Joint Analyticity of the Transformed Field and Dirichlet–Neumann Operator in Periodic Media

Matthew Kehoe

University of Illinois at Chicago

Thesis Defense

Advisor: Professor David Nicholls

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Goals

1. Develop a numerical algorithm to record scattered energy in a two–layer periodic structure.
2. Prove a theorem on the existence and uniqueness of solutions to a system of partial differential equations which model the interaction of linear waves in periodic layered media.

Overview

Maxwell's Equations

As a starting point we consider the time–harmonic Maxwell's equations of electromagnetism in a homogeneous region:

$$
\begin{aligned}
\nabla \times \mathbf{E} &= i\omega\mu_0\mathbf{H}, \\
\nabla \times \mathbf{H} &= -i\omega\epsilon_0\epsilon\mathbf{E}, \\
\nabla \cdot \mathbf{E} &= 0, \\
\nabla \cdot \mathbf{H} &= 0.
\end{aligned}
$$

- $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field.
- ϵ_0 and μ_0 represent the permittivity and permeability in vacuum.
- ϵ is the complex permittivity, ω is the frequency.

Two–Dimensional Simplifications

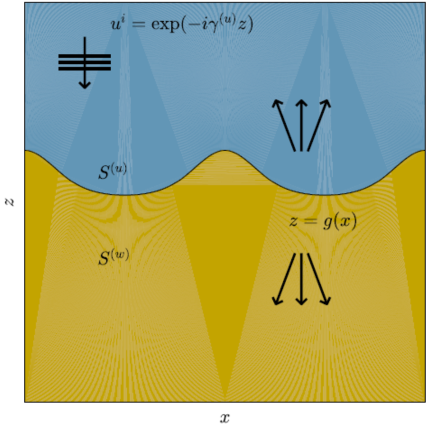
- We choose an interface shaped by $z = g(x, y)$ where the normal is defined by $\mathbf{N} := (-\partial_x g, -\partial_y g, 1)^T$.
- To obtain two-dimensional solutions, we assume the grating shape is invariant in the y–direction:

$$z = g(x),$$

which implies that the interfacial normal becomes

$$\mathbf{N} = \begin{pmatrix} -\partial_x g \\ 0 \\ 1 \end{pmatrix}.$$

The Geometry



A two-layer structure with a periodic interface, $z = g(x)$, separating two material layers, $S^{(u)}$ and $S^{(w)}$.

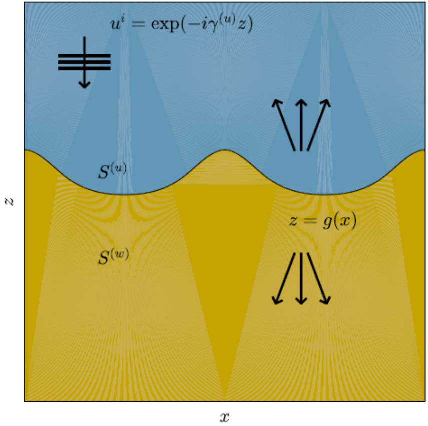
- We consider a y–invariant, doubly layered structure. The interface $z = g(x)$ is d–periodic so that $g(x + d) = g(x)$.
- A dielectric (with refractive index n^u) occupies the domain above the interface

$$S^{(u)} := \{z > g(x)\}.$$

- A material of refractive index n^w is in the lower layer

$$S^{(w)} := \{z < g(x)\}.$$

Incident Radiation



A two-layer structure with a periodic interface, $z = g(x)$, illuminated by plane–wave incidence.

- The structure is illuminated from above by monochromatic plane–wave incident radiation of frequency ω.
- We consider the reduced incident fields

$$\mathbf{E}^i(x,z) = e^{i\omega t}\underline{\mathbf{E}}^i(x,z,t),$$
$$\mathbf{H}^i(x,z) = e^{i\omega t}\underline{\mathbf{H}}^i(x,z,t),$$

  where the time dependence $\exp(-i\omega t)$ is removed.
- The scattered radiation is "outgoing," upward propagating in $S^{(u)}$ and downward propagating in $S^{(w)}$.

