, we can see that if we could show that (i) $f(c^*) = -tr[I_{d^*}(\alpha, \beta)^{-1}]$ and (ii) $f(c) \leq f(c^*)$, then this implies that d^* is an A-optimal design in the entire class $\mathcal{D}$. Lemma 2 will establish (i) and some useful equations for (ii).

Lemma 2. *Let $f(c)$ be defined in (3.9), then we have*
(i) $f(c^*) = -tr[I_{d^*}(\alpha, \beta)^{-1}]$*;*
(ii) $f(c) = \Psi(c)\left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{c^2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{c^2}{\beta^2}(Z_2^* Y_2^*)^2\right) + 2Y_1^* + 2Y_2^*$*;*
(iii) $f'(c^*) = 0$.

Proof. Conclusion (i): by (3.8), (3.3), and (3.1), we have

$$\begin{aligned} Y_1^* + Y_2^* &= -\frac{\xi_1^*(\beta^2 + (\alpha - c^*)^2) + \xi_2^*(\beta^2 + (\alpha + c^*)^2)}{4(c^*)^2 \xi_1^* \xi_2^* \Psi(c^*)} \\ &= -\frac{(\beta^2 + (\alpha - c^*)^2)/\xi_2^* + (\beta^2 + (\alpha + c^*)^2)/\xi_1^*}{4(c^*)^2 \Psi(c^*)} \\ &= -tr[I_{d^*}(\alpha, \beta)^{-1}]. \end{aligned} \tag{3.12}$$

By some simple algebra, we have

$$Z_1^* + \frac{c^* - \alpha}{\beta} = \frac{2c^* \xi_2^*}{\beta}$$

and (3.13)

$$\frac{c^* - \alpha}{\beta} Z_2^* + 1 = \frac{2c^*(c^* + \alpha)\xi_2^*}{\xi_1(\alpha - c^*)^2 + \xi_2(\alpha + c^*)^2}.$$

Utilizing (3.13) and applying some routine algebra, we have

$$\begin{aligned} (Z_1^* + \frac{c^* - \alpha}{\beta})^2 (Y_1^*)^2 + (\frac{c^* - \alpha}{\beta} Z_2^* + 1)^2 (Y_2^*)^2 &= \frac{\beta^2 + (c^* + \alpha)^2}{4(c^*)^2 (\xi_1^*)^2 \Psi^2(c^*)} \\ &= tr[I_{d^*}(\alpha, \beta)^{-1}]/\Psi(c^*). \end{aligned} \tag{3.14}$$

By (3.9), (3.12), and (3.14), conclusion (i) follows.

Conclusion (ii): According to (3.9), the conclusion is sufficient if we can show that

$$(\frac{Z_1^*}{\beta} - \frac{\alpha}{\beta^2})(Y_1^*)^2 + (\frac{Z_2^*}{\beta} - \frac{\alpha (Z_2^*)^2}{\beta^2})(Y_2^*)^2 = 0.$$

By (5.5) of Proposition 2 in the appendix, conclusion (ii) follows.

Conclusion (iii): since there is no explicit expression of c^*, we need to utilize the fact that c^* is the only positive solution of (3.4). By (ii), we have

$$\begin{aligned} f'(c) = \Psi'(c) &\left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{c^2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{c^2}{\beta^2}(Z_2^* Y_2^*)^2 \right) \\ &+ \Psi(c) \left(\frac{2c}{\beta^2}(Y_1^*)^2 + \frac{2c}{\beta^2}(Z_2^* Y_2^*)^2 \right). \end{aligned} \tag{3.15}$$

Notice that $\Psi(c)$ and the second term in the right hand side of (3.15) are positive, (3.15) implies that $f'(c^*) = 0$ is equivalent to

$$\frac{\Psi'(c^*)}{\Psi(c^*)} = -\frac{\frac{2c^*}{\beta^2}(Y_1^*)^2 + \frac{2c^*}{\beta^2}(Z_2^* Y_2^*)^2}{(Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{(c^*)^2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{(c^*)^2}{\beta^2}(Z_2^* Y_2^*)^2}. \tag{3.16}$$

By the definitions of Z_1^*, Z_2^*, Y_1^*, and Y_2^* in (3.8), and the fact that $Z_1^*Y_1^* = Z_2^*Y_2^*$, (3.16) can be simplified to

$$\frac{c^*\Psi'(c^*)}{2\Psi(c^*)} = -\frac{1+(Z_1^*)^2}{(\xi_1^*-\xi_2^*)^2+1+\left(\frac{c^*-\alpha(\xi_1^*-\xi_2^*)}{\beta}\right)^2+(Z_1^*)^2}. \tag{3.17}$$

By (5.6) and (5.7) in Proposition 2 of appendix, (3.17) is equivalent to show

$$\frac{c^*\Psi'(c^*)}{2\Psi(c^*)} = -\frac{4\alpha^2(c^*)^2-(c^*)^2\left(\sqrt{\beta^2+(c^*+\alpha)^2}-\sqrt{\beta^2+(c^*-\alpha)^2}\right)^2}{8\alpha^2(c^*)^2-(\alpha^2+\beta^2+(c^*)^2)\left(\sqrt{\beta^2+(c^*+\alpha)^2}-\sqrt{\beta^2+(c^*-\alpha)^2}\right)^2}. \tag{3.18}$$

On the other hand, by the fact c^* is the only positive solution of (3.4), we have

$$\frac{c^*\Psi'(c^*)}{2\Psi(c^*)} = \frac{(c^*)^2-\alpha^2-\beta^2-\sqrt{\beta^2+(c^*+\alpha)^2}\sqrt{\beta^2+(c^*-\alpha)^2}}{2\sqrt{\beta^2+(c^*+\alpha)^2}\sqrt{\beta^2+(c^*-\alpha)^2}}. \tag{3.19}$$

So $f'(c^*) = 0$ is equivalent to show that the two R.H.S of (3.18) and (3.19) are equal. By Proposition 3 of appendix, the conclusion follows. □

Remark: (i) Z_1^*, Z_2^*, Y_1^*, and Y_2^* are the values which maximizes $G_d(z_1, z_2, y_1, y_2)$ when d is the design d^*. So $f(c^*) = -tr[I_{d^*}(\alpha,\beta)^{-1}]$. Lemma 2 verifies this directly. (ii) $f'(c^*) = 0$ implies that c^* could be a potential point to maximize $f(c)$. But it cannot grantee that. It depends on the specific $\Psi(c)$. In the next section, we shall demonstrate how to verify this through some selected models. Here we assume $f(c)$ satisfies the following condition.

Condition (ii): $f(c)$ is maximized at c^*.

Now we are ready to state our main theorem.

Theorem 2. *Suppose that $\Psi(c)$ satisfies condition (i) and $f(c)$ satisfies condition (ii). Then the design d^*, defined in Theorem 1, is the only A-optimal design in the entire class $\mathcal{D}$.*

Proof. By Lemma 1, the conclusion is sufficient if we can show that $f(c)$ is maximized at c^* and $-c^*$ only. By (ii) of Lemma 2, $f(c)$ is a symmetric function about 0. We could restrict our consideration on $c > 0$. By condition (ii), $f(c)$ is maximized at c^*. Thus the conclusion follows. □

4 Examples

In this section, commonly used regression models are considered. For logistic and probit models, the A-optimal designs are clear-cut, i.e., the d^*, introduced in Theorem 1, is the

A-optimal design in the entire class. For double exponential and double reciprocal models, the A-optimal designs are less clear-cut. We may have the same conclusions as those of logistic and probit models; but the conclusion depends on the value of (α, β). According to Theorem 2, for each model, we only need to verify the two conditions are satisfied. Condition (i) is relatively easy to verify. The difficult part is to verify condition (ii). The following lemma is helpful for this.