Governing Equations for Layered Media

- In this 2D setting the time-harmonic Maxwell equations decouple into two scalar Helmholtz problems: Transverse electric (TE) and transverse magnetic (TM) polarizations.
- We define the invariant (y) directions of the scattered (electric or magnetic) fields by $\{\tilde{u}, \tilde{w}\}$ in $S^{(u)}$ and $S^{(w)}$ and seek outgoing/bounded, periodic solutions of

$$\begin{aligned}
&\Delta \tilde{u} + (k^u)^2 = 0, && z > g(x), \\
&\Delta \tilde{w} + (k^w)^2 = 0, && z < g(x), \\
&\tilde{u} - \tilde{w} = -\tilde{u}^i, && z = g(x), \\
&\partial_N \tilde{u} - \tau^2 \partial_N \tilde{w} = -\partial_N \tilde{u}^i, && z = g(x).
\end{aligned}$$

- $g(x)$ is the grating interface, $\tilde{u}^i$ is the incident radiation.
- $\tau^2 = 1$ in TE, $\tau^2 = (k^u/k^w)^2$ in TM.
- For $q \in \{u, w\}$, $k^q = \omega/c^q$ is the wavenumber.

Governing Equations Without Phase

- We further factor out the phase $\exp(i\alpha x)$ from the fields $\tilde{u}$ and $\tilde{w}$

$$u(x,z) = e^{-i\alpha x}\tilde{u}(x,z), \quad w(x,z) = e^{-i\alpha x}\tilde{w}(x,z).$$

- With these, our governing equations consist of outgoing/bounded, periodic solutions of

$$\begin{aligned}
&\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, && z > g(x),\\
&\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, && z < g(x),\\
&u - w = \zeta, && z = g(x),\\
&\partial_N u - i\alpha(\partial_x g)u - \tau^2\left[\partial_N w - i\alpha(\partial_x g)w\right] = \psi, && z = g(x).
\end{aligned}$$

- $\alpha = k^u \sin(\theta)$, and for $q \in \{u, w\}$, $\gamma^q = k^q \cos(\theta)$.

Artificial Boundaries

- To truncate the bi–infinite problem domain to one of finite size we choose values a and b such that
$$a > |g|_\infty, \quad -b < -|g|_\infty,$$
and define the artificial boundaries $\{z = a\}$ and $\{z = -b\}$.
- In $\{z > a\}$ the Rayleigh expansions tell us that upward propagating solutions of the Helmholtz equation are
$$u(x,z) = \sum_{p=-\infty}^{\infty} \hat{a}_p e^{i\tilde{p}x + i\gamma_p^u z}.$$
- With this we can define the Transparent Boundary Conditions in the following way: we rewrite the solution in the upper layer as
$$u(x,z) = \sum_{p=-\infty}^{\infty} \left(\hat{a}_p e^{i\gamma_p^u a}\right) e^{i\tilde{p}x + i\gamma_p^u (z-a)} = \sum_{p=-\infty}^{\infty} \hat{\xi}_p e^{i\tilde{p}x + i\gamma_p^u (z-a)}.$$

Transparent Boundary Conditions

- We then observe that

$$\partial_z u(x,a) = \sum_{p=-\infty}^{\infty} (i\gamma_p^u)\hat{\xi}_p e^{i\tilde{p}x} =: T^u[\xi(x)],$$

which defines the order–one Fourier multiplier T^u.

- A similar procedure in the lower layer shows that we can write

$$\partial_z w(x,-b) = \sum_{p=-\infty}^{\infty} (-i\gamma_p^w)\hat{\psi}_p e^{i\tilde{p}x} =: T^w[\psi(x)],$$

for the order–one Fourier multiplier T^w.

Upward and Downward Propagating Solutions

- From these we state that upward–propagating solutions of the upper layer satisfy the Transparent Boundary Condition at $z = a$

$$\partial_z u(x,a) - T^u[u(x,a)] = 0, \quad z = a.$$

- Similarly, downward–propagating solutions in the lower layer satisfy the Transparent Boundary Condition at $z = -b$

$$\partial_z w(x,-b) - T^w[w(x,-b)] = 0, \quad z = -b.$$

Full Governing Equations

With these we now state the full set of governing equations as

$$
\begin{aligned}
&\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, && z > g(x),\\
&\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, && z < g(x),\\
&u - w = \zeta, && z = g(x),\\
&\partial_N u - i\alpha(\partial_x g)u - \tau^2\left[\partial_N w - i\alpha(\partial_x g)w\right] = \psi, && z = g(x),\\
&\partial_z u(x,a) - T^u[u(x,a)] = 0, && z = a,\\
&\partial_z w(x,-b) - T^w[w(x,-b)] = 0, && z = -b,\\
&u(x+d,z) = u(x,z),\\
&w(x+d,z) = w(x,z).
\end{aligned}
$$

Domain Decomposition Method

- We now write our governing equations in terms of surface quantities. For this we define the Dirichlet traces and their (outward) Neumann counterparts

$$U(x) := u(x, g(x)), \quad \tilde{U}(x) := -\partial_N u(x, g(x)),$$
$$W(x) := w(x, g(x)), \quad \tilde{W}(x) := \partial_N w(x, g(x)),$$