Lemma 3. *Suppose that $\Psi(c)$ satisfies condition (i). Let c^* be the positive solution of (3.4), Z_1^*, Z_2^*, Y_1^*, and Y_1^* be defined in (3.8), then we have*
(i) $c^*\Psi'(c^*) + 2\Psi(c^*) > 0$;
(ii) $\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2} = -\frac{\Psi'(c^*)}{(c^*)^2\Psi'(c^*)+2c^*\Psi(c^*)}\left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2\right)$;
(iii) if $\Psi(0)(c^*\Psi'(c^*) + 2\Psi(c^*)) \leq 2\Psi^2(c^*)$, then $f(0) \leq f(c^*)$.

Proof. By (5.2) of Proposition 1 in appendix and the fact that c^* is the solution of (3.4), we have

$$-1 < 1 + \frac{c^*\Psi'(c^*)}{\Psi(c^*)} < 1,$$

which implies (i) by the fact that $\Psi(c^*) > 0$.

By (iii) of Lemma 2, we have $f'(c^*) = 0$. Using this fact and (3.15), we can establish (ii).

By (ii) of Lemma 2 and (ii), we have

$$\begin{aligned} f(0) - f(c^*) &= (\Psi(0) - \Psi(c^*))\left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2\right) \\ &\quad - (c^*)^2 \Psi(c^*)\left(\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2}\right) \\ &= \left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2\right) \frac{\Psi(0)(c^*\Psi'(c^*) + 2\Psi(c^*)) - 2\Psi^2(c^*)}{c^*\Psi'(c^*) + 2\Psi(c^*)}. \end{aligned}$$

Since $c^*\Psi'(c^*) + 2\Psi(c^*) > 0$ by (i), conclusion (iii) follows. □

4.1 Logistic regression

For logistic model, $P(c) = \frac{e^c}{1+e^c}$. Thus

$$P'(c) = \Psi(c) = \frac{e^c}{(1+e^c)^2} = \frac{1}{(e^{\frac{c}{2}} + e^{-\frac{c}{2}})^2}. \tag{4.1}$$

From (4.1), we can easily verify that $P(c)$ satisfies condition (i). By Theorem 1, the A-optimal design in $\mathcal{D}_1$ is

$$d^* = \{(x_i^*, \xi_i^*), i = 1, 2\}. \tag{4.2}$$

Here $x_1^* = (c^* - \alpha)/\beta$, $x_2^* = (-c^* - \alpha)/\beta$, $\xi_1^* = \xi_{c^*,\alpha,\beta}$, and $\xi_2^* = 1 - \xi_1^*$, where $\xi_{c^*,\alpha,\beta}$ is defined in (3.3) and $c^* > 0$ is the only positive solution of the following equation

$$\frac{c^2 - \alpha^2 - \beta^2}{\sqrt{\beta^2 + (c+\alpha)^2}\sqrt{\beta^2 + (c-\alpha)^2}} = 1 + \frac{c(1-e^c)}{1+e^c}. \tag{4.3}$$

Next we shall show that d^* is the only A-optimal design in the entire class. Before we do that, we need the following lemma.

Lemma 4. *Let c^* be the positive solution of (4.3), Z_1^*, Z_2^*, Y_1^*, and Y_1^* be defined in (3.8), then we have*

$$\frac{4(Y_1^*)^2}{\beta^2} + \frac{4(Z_2^* Y_2^*)^2}{\beta^2} - (Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 - (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 > 0. \tag{4.4}$$

Proof. Notice that α, β, and c^* all are positive, this implies that $(Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 > 0$. By this fact and (ii) of Lemma 3, it is clear that (4.4) is equivalent to

$$\frac{4\Psi'(c^*) + (c^*)^2 \Psi'(c^*) + 2c^* \Psi(c^*)}{(c^*)^2 \Psi'(c^*) + 2c^* \Psi(c^*)} < 0. \tag{4.5}$$

On the other hand,

$$\begin{aligned} 4\Psi'(c^*) + (c^*)^2 \Psi'(c^*) + 2c^* \Psi(c^*) &= (4 + (c^*)^2) \frac{e^{c^*}(1 - e^{c^*})}{(1 + e^{c^*})^3} + \frac{2c^* e^{c^*}}{(1 + e^{c^*})^2} \\ &= \frac{e^{c^*}}{(1 + e^{c^*})^3} \Big((c^*)^2 + 2c^* + 4 - e^{c^*} ((c^*)^2 - 2c^* + 4) \Big) \\ &< 0. \end{aligned} \tag{4.6}$$

The preceding inequality holds because $c^2 + 2c + 4 - e^c(c^2 - 2c + 4) < 0$ for all $c > 0$, which can be verified by routine algebra. By (i) of Lemma 3 and (4.6), (4.5) follows. □

Next, we shall verify that condition (ii) is satisfied. By (3.15), we have

$$f'(c) = \frac{e^c}{(1+e^c)^3} f_1(c),$$

where

$$\begin{aligned} f_1(c) = (1 - e^c) &\left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{c^2}{\beta^2} (Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{c^2}{\beta^2} (Z_2^* Y_2^*)^2 \right) \\ &+ (1 + e^c) \left(\frac{2c}{\beta^2} (Y_1^*)^2 + \frac{2c}{\beta^2} (Z_2^* Y_2^*)^2 \right). \end{aligned} \tag{4.7}$$

Next we shall investigate the property of $f_1(c)$. We have

$$f_1'(c) = \frac{2c+2}{\beta^2}(Y_1^*)^2 + \frac{2c+2}{\beta^2}(Z_2^*Y_2^*)^2 - e^c\left((Z_1^* - \frac{\alpha}{\beta})^2(Y_1^*)^2 + \frac{c^2-2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta}Z_2^* - 1)^2(Y_2^*)^2 + \frac{c^2-2}{\beta^2}(Z_2^*Y_2^*)^2\right), \tag{4.8}$$

$$f_1''(c) = \frac{2}{\beta^2}(Y_1^*)^2 + \frac{2}{\beta^2}(Z_2^*Y_2^*)^2 - e^c\left((Z_1^* - \frac{\alpha}{\beta})^2(Y_1^*)^2 + \frac{c^2+2c-2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta}Z_2^* - 1)^2(Y_2^*)^2 + \frac{c^2+2c-2}{\beta^2}(Z_2^*Y_2^*)^2\right), \tag{4.9}$$

and

$$f_1'''(c) = -e^c\left((Z_1^* - \frac{\alpha}{\beta})^2(Y_1^*)^2 + \frac{c^2+4c}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta}Z_2^* - 1)^2(Y_2^*)^2 + \frac{c^2+4c}{\beta^2}(Z_2^*Y_2^*)^2\right). \tag{4.10}$$

By (4.7), we have $f_1(0) = 0$. Since $f'(c^*) = 0$ by (iii) of Lemma 2, we have $f_1(c^*) = 0$. By (4.4) and (4.8) we have $f_1'(0) > 0$ and $\lim_{c\to+\infty} f_1'(c) = -\infty$. Similarly, we have $f_1''(0) > 0$ and $\lim_{c\to+\infty} f_1''(c) = -\infty$. By (4.10), we have $f_1'''(c) < 0$ for $c > 0$. By Proposition 4, we have the conclusion that $f(c)$ is maximized at c^*. Immediately we have following theorem by Theorem 2.

Theorem 3. *For logistic regression model, design d^* (4.2) is the only A-optimal design in the entire class $\mathcal{D}$.*

Remark: Theorem 3 proves that the conjecture by Mathew and Sinha (2001) is correct.