- In terms of these our full governing equations are equivalent to the pair of boundary conditions,

$$U - W = \zeta,$$
$$-\tilde{U} - (i\alpha)(\partial_x g)U - \tau^2 \left[\tilde{W} - (i\alpha)(\partial_x g)W\right] = \psi.$$

- The set of two equations and four unknowns can be closed by noting that the pairs $\{U, \tilde{U}\}$ and $\{W, \tilde{W}\}$ are connected, e.g., by DNOs

$$G : U \to \tilde{U}, \quad J : W \to \tilde{W}.$$

Interfacial Reformulation

The interfacial reformulation of our governing equations becomes

$$\mathbf{AV} = \mathbf{R},$$

where

$$\mathbf{A} = \begin{pmatrix} I & -I \\ G + (\partial_x g)(i\alpha) & \tau^2 J - \tau^2 (\partial_x g)(i\alpha) \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix}.$$

Numerical Methods

- A variety of numerical algorithms have been devised for the simulation of these problems including Finite Difference, Finite Element, and Spectral Element methods.
- These methods suffer from the requirement that they discretize the full volume of the problem domain.
- We advocate the use of surface methods, especially the High–Order Perturbation of Surfaces (HOPS) methods:
  - provide the solution at the interface.
  - only discretize the layer interfaces.
  - are highly accurate, rapid, and robust.
- The HOPS methods are based on the foundational contributions of
  - Field Expansion (FE) method: Bruno & Reitich (1993).
  - Transformed Field Expansion (TFE) method: Nicholls & Reitich (1999).

Boundary and Frequency Perturbations

- We take a perturbative approach which makes two smallness assumptions:
  1. Boundary Perturbation: $g(x) = \varepsilon f(x)$, $\varepsilon \in \mathbf{R}$, $\varepsilon \ll 1$,
  2. Frequency Perturbation: $\omega = (1+\delta)\underline{\omega}$, $\delta \in \mathbf{R}$, $\delta \ll 1$.
- The second of these assumptions has the following important consequences

$$k^q = (1+\delta)\underline{k}^q, \quad \alpha = (1+\delta)\underline{\alpha}, \quad \gamma^q = (1+\delta)\underline{\gamma}^q,$$

for $q \in \{u, w\}$.

Transformed Field Expansions Method

- The method of Transformed Field Expansions (TFE) proceeds a domain–flattening change of variables prior to perturbation expansion.
- Focusing on the upper layer, the change of variable is

$$x' = x, \quad z' = a\left(\frac{z - g(x)}{a - g(x)}\right),$$

  which maps the perturbed domain $\{g(x) < z < a\}$ to the separable domain $\{0 < z' < a\}$.
- A similar transformation occurs in the lower layer where the perturbed domain $\{-b < z < g(x)\}$ becomes $\{-b < z' < 0\}$.

Perturbation Expansions

- Provided f is sufficiently smooth, we will later show we will show the joint analytic dependence of $\mathbf{A} = \mathbf{A}(\varepsilon, \delta)$ and $\mathbf{R} = \mathbf{R}(\varepsilon, \delta)$ upon ε and δ, will induce a jointly analytic solution, $\mathbf{V} = \mathbf{V}(\varepsilon, \delta)$.
- In this case we may expand

$$\{\mathbf{A}, \mathbf{V}, \mathbf{R}\}(\varepsilon, \delta) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \{\mathbf{A}_{n,m}, \mathbf{V}_{n,m}, \mathbf{R}_{n,m}\} \varepsilon^n \delta^m,$$

and a calculation reveals that at every perturbation order (n, m), we can find the $\mathbf{V}_{n,m}$ by solving

$$\mathbf{A}_{0,0} \mathbf{V}_{n,m} = \mathbf{R}_{n,m} - \sum_{\ell=0}^{n-1} \mathbf{A}_{n-\ell,0} \mathbf{V}_{\ell,m} - \sum_{r=0}^{m-1} \mathbf{A}_{0,m-r} \mathbf{V}_{n,r} - \sum_{\ell=0}^{n-1} \sum_{r=0}^{m-1} \mathbf{A}_{n-\ell,m-r} \mathbf{V}_{\ell,r}.$$

Order (n, m)

- A brief inspection of the formulas for **A** and **R**, reveals that

$$\mathbf{A}_{0,0} = \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix},$$

$$\mathbf{A}_{n,m} = \begin{pmatrix} 0 & 0 \\ G_{n,m} & \tau^2 J_{n,m} \end{pmatrix} + \delta_{n,1} \left\{ 1 + \delta_{m,1} \right\} (\partial_x f)(i\underline{\alpha}) \begin{pmatrix} 0 & 0 \\ 1 & -\tau^2 \end{pmatrix}, \quad n \neq 0 \text{ or } m \neq 0,$$

$$\mathbf{R}_{n,m} = \begin{pmatrix} \zeta_{n,m} \\ -\psi_{n,m} \end{pmatrix}.$$

- $\delta_{n,m}$ is the Kronecker delta function and the forms for $\zeta_{n,m}$ and $\psi_{n,m}$ are known.

Numerical Approximation

- In our approximation we begin by truncating the Taylor series

$$\{\mathbf{A},\mathbf{V},\mathbf{R}\}(\varepsilon,\delta) \approx \{\mathbf{A}^{N,M},\mathbf{V}^{N,M},\mathbf{R}^{N,M}\}(\varepsilon,\delta) := \sum_{n=0}^{N}\sum_{m=0}^{M}\{\mathbf{A}_{n,m},\mathbf{V}_{n,m},\mathbf{R}_{n,m}\}\varepsilon^n\delta^m,$$

  where we must specify (i.) how the forms $\mathbf{A}_{n,m}$ are simulated, and (ii.) how the operator $\mathbf{A}_{0,0}$ is to be inverted.

- Regarding the forms $\mathbf{A}_{n,m}$, these boil down to the (n,m)–th corrections of the DNOs G and J, respectively, in a Taylor series expansion of each jointly in ε and δ. We will simulate these numerically.
- The inversion of $\mathbf{A}_{0,0}$ will follow from the proof of existence and uniqueness.

A Fourier/Chebyshev Collocation Discretization

- To show how we simulate $\mathbf{A}_{n,m}$, we will focus on the upper layer DNO, G. We begin by approximating

$$u(x,z;\varepsilon,\delta) \approx u^{N,M}(x,z;\varepsilon,\delta) := \sum_{n=0}^{N} \sum_{m=0}^{M} u_{n,m}(x,z) \varepsilon^n \delta^m.$$

- Each of these $u_{n,m}(x,z)$ are then simulated by a Fourier–Chebyshev approach which posits the form

$$u_{n,m}(x,z) \approx u_{n,m}^{N_x,N_z}(x,z) := \sum_{p=-N_x/2}^{N_x/2-1} \sum_{\ell=0}^{N_z} \hat{u}_{n,m,p,\ell} e^{i\tilde{p}x} T_\ell \left(\frac{2z-a}{a} \right),$$

where T_ℓ is the ℓ–th Cheybshev polynomial. The unknowns $\hat{u}_{n,m,p,\ell}$ are recovered by the collocation approach.

Equispaced Grid Points / Collocation Points

- As mentioned previously, the Fourier–Chebyshev approach posits the form

$$u_{n,m}(x,z) \approx u_{n,m}^{N_x,N_z}(x,z) := \sum_{p=-N_x/2}^{N_x/2-1} \sum_{\ell=0}^{N_z} \hat{u}_{n,m,p,\ell} e^{i\tilde{p}x} T_\ell\left(\frac{2z-a}{a}\right).$$

- More specifically, our HOPS/AWE algorithm requires $N_x \times N_z$ unknowns at every perturbation order, (n,m).
- As our problem is d–periodic in the lateral direction, we will expand using a Fourier spectral method where we require N_x equally–spaced gridpoints.
- However, our problem is not z–periodic, so our strategy is to use a Chebyshev spectral method in the vertical direction. For this, we select $N_z + 1$ collocation points.

Simulation of DNOs

- With this we can simulate the upper layer DNO through

$$G(x;\varepsilon,\delta) \approx G^{N,M}(x;\varepsilon,\delta) := \sum_{n=0}^{N}\sum_{m=0}^{M} G_{n,m}(x)\varepsilon^n\delta^m.$$

- Here

$$G_{n,m}(x) \approx G_{n,m}^{N_x}(x) := \sum_{p=-N_x/2}^{N_x/2-1} \hat{G}_{n,m,p} e^{i\tilde{p}x},$$

  and the $\hat{G}_{n,m,p}$ are recovered from the $\hat{u}_{n,m,p,\ell}$.
- We apply the same procedure to the lower layer DNO, J.