Example 1: suppose $\alpha = -2$ and $\beta = 1/2$. A simple computer program shows that $c^* = 1.8710$, $d^*=\{(x_1^* = 0.2579, \xi_1^* = 0.1168),\ (x_2^* = 7.7421,\ \xi_2^* = 0.8832)\}$. d^* is the A-optimal design in the entire class $\mathcal{D}$.

4.2 Probit regression

For probit model, $P(c) = \Phi(c) = \int_{-\infty}^{c}\phi(u)du$, where $\phi(u) = (1/\sqrt{2\pi})\exp(-\mathrm{u}^2/2)$. So we have $\Psi(c) = \phi^2(c)/[\Phi(c)(1-\Phi(c))]$. It is straight forward to show that $\lim_{c\to+\infty}\Psi(c) = 0$. On the other hand, it is clear that

$$\left(\Psi^{-1/2}(c)\right)' = -\frac{\Psi'(c)}{2\Psi^{3/2}(c)} \tag{4.11}$$

and

$$\begin{aligned}\left(\Psi^{-1/2}(c)\right)'' &= \frac{1}{2\Psi^{5/2}(c)}\left(\frac{3}{2}\left(\Psi'(c)\right)^2 - \Psi(c)\Psi''(c)\right) \\ &\geq \frac{1}{2\Psi^{5/2}(c)}\left(\left(\Psi'(c)\right)^2 - \Psi(c)\Psi''(c)\right).\end{aligned} \tag{4.12}$$

By the conclusion (iii) of Proposition 5 in appendix, both $\left(\Psi^{-1/2}(c)\right)'$ and $\left(\Psi^{-1/2}(c)\right)''$ are positive. So $P(c)$ satisfies condition (i). By Theorem 1, the A-optimal design in $\mathcal{D}_1$ is

$$d^* = \{(x_i^*, \xi_i^*), i = 1, 2\}. \tag{4.13}$$

Here $x_1^* = (c^* - \alpha)/\beta$, $x_2^* = (-c^* - \alpha)/\beta$, $\xi_1^* = \xi_{c^*,\alpha,\beta}$, and $\xi_2^* = 1 - \xi_1^*$, where $\xi_{c^*,\alpha,\beta}$ is defined in (3.3) and $c^* > 0$ is the only positive solution of the following equation

$$\frac{c^2 - \alpha^2 - \beta^2}{\sqrt{\beta^2 + (c+\alpha)^2}\sqrt{\beta^2 + (c-\alpha)^2}} = 1 + ch(c). \tag{4.14}$$

Here $h(c) = \Psi'(c)/\Psi(c)$.

Next we shall show that d^* is the only A-optimal design in the entire class. Before we do that, we need the following lemma.

Lemma 5. *Let c^* be the positive solution of (4.14), then we have (i) $c^* < 2$; (ii) $c^2h(c) + 2c < 0$ for $c \geq 2$; and (iii) $\frac{h(c^*)}{(c^*)^2h(c^*)+2c^*} < \frac{h'(0)}{2}$.*

Proof. By Lemma 3, c^* must satisfy (i) of Lemma 3, i.e., $c^*h(c^*) + 2 > 0$. By (i) of proposition 5 in appendix, $(ch(c) + 2)' = ch'(c) + h(c) < 0$ for $c > 0$. Thus $ch(c) + 2$ is a decreasing function of c. Direct computation shows that $2h(2) + 2 < 0$, thus we have $ch(c) + 2 < 0$ for $c \geq 2$. And $c^* < 2$ follows immediately.

Since $c^*h(c^*) + 2 > 0$, (iii) is equivalent to $h_1(c^*) > 0$, where $h_1(c) = h'(0)c^2h(c) - 2h(c) + 2h'(0)c$. Notice that

$$h_1'(c) = 2ch'(0)h(c) + c^2h'(0)h'(c) + 2h'(0) - 2h'(c). \tag{4.15}$$

By (i) of proposition 5 in appendix, the first two terms of right hand side of (4.15) are positive. By (ii) of proposition 5 in appendix, $2h'(0) - 2h'(c) > 0$ when $c < 2$. Thus $h_1'(c) > 0$ when $c < 2$. This implies that $h_1(c) > h_1(0) = 0$ for $c < 2$. The conclusion follows since $c^* < 2$. □

Next we shall show that $f(c)$ is maximized at c^*. By (3.15), (ii) of Lemma 3, and the definition of $h(c)$, we have

$$f'(c) = \left((Z_1^* - \frac{\alpha}{\beta})^2(Y_1^*)^2 + (\frac{\alpha}{\beta}Z_2^* - 1)^2(Y_2^*)^2\right)\Psi(c)f_1(c).$$

where

$$f_1(c) = h(c) - \left(c^2 h(c) + 2c\right) \frac{h(c^*)}{(c^*)^2 h(c^*) + 2c^*}. \tag{4.16}$$

It can be directly verified that $f_1(0) = 0$. $f(c)$ is maximized at c^* is sufficient if we can show that $f_1(c) > 0$ for $0 < c < c^*$ and $f_1(c) < 0$ for $c > c^*$. When $c \geq 2$, by (ii) of Lemma 5, $c^2 h(c) + 2c < 0$. On the other hand, $\frac{h(c^*)}{(c^*)^2 h(c^*)+2c^*} < 0$ by (iii) of Lemma 5 and the fact $h'(0) < 0$ ((i) of proposition 5)). So the last term of right hand side of (4.16) is positive. Notice that $h(c) < 0$ by (i) of Proposition 5 in appendix, $f_1(c) < 0$ when $c \geq 2$. By (i) of Lemma 5, we have $c^* < 2$. So the conclusion is sufficient if we can show that $f_1(c) > 0$ for $0 < c < c^*$ and $f_1(c) < 0$ for $c^* < c < 2$.

It is clear that

$$f_1'(c) = h'(c) - \left(c^2 h'(c) + 2ch(c) + 2\right) \frac{h(c^*)}{(c^*)^2 h(c^*) + 2c^*} \tag{4.17}$$

and

$$f_1''(c) = h''(c) - \left(c^2 h''(c) + 4ch'(c) + 2h(c)\right) \frac{h(c^*)}{(c^*)^2 h(c^*) + 2c^*}. \tag{4.18}$$

By Proposition 5 in appendix, when $0 < c < 2$, we have $h''(c) < 0$, $h'(c) < 0$, and $h(c) < 0$. Thus $c^2 h''(c) + 4ch'(c) + 2h(c) < 0$. Also notice that $\frac{h(c^*)}{(c^*)^2 h(c^*)+2c^*} < 0$ by (iii) of Lemma 5 and the fact $h'(0) < 0$. The last term of right hand side of (4.18) is positive. Applying this fact, $h''(c) < 0$, and (4.18), we have $f_1''(c) < 0$ when $0 < c < 2$. This implies that $f_1'(c)$ is decreasing function of c. On the other hand, applying (iii) of Lemma 5 again, we have $f_1'(0) > 0$. $f_1'(c)$ will be either (i) always positive when $0 < c < 2$ or (ii) first positive then negative. Case (i) cannot happen since it implies $f_1(c)$ is an increasing function when $0 < c < 2$. It is contradiction to the fact that $f_1(0) = 0$ and $f_1(c^*) = 0$. Thus Case (ii) holds. This means for some $\tilde{c} > 0$, $f_1(c)$ is an increasing function when $0 < c < \tilde{c}$ and decreasing function when $\tilde{c} < c < 2$. By the fact that $f_1(0) = 0$ and $f_1(c^*) = 0$, we can easily see that $f_1(c) > 0$ for $0 < c < c^*$ and $f_1(c) < 0$ for $c^* < c < 2$. Thus $f(c)$ is maximized at c^*. Immediately, we have the following theorem.