The Rayleigh Expansions

- Previously, we observed that solutions to the Helmholtz problem in the upper layer can be expressed in terms of Rayleigh expansions

$$u(x,z) = \sum_{p=-\infty}^{\infty} \hat{a}_p e^{i\tilde{p}x + i\gamma_p^u z}.$$

- For $p \in \mathbf{Z}$ we define

$$\tilde{p} := \frac{2\pi p}{d}, \quad \alpha_p := \alpha + \tilde{p}, \quad \gamma_p^u := \begin{cases} \sqrt{(k^u)^2 - \alpha_p^2}, & p \in \mathcal{U}^u, \\ i\sqrt{\alpha_p^2 - (k^u)^2}, & p \notin \mathcal{U}^u. \end{cases}$$

Propagating Modes

- We have

$$\gamma_p^u := \begin{cases} \sqrt{(k^u)^2 - \alpha_p^2}, & p \in \mathcal{U}^u, \\ i\sqrt{\alpha_p^2 - (k^u)^2}, & p \notin \mathcal{U}^u, \end{cases} \quad \mathcal{U}^u := \left\{ p \in \mathbf{Z} \mid \alpha_p^2 < (k^u)^2 \right\}.$$

- Components of $u(x, z)$ corresponding to $p \in \mathcal{U}^u$ propagate away from the layer interface, while those not in this set decay exponentially from $z = g(x)$.
- The latter are called evanescent waves while the former are propagating (defining the set of propagating modes $\mathcal{U}^u$) and carry energy away from the grating.

The Reflectivity Map

- With this in mind one defines the efficiencies

$$e_p^u := (\gamma_p^u/\gamma^u)\,|\hat{a}_p|^2, \quad p \in \mathcal{U}^u,$$

- and the Reflectivity Map as the sum of efficiencies in the upper layer

$$R := \sum_{p \in \mathcal{U}^u} e_p^u.$$

- Similar quantities can be defined in the lower layer, and with these the principle of conservation of energy can be stated for structures composed entirely of dielectrics

$$\sum_{p \in \mathcal{U}^u} e_p^u + \tau^2 \sum_{p \in \mathcal{U}^w} e_p^w = 1.$$

Energy Defect

- In this situation a useful diagnostic of convergence for a numerical scheme is the "Energy Defect"

$$D := 1 - \sum_{p \in \mathcal{U}^u} e_p^u - \tau^2 \sum_{p \in \mathcal{U}^w} e_p^w,$$

which should be zero for a purely dielectric structure.

Rayleigh Singularities (Wood's Anomalies)

- The Taylor series expansion for γ_p^q, $q \in \{u, w\}$, is

$$\gamma_p^q = \gamma_p^q(\delta) = \sum_{m=0}^{\infty} \gamma_{p,m}^q \delta^m.$$

- Recalling $\gamma_p^q = (1+\delta)\underline{\gamma}_p^q$, $k^q = (1+\delta)\underline{k}^q$ one finds

$$\underline{\alpha}_p^2 + (\underline{\gamma}_p^q)^2 = (\underline{k}^q)^2.$$

- When $\underline{\gamma}_p^q = 0$, the Taylor series expansion of $\gamma_p^q(\delta)$ is invalid. A Rayleigh singularity (or Wood's anamoly) occurs when $\underline{\alpha}_p^2 = (\underline{k}^q)^2$.
- Therefore, the permissible values of δ are constrained by this.

The Domain of Analyticity

- To guide our computations we explore this restriction on δ.
- In the upper layer, Rayleigh singularities occur when $\underline{\alpha}_p^2 = (\underline{k}^u)^2$ which implies

$$\underline{\omega} = \pm \frac{c_0}{n^u} \left\{ \underline{\alpha} + \frac{2\pi p}{d} \right\}, \quad \text{for any } p \in \mathbf{Z}.$$

- In the interest of maximizing our choice of δ we select a "mid–point" value of $\underline{\omega}$ which is as far away as possible from consecutive Rayleigh singularities

$$\underline{\omega}_q := \frac{c_0}{n^u} \left\{ \underline{\alpha} + \frac{2\pi(q + 1/2)}{d} \right\}.$$

- Our algorithm will expand in δ at the "mid–points" away from Rayleigh singularities.

Simulation: Reflectivity Map for Vacuum over Dielectric

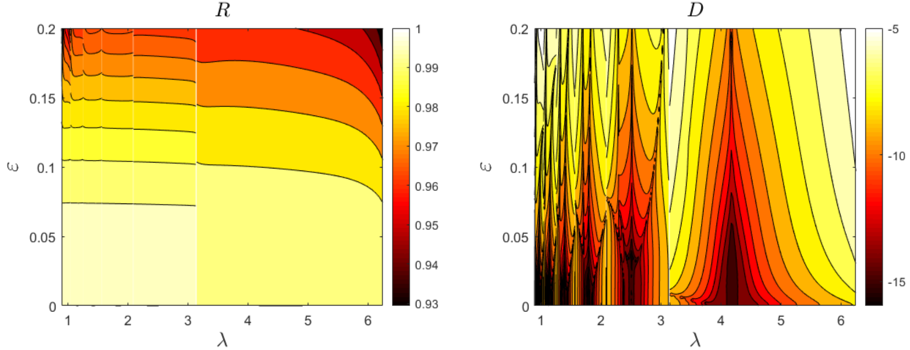


Figure 1: The Reflectivity Map, $R(\varepsilon, \delta)$, and energy defect D computed with our HOPS/AWE algorithm with Taylor summation. We set $N = M = 16$ and the parameter choices were $\alpha = 0$, $n^u = 1$, and $n^w = 1.1$.