Theorem 4. *For probit regression model, design d^* (4.13) is the only A-optimal design in the entire class $\mathcal{D}$.*

Example 2: suppose $\alpha = 1/2$ and $\beta = -1$. A simple computer program shows that $c^* = 1.1311$, d^*=$\{(x_1^* = -0.6311, \xi_1^* = 0.3820), (x_2^* = 1.6311, \xi_2^* = 0.6180)\}$. d^* is the A-optimal design in the entire class $\mathcal{D}$.

4.3 Double exponential regression

For double exponential model, $P(c) = 1 - \frac{1}{2}e^{-|c|}$ when $c > 0$ and $\frac{1}{2}e^{-|c|}$ when $c \le 0$. Thus $P'(c) = \frac{1}{2}e^{-|c|}$ and $\Psi(c) = \frac{1}{2e^{|c|}-1}$. For $c > 0$, we have

$$\left(\Psi^{-1/2}(c)\right)' = \frac{e^c}{\sqrt{2e^c - 1}} \quad \text{and} \quad \left(\Psi^{-1/2}(c)\right)'' = \frac{e^c(e^c-1)}{(2e^c-1)^{3/2}}.$$

So $P(c)$ satisfies condition (i). By Theorem 1, the A-optimal design in $\mathcal{D}_1$ is

$$d^* = \{(x_i^*, \xi_i^*), i = 1, 2\}. \tag{4.19}$$

Here $x_1^* = (c^* - \alpha)/\beta$, $x_2^* = (-c^* - \alpha)/\beta$, $\xi_1^* = \xi_{c^*,\alpha,\beta}$, and $\xi_2^* = 1 - \xi_1^*$, where $\xi_{c^*,\alpha,\beta}$ is defined in (3.3) and $c^* > 0$ is the only positive solution of the following equation

$$\frac{c^2 - \alpha^2 - \beta^2}{\sqrt{\beta^2 + (c+\alpha)^2}\sqrt{\beta^2 + (c-\alpha)^2}} = 1 - \frac{2ce^c}{2e^c - 1}. \tag{4.20}$$

Next we shall show that d^* is the only A-optimal design in the entire class if $(2-c^*)e^{c^*} - 2 \le 0$. Before we do that, we need the following lemma.

Lemma 6. *Let c^* be the positive solution of (4.20), Z_1^*, Z_2^*, Y_1^*, and Y_1^* be defined in (3.8), then we have*
(i) $\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2} - (Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 - (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 > 0$;
(ii) if $(2 - c^*)e^{c^*} - 2 \le 0$, *then* $f(0) \le f(c^*)$.

Proof. By the similar argument as that of Lemma 4, (i) is equivalent to

$$\frac{\Psi'(c^*) + (c^*)^2\Psi'(c^*) + 2c^*\Psi(c^*)}{(c^*)^2\Psi'(c^*) + 2c^*\Psi(c^*)} < 0. \tag{4.21}$$

On the other hand,

$$\begin{aligned} \Psi'(c^*) + (c^*)^2\Psi'(c^*) + 2c^*\Psi(c^*) &= -(1 + (c^*)^2)\frac{2e^{c^*}}{(2e^{c^*}-1)^2} + \frac{2c^*}{2e^{c^*}-1} \\ &= -\frac{2}{(2e^{c^*}-1)^2}\left(e^{c^*}(c^*-1)^2 + c^*\right) \\ &< 0 \end{aligned} \tag{4.22}$$

By (i) of Lemma 3 and (4.22), (4.21) follows.

As for (ii), by directly computation, we have

$$\Psi(0)(c^*\Psi'(c^*) + 2\Psi(c^*)) - 2\Psi^2(c^*) = \frac{2\left((2-c^*)e^{c^*} - 2\right)}{(2e^{c^*}-1)^2}.$$

By (iii) of Lemma 3, conclusion (ii) follows. □

Next, we shall study the property of $f(c)$. By (3.15), we have

$$f'(c) = \frac{1}{(2e^c - 1)^2} f_1(c),$$

where

$$\begin{aligned} f_1(c) = -2e^c & \left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{c^2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{c^2}{\beta^2}(Z_2^* Y_2^*)^2 \right) \\ & + (2e^c - 1) \left(\frac{2c}{\beta^2}(Y_1^*)^2 + \frac{2c}{\beta^2}(Z_2^* Y_2^*)^2 \right). \end{aligned} \tag{4.23}$$

It is clear that $f_1(0) < 0$. Next we shall investigate the property of $f_1(c)$. We have

$$\begin{aligned} f_1'(c) = & - \frac{2}{\beta^2}(Y_1^*)^2 - \frac{2}{\beta^2}(Z_2^* Y_2^*)^2 \\ & - 2e^c \left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{c^2 - 2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{c^2 - 2}{\beta^2}(Z_2^* Y_2^*)^2 \right), \end{aligned} \tag{4.24}$$

$$f_1''(c) = -2e^c \left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{c^2 + 2c - 2}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{c^2 + 2c - 2}{\beta^2}(Z_2^* Y_2^*)^2 \right), \tag{4.25}$$

and

$$f_1'''(c) = - 2e^c \left((Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 + \frac{c^2 + 4c}{\beta^2}(Y_1^*)^2 + (\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 + \frac{c^2 + 4c}{\beta^2}(Z_2^* Y_2^*)^2 \right). \tag{4.26}$$

Since $f'(c^*) = 0$ by (iii) of Lemma 2, we have $f_1(c^*) = 0$. By (i) of Lemma 6 and (4.24) we have $f_1'(0) > 0$ and $\lim_{c \to +\infty} f_1'(c) = -\infty$. Similarly, we have $f_1''(0) > 0$ and $\lim_{c \to +\infty} f_1''(c) = -\infty$. By (4.26), we have $f_1'''(c) < 0$ for $c > 0$. If we assume that $(2 - c^*)e^{c^*} - 2 \leq 0$, then by (ii) of Lemma 6, we have $f(0) \leq f(c^*)$. Notice that $f_1(0) < 0$, then by Proposition 4, $f(c)$ is maximized at c^*. Immediately we have following theorem by Theorem 2.

Theorem 5. *For double exponential regression model, design d^* (4.19) is the only A-optimal design in the entire class $\mathcal{D}$ if $(2 - c^*)e^{c^*} - 2 \leq 0$.*

Example 3: suppose $\alpha = 3$ and $\beta = 1$. A simple computer program shows that $c^* = 1.7285$, d^*=$\{(x_1^* = -1.2715, \xi_1^* = 0.2508), (x_2^* = -4.7285, \xi_2^* = 0.7492)\}$, and $(2 - c^*)e^{c^*} - 2 \leq 0$. Thus d^* is the A-optimal design in the entire class $\mathcal{D}$.

4.4 Double reciprocal regression

For double reciprocal model, $P(c) = 1 - \frac{1}{2(1+c)}$ when $c > 0$ and $\frac{1}{2(1-c)}$ when $c \leq 0$. Thus $P'(c) = \frac{1}{2(1+|c|)^2}$ and $\Psi(c) = \frac{1}{(1+|c|)^2(1+2|c|)}$. For $c > 0$, we have

$$\left(\Psi^{-1/2}(c)\right)' = \frac{3c+2}{\sqrt{2c+1}} \quad \text{and} \quad \left(\Psi^{-1/2}(c)\right)'' = \frac{3c+1}{(2c+1)^{3/2}}.$$

So $P(c)$ satisfies condition (i). By Theorem 1, the A-optimal design in $\mathcal{D}_1$ is

$$d^* = \{(x_i^*, \xi_i^*), i = 1, 2\}. \tag{4.27}$$

Here $x_1^* = (c^* - \alpha)/\beta$, $x_2^* = (-c^* - \alpha)/\beta$, $\xi_1^* = \xi_{c^*,\alpha,\beta}$, and $\xi_2^* = 1 - \xi_1^*$, where $\xi_{c^*,\alpha,\beta}$ is defined in (3.3) and $c^* > 0$ is the only positive solution of the following equation

$$\frac{c^2 - \alpha^2 - \beta^2}{\sqrt{\beta^2 + (c+\alpha)^2}\sqrt{\beta^2 + (c-\alpha)^2}} = -\frac{4c^2 + c - 1}{2c^2 + 3c + 1}. \tag{4.28}$$

Next we shall show that d^* is the only A-optimal design in the entire class if $c^* \geq \sqrt{2}$. Before we do that, we need the following lemma.

Lemma 7. *Let c^* be the positive solution of (4.28), Z_1^*, Z_2^*, Y_1^*, and Y_1^* be defined in (3.8), then we have*
(i) $\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2} - 3(Z_1^* - \frac{\alpha}{\beta})^2 (Y_1^*)^2 - 3(\frac{\alpha}{\beta} Z_2^* - 1)^2 (Y_2^*)^2 > 0$;
(ii) if $c^* \geq \sqrt{2}$, then $f(0) \leq f(c^*)$.

Proof. By the similar argument as that of Lemma 4, (i) is equivalent to

$$\frac{\Psi'(c^*) + 3(c^*)^2\Psi'(c^*) + 6c^*\Psi(c^*)}{(c^*)^2\Psi'(c^*) + 2c^*\Psi(c^*)} < 0. \tag{4.29}$$

On the other hand,

$$\begin{aligned} \Psi'(c^*) + 3(c^*)^2\Psi'(c^*) + 6c^*\Psi(c^*) &= -\frac{6(c^*)^3 - 6(c^*)^2 + 4}{(2c^*+1)^2(c^*+1)^3} \\ &< 0. \end{aligned} \tag{4.30}$$

By (i) of Lemma 3 and (4.30), (4.29) follows.

As for (ii), by directly computation, we have

$$\Psi(0)(c^*\Psi'(c^*) + 2\Psi(c^*)) - 2\Psi^2(c^*) = \frac{2c^*(2 - (c^*)^2)}{(1+c^*)^4(1+2c^*)^2}.$$

By (iii) of Lemma 3, conclusion (ii) follows. □

Next, we shall study the property of $f(c)$. By (3.15), we have

$$f'(c) = \frac{1}{(2c+1)^2(c+1)^3} f_1(c),$$

where

$$f_1(c) = -(6c+4)\left((Z_1^* - \frac{\alpha}{\beta})^2(Y_1^*)^2 + (\frac{\alpha}{\beta}Z_2^* - 1)^2(Y_2^*)^2\right) \\ - (2c^3 - 2c^2 - 2c)\left(\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2}\right). \tag{4.31}$$

It is clear that $f_1(0) < 0$. Next we shall investigate the property of $f_1(c)$. We have

$$f_1'(c) = -6\left((Z_1^* - \frac{\alpha}{\beta})^2(Y_1^*)^2 + (\frac{\alpha}{\beta}Z_2^* - 1)^2(Y_2^*)^2\right) - (6c^2 - 4c - 2)\left(\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2}\right), \tag{4.32}$$

$$f_1''(c) = -(12c - 4)\left(\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2}\right), \tag{4.33}$$

and

$$f_1'''(c) = -12\left(\frac{(Y_1^*)^2}{\beta^2} + \frac{(Z_2^* Y_2^*)^2}{\beta^2}\right). \tag{4.34}$$

Since $f'(c^*) = 0$ by (iii) of Lemma 2, we have $f_1(c^*) = 0$. By (i) of Lemma 7 and (4.32) we have $f_1'(0) > 0$ and $\lim_{c\to+\infty} f_1'(c) = -\infty$. Similarly, we have $f_1''(0) > 0$ and $\lim_{c\to+\infty} f_1''(c) = -\infty$. By (4.34), we have $f_1'''(c) < 0$ for $c > 0$. If we assume that $c^* \geq \sqrt{2}$, then by (ii) of Lemma 7, we have $f(0) \leq f(c^*)$. Notice that $f_1(0) < 0$, then by Proposition 4, $f(c)$ is maximized at c^*. Immediately we have following theorem by Theorem 2.

Theorem 6. *For double reciprocal regression model, design d^* (4.27) is the only A-optimal design in the entire class $\mathcal{D}$ if $c^* \geq \sqrt{2}$.*

Example 4: suppose $\alpha = -5$ and $\beta = -1$. A simple computer program shows that $c^* = 1.5956$, d^*=$\{(x_1^* = -3.4044,\ \xi_1^* = 0.3472),\ (x_2^* = -6.5956,\ \xi_2^* = 0.6528)\}$. Thus d^* is the A-optimal design in the entire class $\mathcal{D}$.

5 Discussion

Many results for optimal designs related to binary data are based on the geometry approach. The main purpose of this paper is to develop an algebraic approach for constructing A-optimal design for generalized linear two-parameter models. This approach gives two sufficient conditions to identify the A-optimal design. It shows that if the two conditions are satisfied, the A-optimal design is unique and has exactly two support points. The two points are symmetric, but not weight symmetric. For a given model, condition (i) is straight forward to verify; condition (ii) is relative difficult to verify. The commonly used models, such as logistic and probit models satisfies the two conditions. Thus the A-optimal designs are based on two symmetric points with different weights. This result show that the conjecture of Mathew and Sinha (2001) is true for both logistic and probit models. It is interesting to see that, for some parameters, the A-optimal designs for double exponential and double reciprocal is based on two symmetric points with different weights. Sitter and Wu (1993a) shows that the D-optimal designs of double exponential and double reciprocal models are based on two symmetric points plus point 0. Holger and Haines (1994) shows that, for some parameters, the E-optimal design of double reciprocal is also based on two symmetric points plus point 0. Could A-optimal designs be based on two symmetric point plus point 0 for the parameters that condition (ii) is not satisfied? The research effort in this area is continuing.

Point c^* is the only solution of (3.4). There is no explicit expression of the solution in general. The solution has to be solved numerically. However c^* is also the only solution of (3.5), which is a increasing function of c. This is very helpful for us to find the solution numerically. Once the solution is obtained, the two weights can be obtained immediately by (3.3). A SAS macro program to derive the corresponding A-optimal design based on the approach is available from the author as an addendum to this paper.

Appendix

Proposition 1. *For any positive α, $\beta > 0$, and c, we have*

$$\frac{\frac{c^2+\alpha c}{\sqrt{\beta^2+(c+\alpha)^2}} + \frac{c^2-\alpha c}{\sqrt{\beta^2+(c-\alpha)^2}}}{\sqrt{\beta^2+(c+\alpha)^2}+\sqrt{\beta^2+(c-\alpha)^2}} = \frac{1}{2} + \frac{c^2-\alpha^2-\beta^2}{2\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}}, \tag{5.1}$$

and

$$\frac{|c^2-\alpha^2-\beta^2|}{\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}} < 1. \tag{5.2}$$

Proof. Apply some routine algebra, we have

$$\frac{\sqrt{\beta^2+(c+\alpha)^2}-\sqrt{\beta^2+(c-\alpha)^2}}{\sqrt{\beta^2+(c+\alpha)^2}+\sqrt{\beta^2+(c-\alpha)^2}} = \frac{1}{2\alpha c}\left(\beta^2+\alpha^2+c^2-\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}\right) \tag{5.3}$$

On the other hand, we have

$$\begin{aligned}&\frac{c^2+\alpha c}{\sqrt{\beta^2+(c+\alpha)^2}}+\frac{c^2-\alpha c}{\sqrt{\beta^2+(c-\alpha)^2}}\\&=\frac{c^2(\sqrt{\beta^2+(c+\alpha)^2}+\sqrt{\beta^2+(c-\alpha)^2})-ac(\sqrt{\beta^2+(c+\alpha)^2}-\sqrt{\beta^2+(c-\alpha)^2})}{\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}}.\end{aligned}\tag{5.4}$$

Applying (5.3) and (5.4), we can obtain (5.1) after some simple algebra.