Simulation: Reflectivity Map for Vacuum over Silver and Gold

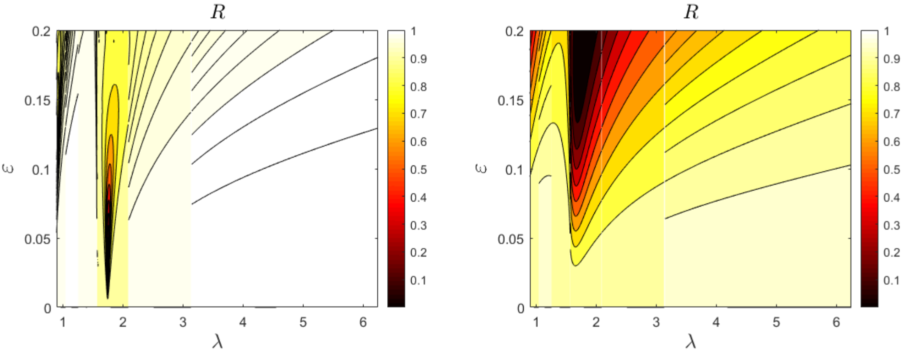


Figure 2: The Reflectivity Map, $R(\varepsilon, \delta)$, for silver (left) and gold (right) with Padé summation. We set $N = M = 15$ and parameter choices were $\alpha = 0$, $n^u = 1$, $n^w = 0.05 + 2.275i$ (left) and $n^w = 1.48 + 1.883i$ (right).

- The interfacial reformulation of our governing equations is $\mathbf{A}\mathbf{V} = \mathbf{R}$ and the formulas for $\mathbf{A}$ and $\mathbf{R}$ at order (n, m) are

$$
\begin{aligned}
\mathbf{A}_{0,0} &= \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix}, \\
\mathbf{A}_{n,m} &= \begin{pmatrix} 0 & 0 \\ G_{n,m} & \tau^2 J_{n,m} \end{pmatrix} \\
&\qquad + \delta_{n,1} \left\{ 1 + \delta_{m,1} \right\} (\partial_x f)(i\underline{\alpha}) \begin{pmatrix} 0 & 0 \\ 1 & -\tau^2 \end{pmatrix}, \quad n \neq 0 \text{ or } m \neq 0, \\
\mathbf{R}_{n,m} &= \begin{pmatrix} \zeta_{n,m} \\ -\psi_{n,m} \end{pmatrix}.
\end{aligned}
$$

- We will now establish the existence, uniqueness, and analyticity of solutions to $\mathbf{A}\mathbf{V} = \mathbf{R}$.
- To accomplish this we will show the joint analytic dependence of $\mathbf{A} = \mathbf{A}(\varepsilon, \delta)$ and $\mathbf{R} = \mathbf{R}(\varepsilon, \delta)$ upon ε and δ, will induce a jointly analytic solution, $\mathbf{V} = \mathbf{V}(\varepsilon, \delta)$.

Theorem: Analyticity of Solutions [Kehoe,Nicholls 22]

Theorem

Given two Banach spaces X and Y, suppose that

H1 $\mathbf{R}_{n,m} \in Y$ *for all* $n, m \geq 0$*, and there exists constants* $B_R > 0, C_{R,N} > 0, C_{R,M} > 0, D_R > 0$ *such that*

$$\|\mathbf{R}_{n,m}\|_Y \leq C_{R,N} C_{R,M} B_R^n D_R^m,$$

H2 $\mathbf{A}_{n,m} : X \to Y$ *for all* $n, m \geq 0$*, and there exists constants* $B_A > 0, C_{A,N} > 0, C_{A,M} > 0, D_A > 0$ *such that*

$$\|\mathbf{A}_{n,m}\|_{X \to Y} \leq C_{A,N} C_{A,M} B_A^n D_A^m,$$

H3 $\mathbf{A}_{0,0}^{-1} : Y \to X$ *for all* $n, m \geq 0$*, and there exists a constant* $C_e > 0$ *such that*

$$\|\mathbf{A}_{0,0}^{-1}\|_{Y \to X} \leq C_e.$$

Theorem: Analyticity of Solutions (Continued)