As for (5.2), we have

$$\frac{|c^2-\alpha^2-\beta^2|}{\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}}=\frac{|c^2-\alpha^2-\beta^2|}{\sqrt{(c^2-\alpha^2-\beta^2)^2+4\beta^2c^2}}.$$

Thus the conclusion follows. □

Proposition 2. *For the Z_1^*, Z_2^*, Y_1^*, and Y_2^* defined in (3.8), we have*

$$(\frac{Z_1^*}{\beta}-\frac{\alpha}{\beta^2})(Y_1^*)^2+(\frac{Z_2^*}{\beta}-\frac{\alpha(Z_2^*)^2}{\beta^2})(Y_2^*)^2=0,\tag{5.5}$$

$$1+(Z_1^*)^2=\frac{\sqrt{\beta^2+(c^*+\alpha)^2}\sqrt{\beta^2+(c^*-\alpha)^2}}{4\alpha^2\beta^2}\left(4\alpha^2-\left(\sqrt{\beta^2+(c^*+\alpha)^2}-\sqrt{\beta^2+(c^*-\alpha)^2}\right)^2\right)\tag{5.6}$$

$$(\xi_1^*-\xi_2^*)^2+\left(\frac{c^*-\alpha(\xi_1^*-\xi_2^*)}{\beta}\right)^2+1+(Z_1^*)^2=\frac{\sqrt{\beta^2+(c^*+\alpha)^2}\sqrt{\beta^2+(c^*-\alpha)^2}}{4\alpha^2\beta^2(c^*)^2}\left(8\alpha^2(c^*)^2-(\alpha^2+\beta^2+(c^*)^2)\left(\sqrt{\beta^2+(c^*+\alpha)^2}-\sqrt{\beta^2+(c^*-\alpha)^2}\right)^2\right)\tag{5.7}$$

Proof. By (3.8), it is clear that $Y_2^*=\frac{Y_1^*Z_1^*}{Z_2^*}$. Applying this fact, by some routine algebra, we have

$$\begin{aligned}(\frac{Z_1^*}{\beta}-\frac{\alpha}{\beta^2})(Y_1^*)^2+(\frac{Z_2^*}{\beta}-\frac{\alpha(Z_2^*)^2}{\beta^2})(Y_2^*)^2&=\frac{(Y_1^*)^2}{\beta^2}\left(Z_1^*(\beta+\frac{\beta Z_1^*}{Z_2^*}-\alpha Z_1^*)-\alpha\right)\\&=\frac{(Y_1^*)^2}{\beta^2}\left(\frac{Z_1^*}{\beta}\left(\beta^2+(c^*)^2-\alpha c^*(\xi_1^*-\xi_2^*)\right)-\alpha\right).\end{aligned}\tag{5.8}$$

By the definitions of ξ_1^* and ξ_2^*, by the similar approach as that of (5.3), we have

$$\xi_1^*-\xi_2^*=\frac{1}{2\alpha c^*}\left(\beta^2+\alpha^2+(c^*)^2-\sqrt{\beta^2+(c^*+\alpha)^2}\sqrt{\beta^2+(c^*-\alpha)^2}\right).\tag{5.9}$$

Applying (5.9), we have

$$\beta^2+(c^*)^2-\alpha c^*(\xi_1^*-\xi_2^*)=\frac{1}{2}\left(\beta^2+(c^*)^2-\alpha^2+\sqrt{\beta^2+(\alpha+c^*)^2}\sqrt{\beta^2+(\alpha-c^*)^2}\right)\tag{5.10}$$

and

$$Z_1^* = \frac{1}{2\alpha\beta}\left(-\beta^2 - (c^*)^2 + \alpha^2 + \sqrt{\beta^2 + (\alpha + c^*)^2}\sqrt{\beta^2 + (\alpha - c^*)^2}\right). \tag{5.11}$$

Apply (5.10) and (5.11) into (5.8), by some routine algebra, we can establish (5.5).

Applying (5.9) again, we have

$$\begin{aligned}
(\xi_1^* - \xi_2^*)^2 - 1 &= \frac{\left(\beta^2 + \alpha^2 + (c^*)^2 - \sqrt{\beta^2 + (\alpha + c^*)^2}\sqrt{\beta^2 + (\alpha - c^*)^2}\right)^2 - 4\alpha^2(c^*)^2}{4\alpha^2(c^*)^2} \\
&= \frac{1}{4\alpha^2(c^*)^2}\left(\beta^2 + (\alpha^2 - c^*)^2 - \sqrt{\beta^2 + (\alpha + c^*)^2}\sqrt{\beta^2 + (\alpha - c^*)^2}\right) \\
&\qquad \left(\beta^2 + (\alpha^2 + c^*)^2 - \sqrt{\beta^2 + (\alpha + c^*)^2}\sqrt{\beta^2 + (\alpha - c^*)^2}\right) \\
&= -\frac{\sqrt{\beta^2 + (\alpha + c^*)^2}\sqrt{\beta^2 + (\alpha - c^*)^2}\left(\sqrt{\beta^2 + (\alpha + c^*)^2} - \sqrt{\beta^2 + (\alpha - c^*)^2}\right)^2}{4\alpha^2(c^*)^2}.
\end{aligned} \tag{5.12}$$

By the definition of Z_1^* and (5.9), we have

$$\begin{aligned}
1 + (Z_1^*)^2 &= \frac{1}{\beta^2}\left(\beta^2 + \alpha^2 + (c^*)^2(\xi_1^* - \xi_2^*)^2 - 2\alpha c^*(\xi_1^* - \xi_2^*)\right) \\
&= \frac{1}{\beta^2}\left((c^*)^2[(\xi_1^* - \xi_2^*)^2 - 1] + \sqrt{\beta^2 + (\alpha + c^*)^2}\sqrt{\beta^2 + (\alpha - c^*)^2}\right).
\end{aligned} \tag{5.13}$$

Applying (5.12) into (5.13), we can establish (5.6).

By the similar approach, we have

$$\begin{aligned}
\left(\frac{c^* - \alpha(\xi_1^* - \xi_2^*)}{\beta}\right)^2 + 1 &= \frac{1}{\beta^2}\left(\alpha^2[(\xi_1^* - \xi_2^*)^2 - 1] + \sqrt{\beta^2 + (\alpha + c^*)^2}\sqrt{\beta^2 + (\alpha - c^*)^2}\right) \\
&= \frac{\sqrt{\beta^2 + (c^* + \alpha)^2}\sqrt{\beta^2 + (c^* - \alpha)^2}}{4\beta^2(c^*)^2} \\
&\qquad \left(4(c^*)^2 - \left(\sqrt{\beta^2 + (c^* + \alpha)^2} - \sqrt{\beta^2 + (c^* - \alpha)^2}\right)^2\right)
\end{aligned} \tag{5.14}$$

By (5.12), (5.14), and established (5.6), we can establish (5.7). □

Proposition 3. *For any positive α, β, and c,*

$$\begin{aligned}
&\frac{c^2 - \alpha^2 - \beta^2 - \sqrt{\beta^2 + (c + \alpha)^2}\sqrt{\beta^2 + (c - \alpha)^2}}{2\sqrt{\beta^2 + (c + \alpha)^2}\sqrt{\beta^2 + (c - \alpha)^2}} \\
&= -\frac{4\alpha^2 c^2 - c^2\left(\sqrt{\beta^2 + (c + \alpha)^2} - \sqrt{\beta^2 + (c - \alpha)^2}\right)^2}{8\alpha^2 c^2 - (\alpha^2 + \beta^2 + c^2)\left(\sqrt{\beta^2 + (c + \alpha)^2} - \sqrt{\beta^2 + (c - \alpha)^2}\right)^2}.
\end{aligned} \tag{5.15}$$

Proof. Notice that

$$\left(\sqrt{\beta^2+(c+\alpha)^2}-\sqrt{\beta^2+(c-\alpha)^2}\right)^2 = 2(\alpha^2+\beta^2+c^2) - 2\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}. \tag{5.16}$$

By (5.16) and some routine algebra, we have

$$-\frac{4\alpha^2c^2 - c^2\left(\sqrt{\beta^2+(c+\alpha)^2}-\sqrt{\beta^2+(c-\alpha)^2}\right)^2}{8\alpha^2c^2-(\alpha^2+\beta^2+c^2)\left(\sqrt{\beta^2+(c+\alpha)^2}-\sqrt{\beta^2+(c-\alpha)^2}\right)^2}$$
$$= -\frac{c^2(\alpha^2-\beta^2-c^2+\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2})}{\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}(\alpha^2+\beta^2+c^2-\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2})}. \tag{5.17}$$

So (5.15) is sufficient if we can show

$$c^2-\alpha^2-\beta^2-\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2} = -\frac{2c^2(\alpha^2-\beta^2-c^2+\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2})}{\alpha^2+\beta^2+c^2-\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}},$$

which can be verified directly by the fact

$$\left(\sqrt{\beta^2+(c+\alpha)^2}\sqrt{\beta^2+(c-\alpha)^2}\right)^2 = \alpha^4+\beta^4+c^4+2\alpha^2\beta^2+2\beta^2c^2-2\alpha^2c^2.$$

So the conclusion follows. □

Proposition 4. *Let $g(c)$ be a function defined in $[0,+\infty)$ and $g'(c)=g_0(c)g_1(c)$, where $g_0(c)>0$ for $c\in[0,+\infty)$. Suppose that $g_1(c^*)=0$ for some $c^*>0$; $g_1'(0)>0$ and $\lim_{c\to+\infty} g_1'(c)=-\infty$; $g_1''(0)>0$ and $\lim_{c\to+\infty} g_1''(c)=-\infty$; $g_1'''(c)<0$ for $c>0$. Then $g(c)$ is maximized at c^* if one of the following conditions holds: (i) $g_1(0)=0$ or (ii) $g_1(0)<0$ and $g(0)\le g(c^*)$.*

Proof. $g_1'''(c)<0$ implies that $g_1''(c)$ is a decreasing function of c. On the other hand, $g_1''(0)>0$ and $\lim_{c\to+\infty} g_1''(c)=-\infty$. So there exist $c_1>0$, such that $g_1''(c)>0$ when $c\in(0,c_1)$ and $g_1''(c)<0$ when $c\in(c_1,+\infty)$. This implies that $g_1'(c)$ is an increasing function when $c\in(0,c_1)$ and a decreasing function when $c\in(c_1,+\infty)$. With this fact and $g_1'(0)>0$ and $\lim_{c\to+\infty} g_1'(c)=-\infty$, there exist $c_2>0$, such that $g_1'(c)>0$ when $c\in(0,c_2)$ and $g_1'(c)<0$ when $c\in(c_2,+\infty)$. Thus $g_1(c)$ is an increasing function when $c\in(0,c_2)$ and a decreasing function when $c\in(c_2,+\infty)$.

Case (i): when $g_1(0)=0$. If $c_2\ge c^*$, then $g_1(c^*)>0$ since $g_1(0)=0$ and $g_1(c)$ is an increasing function for $c\in(0,c^*)$. This is contradiction to $g_1(c^*)=0$. So we must have $c_2 < c^*$. $g_1(0) = 0$ and the fact that $g_1(c)$ is an increasing function when $c \in (0, c_2)$ implies that $g_1(c) > 0$ when $c \in (0, c_2]$. $g_1(c^*) = 0$ and the fact that $g_1(c)$ is a decreasing function when $c \in (c_2, +\infty)$ implies that $g_1(c) > 0$ when $c \in [c_2, c^*)$ and $g_1(c) < 0$ when $c \in (c^*, +\infty)$. So we have $g_1(c) > 0$ when $c \in (0, c^*)$ and $g_1(c) < 0$ when $c \in (c^*, +\infty)$. This is equivalent to $g'(c) > 0$ when $c \in (0, c^*)$ and $g'(c) < 0$ when $c \in (c^*, +\infty)$ since $g_0(c) > 0$. This implies that $g(c)$ is maximized at c^*.

Case (ii): when $g_1(0) < 0$ and $g(0) \leq g(c^*)$. If $c_2 \geq c^*$, then $g_1(c) < 0$ for $c \in (0, c^*)$ since $g_1(c^*) = 0$ and $g_1(c)$ is an increasing function for $c \in (0, c^*)$. This is equivalent to $g'(c) < 0$ for $c \in (0, c^*)$, which means that $g(0) > g(c^*)$. This is contradiction to $g(0) \leq g(c^*)$. So we must have $c_2 < c^*$. Similarly as case (i), we have $g_1(c) > 0$ when $c \in [c_2, c^*)$ and $g_1(c) < 0$ when $c \in (c^*, +\infty)$. On the other hand, $g_1(0) < 0$, $g_1(c_2) > 0$, and the fact that $g_1(c)$ is an increasing function when $c \in (0, c_2)$ implies that there exist a point $c_3 \in (0, c_2)$ such that $g_1(c) < 0$ for $c \in (0, c_3)$ and $g_1(c) > 0$ for $c \in (c_3, c_2)$. So we have $g_1(c) < 0$ for $c \in (0, c_3)$, $g_1(c) > 0$ for $c \in (c_3, c^*)$, and $g_1(c) < 0$ for $c \in (c^*, +\infty)$. Again, $g'(c)$ has the same pattern by the fact that $g_0(c) > 0$. This implies that $g(c)$ is maximized at either 0 or c^*. However we have $g(0) \leq g(c^*)$, thus the conclusion follows. □

Proposition 5. *Let* $\Phi(c) = \int_{-\infty}^{c} \phi(u)du$, *where* $\phi(u) = (1/\sqrt{2\pi})\exp(-u^2/2)$. *Define* $\Psi(c) = \phi^2(c)/[\Phi(c)(1 - \Phi(c))]$ *and* $h(c) = \Psi'(c)/\Psi(c)$. *Then we have*

- *(i)* $h(c) < 0$ *and* $h'(c) < 0$ *for* $c > 0$*;* $h'(0) < 0$.
- *(ii)* $h''(c) < 0$ *for* $0 < c < 2$*;*
- *(iii)* $\Psi'(c) < 0$ *and* $\Psi(c)\Psi''(c) - (\Psi'(c))^2 < 0$ *for* $c > 0$.

Proof. Since $\Psi(c) > 0$, conclusion (iii) is equivalent to conclusion (i). We only need to show conclusion (i) and (ii).