Theorem (continued)

Then, given an integer $s \geq 0$*, if* $f \in C^{s+2}([0,d])$ *then the linear system* $\mathbf{AV} = \mathbf{R}$ *has a unique solution,* $\sum_{n,m} \mathbf{V}_{n,m}\varepsilon^n\delta^m$*, and there exist constants* $B, C, D > 0$ *such that*

$$\|\mathbf{V}_{n,m}\|_{X^s} \leq CB^nD^m,$$

for all $n, m \geq 0$*. This implies that for any* $0 \leq \rho, \sigma < 1$*,* $\sum_{n,m} \mathbf{V}_{n,m}\varepsilon^n\delta^m$ *converges for all* ε *such that* $B\varepsilon < \rho$*, i.e.,* $\varepsilon < \rho/B$ *and all* δ *such that* $D\delta < \sigma$*, i.e.,* $\delta < \sigma/D$*.*

Sketch of Proof

- First, we define the vector–valued spaces for $s \geq 0$

$$X^s := \left\{ \mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix} \middle| \, U, W \in H^{s+3/2}([0,d]) \right\},$$

$$Y^s := \left\{ \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \middle| \, \zeta \in H^{s+3/2}([0,d]), \psi \in H^{s+1/2}([0,d]) \right\}.$$

- These have the norms

$$\|\mathbf{V}\|_{X^s}^2 = \left\| \begin{pmatrix} U \\ W \end{pmatrix} \right\|_{X^s}^2 := \|U\|_{H^{s+3/2}}^2 + \|W\|_{H^{s+3/2}}^2 ,$$

$$\|\mathbf{R}\|_{Y^s}^2 = \left\| \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \right\|_{Y^s}^2 := \|\zeta\|_{H^{s+3/2}}^2 + \|\psi\|_{H^{s+1/2}}^2 .$$

Sketch of Proof (Continued)

- **Hypothesis H1**: Consider the Banach spaces $X = X^s$ and $Y = Y^s$. Our first task is to show that
$$\mathbf{R}_{n,m} = \begin{pmatrix} \zeta_{n,m} \\ -\psi_{n,m} \end{pmatrix},$$
is bounded in Y^s for any $s \geq 0$.
- Upon performing the boundary/frequency perturbations, we define
$$\mathcal{E}(x; \varepsilon, \delta) := e^{-i(1+\delta)\underline{\gamma}^u \varepsilon f(x)},$$
so that
$$\zeta(x) = \zeta(x; \varepsilon, \delta) = -\mathcal{E}(x; \varepsilon, \delta),$$
$$\psi(x) = \psi(x; \varepsilon, \delta) = \left\{ i(1+\delta)\underline{\gamma}^u + i(1+\delta)\underline{\alpha}(\varepsilon \partial_x f) \right\} \mathcal{E}(x; \varepsilon, \delta).$$
- A joint Taylor expansion followed by an induction argument shows that $\|\zeta_{n,m}\|_{H^{s+3/2}}$ and $\|\psi_{n,m}\|_{H^{s+1/2}}$ are bounded. Therefore, $\|\mathbf{R}_{n,m}\|_{Y^s}$ is bounded.

Sketch of Proof (Continued)

- **Hypothesis H2**: Our next task is to show that the operators $G_{n,m}$ and $J_{n,m}$ in
$$\mathbf{A}'_{n,m} = \begin{pmatrix} 0 & 0 \\ G_{n,m} & \tau^2 J_{n,m} \end{pmatrix},$$
for the Taylor series expansions of the DNOs satisfy the appropriate bounds.
- For brevity, we will outline our technique for the Taylor expansion of the upper layer DNO, $G_{n,m}$.
- **Lemma (Algebra Property)**: Given an integer $s \geq 0$, there exists a constant $\mathcal{M} = \mathcal{M}(s)$ such that if $f \in C^s([0,d])$ and $u \in H^s([0,d] \times [0,a])$ then
$$\|fu\|_{H^s} \leq \mathcal{M} \, |f|_{C^s} \, \|u\|_{H^s} .$$

Sketch of Proof (Continued)