Notice that $\phi'(c) = -c\phi(c)$ and $\Phi'(c) = \phi(c)$, by the straight algebra, we have

$$h(c) = -2c - \frac{\phi(c)}{\Phi(c)} + \frac{\phi(c)}{1 - \Phi(c)}, \tag{5.18}$$

$$h'(c) = -2 + \frac{c\phi(c)\Phi(c) + \phi^2(c)}{\Phi^2(c)} + \frac{\phi^2(c) - c\phi(c)\left(1 - \Phi(c)\right)}{\left(1 - \Phi(c)\right)^2}, \tag{5.19}$$

$$\begin{aligned} h''(c) = {} & \frac{(1-c^2)\phi(c)}{\Phi(c)} - \frac{3c\phi^2(c)}{\Phi^2(c)} - \frac{2\phi^3(c)}{\Phi^3(c)} \\ & - \frac{(1-c^2)\phi(c)}{1 - \Phi(c)} - \frac{3c\phi^2(c)}{(1 - \Phi(c))^2} + \frac{2\phi^3(c)}{(1 - \Phi(c))^3}, \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} h'''(c) = {} & \frac{(c^3 - 3c)\phi(c)}{\Phi(c)} + \frac{(7c^2 - 4)\phi^2(c)}{\Phi^2(c)} + \frac{12c\phi^3(c)}{\Phi^3(c)} + \frac{6\phi^4(c)}{\Phi^4(c)} \\ & + \frac{(3c - c^3)\phi(c)}{1 - \Phi(c)} + \frac{(7c^2 - 4)\phi^2(c)}{(1 - \Phi(c))^2} - \frac{12c\phi^3(c)}{(1 - \Phi(c))^3} + \frac{6\phi^4(c)}{(1 - \Phi(c))^4}. \end{aligned} \tag{5.21}$$

Next, we shall show that $h'''(c) < 0$ for $0 < c \leq 1/2$. Since $(0, 1/2] = \bigcup_{i=1}^{500}((i-1)/1000, i/1000]$, it is sufficient to show $h'''(c) < 0$ for $c \in ((i-1)/1000, i/1000]$, $i = 1, ..., 500$. For $c \in (c', c'']$, where $0 \leq c' < c'' \leq 1/2$, we have $\frac{(c^3-3c)\phi(c)}{\Phi(c)} \leq \frac{(c'^3-3c')\phi(c'')}{\Phi(c'')}$; $\frac{(7c^2-4)\phi^2(c)}{\Phi^2(c)} \leq \frac{(7c''^2-4)\phi^2(c'')}{\Phi^2(c'')}$; $\frac{12c\phi^3(c)}{\Phi^3(c)} \leq \frac{12c''\phi^3(c')}{\Phi^3(c')}$; $\frac{6\phi^4(c)}{\Phi^4(c)} \leq \frac{6\phi^4(c')}{\Phi^4(c')}$; $\frac{(3c-c^3)\phi(c)}{1-\Phi(c)} \leq \frac{(3c''-c''^3)\phi(c')}{1-\Phi(c'')}$; $\frac{(7c^2-4)\phi^2(c)}{(1-\Phi(c))^2} \leq \frac{(7c''^2-4)\phi^2(c'')}{(1-\Phi(c'))^2}$; $-\frac{12c\phi^3(c)}{(1-\Phi(c))^3} \leq -\frac{12c'\phi^3(c'')}{(1-\Phi(c'))^3}$; and $\frac{6\phi^4(c)}{(1-\Phi(c))^4} \leq \frac{6\phi^4(c')}{(1-\Phi(c''))^4}$. By (5.21) and a simple computer program can verify that $h'''(c) < 0$ for $c \in ((i-1)/1000, i/1000]$, $i = 1, ..., 500$.

By (5.20), we have $h''(0) = 0$. Since $h'''(c) < 0$ for $0 < c \leq 1/2$, this implies that $h''(c) < 0$ for $0 < c \leq 1/2$. We shall show that $h''(c) < 0$ for $1/2 < c < 2$. By the similar argument as that of $h'''(c)$, we only need to show $h''(c) < 0$ for $c \in ((i-1)/10000, i/10000]$, $i = 5001, ..., 20000$. For $c \in (c', c'']$, where $1/2 < c' < c'' < 2$, we have $-\frac{3c\phi^2(c)}{\Phi^2(c)} \leq -\frac{3c'\phi^2(c'')}{\Phi^2(c'')}$; $-\frac{2\phi^3(c)}{\Phi^3(c)} \leq -\frac{2\phi^3(c'')}{\Phi^3(c'')}$; $-\frac{3c\phi^2(c)}{(1-\Phi(c))^2} \leq -\frac{3c'\phi^2(c'')}{(1-\Phi(c'))^2}$; and $\frac{2\phi^3(c)}{(1-\Phi(c))^3} \leq \frac{2\phi^3(c')}{(1-\Phi(c''))^3}$. When $1/2 < c' < c'' < 1$, we have $\frac{(1-c^2)\phi(c)}{\Phi(c)} \leq \frac{(1-c'^2)\phi(c')}{\Phi(c')}$ and $-\frac{(1-c^2)\phi(c)}{1-\Phi(c)} \leq -\frac{(1-c''^2)\phi(c'')}{1-\Phi(c')}$. When $1 \leq c' < c''$, we have $\frac{(1-c^2)\phi(c)}{\Phi(c)} \leq \frac{(1-c'^2)\phi(c'')}{\Phi(c'')}$ and $-\frac{(1-c^2)\phi(c)}{1-\Phi(c)} \leq -\frac{(1-c''^2)\phi(c')}{1-\Phi(c'')}$. By (5.20) and a simple computer program can verify that $h''(c) < 0$ for $c \in ((i-1)/10000, i/10000]$, $i = 5001, ..., 20000$. Thus we have the conclusion (ii).

By (5.19), we have $h'(0) = -2 + 4/\pi < 0$. Since $h''(c) < 0$ for $0 < c < 2$, this implies that $h'(c) < 0$ for $0 < c < 2$. We shall show that $h'(c) < 0$ for $c \geq 2$. By integration by parts, for any $c > 0$, we have

$$1 - \Phi(c) > (\frac{1}{c} - \frac{1}{c^3})\phi(c). \tag{5.22}$$

By (5.22) and $c \geq 2$, we have

$$\begin{aligned} \frac{\phi^2(c) - c\phi(c)\,(1-\Phi(c))}{(1-\Phi(c))^2} &\leq \frac{\frac{1}{c^2}\phi^2(c)}{(1-\Phi(c))^2} \\ &\leq \frac{1}{(1-\frac{1}{c^2})^2} \leq \frac{16}{9}. \end{aligned} \tag{5.23}$$

On the other hand, it can be verified that $c\phi(c)$ is a decreasing function of c for $c \geq 2$. Thus for $c \geq 2$, we have

$$\begin{aligned} \frac{c\phi(c)\Phi(c) + \phi^2(c)}{\Phi^2(c)} &= \frac{c\phi(c)}{\Phi(c)} + \left(\frac{\phi(c)}{\Phi(c)}\right)^2 \\ &\leq \frac{2\phi(2)}{\Phi(2)} + \left(\frac{\phi(2)}{\Phi(2)}\right)^2. \end{aligned} \tag{5.24}$$

By (5.19), (5.23), and (5.24), we can directly show that $h'(c) < 0$ when $c \geq 2$. Thus we have the conclusion that $h'(c) < 0$ for $c > 0$. Immediately we have $h(c) < 0$ for $c > 0$ since $h(c) = 0$ by (5.18). So we have the conclusion (i). □

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