- The bound on $G_{n,m}$ follows from
  1. Applying the boundary and frequency perturbations followed by the TFE method results in the upper layer DNO problem

$$\begin{aligned} &\Delta u_{n,m} + 2i\underline{\alpha}\partial_x u_{n,m} + (\underline{\gamma}^u)^2 u_{n,m} = F_{n,m}(x,z), && 0 < z < a, \\ &u_{n,m}(x,0) = U_{n,m}(x), && z = 0, \\ &\partial_z u_{n,m}(x,a) - T^u[u_{n,m}(x,a)] = P_{n,m}(x), && z = a, \end{aligned}$$

     where

$$G_{n,m}(f) = -\partial_z u_{n,m}(x,0) + H_{n,m}(x).$$

  2. The **Algebra Property** establishes bounds on the non–homogeneous terms $F_{n,m}$, $P_{n,m}$, and $H_{n,m}$.
  3. With these, the **Elliptic Estimate** and an induction argument establishes

$$\left\| u_{n,m} \right\|_{H^{s+2}} \leq K B^n D^m,$$

     for constants $K, B, D > 0$. This shows that the transformed upper field is jointly analytic with respect a boundary/frequency perturbation.

Sketch of Proof (Continued)

- The bound on $G_{n,m}$ follows from (continued)
  4. The bound on the upper layer DNO

$$G_{n,m}(f) = -\partial_z u_{n,m}(x,0) + H_{n,m}(x),$$

  then follows from the joint analyticity of the transformed upper field, $u_{n,m}$, an induction argument, and the fact that $H_{n,m}$ is bounded.
  5. One finds

$$\|G_{n,m}\|_{H^{s+1/2}} \leq \tilde{K}\tilde{B}^n\tilde{D}^m,$$

  for constants $\tilde{K}, \tilde{B}, \tilde{D} > 0$ which shows that $G_{n,m}$ is bounded. A similar argument works for the lower layer DNO, $J_{n,m}$, so that $\mathbf{A}_{n,m}$ is bounded and **H2** is satisfied.

Sketch of Proof (Continued)

- **Hypothesis H3**: Our final task is show that $\mathbf{A}_{0,0}^{-1}$ exists and the estimates and mapping properties of $\mathbf{A}_{0,0}^{-1}$ hold where $\mathbf{A}_{0,0}$ is defined by

$$\mathbf{A}_{0,0} = \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix}.$$

- We define the operator

$$\Delta := G_{0,0} + \tau^2 J_{0,0} = (-i\gamma_D^u) + \tau^2(-i\gamma_D^w),$$

so that Δ^{-1} exists and that

$$\Delta : H^{s+3/2} \to H^{s+1/2}, \quad \Delta^{-1} : H^{s+1/2} \to H^{s+3/2}.$$

Sketch of Proof (Continued)

- Next, we write generic elements of X^s and Y^s as

$$\mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix} \in X^s, \quad \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \in Y^s.$$

- Using the definitions of the norms of X^s and Y^s we find

$$\|\mathbf{A}_{0,0}\mathbf{V}\|_{Y^s}^2 \leq C \|\mathbf{V}\|_{X^s}^2 ,$$

  so that $\mathbf{A}_{0,0}$ maps X^s to Y^s. Furthermore,

$$\left\|\mathbf{A}_{0,0}^{-1}\mathbf{R}\right\|_{X^s}^2 \leq C_{\Delta^{-1}} \|\mathbf{R}\|_{Y^s}^2 ,$$

  which shows that $\mathbf{A}_{0,0}^{-1}$ maps Y^s to X^s.
- Thus, $\|\mathbf{A}_{0,0}^{-1}\|_{Y^s \to X^s}$ is bounded and the mapping properties hold.

Conclusion

We seek outgoing/bounded, periodic solutions of the scattering problem

$$\begin{aligned}
&\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, && z > g(x),\\
&\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, && z < g(x),\\
&u - w = \zeta, && z = g(x),\\
&\partial_N u - i\alpha(\partial_x g)u - \tau^2 \left[\partial_N w - i\alpha(\partial_x g)w\right] = \psi, && z = g(x).
\end{aligned}$$

1. Numerical Algorithm
   - DNOs, boundary/frequency perturbations, and COV through TFE
   - Joint Taylor expansion followed by Fourier/Chebyshev collocation
   - Simulated scattered energy through Reflectivity map
2. Joint Analyticity of Solutions
   - Reformulate governing equations in terms of a linear system
   - Sobolev space theory: Algebra Property and Elliptic Estimate

Future Work

1. Extend HOPS/AWE algorithm to multilayered surfaces with different material layers. Introduce a new DNO to handle the intermediate layers.
2. Implement parallel programming techniques to handle the computation of the intermediate layers.
3. Introduce multiple small perturbation parameters outside of an interfacial perturbation and a frequency perturbation. Extend the proof of analyticity to handle any finite number of perturbation parameters.
4. Develop techniques to expand around Rayleigh singularities where the Taylor series expansion is invalid.

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