Math 215 Spring 2012, Test 1 Solutions

1) Prove or disprove the following statements concerning integers a, b, c
(i) (a divides b) and (a divides c) $\Rightarrow$ a divides $(3b - 2c)$
(ii) (a does not divide b) and (a divides c) $\Rightarrow$ a divides (bc)
Solution: (i) If $a|b$ and $a|c$, then there are integers r, s, such that $b = ar$ and $c = as$. Then $3b - 2c = 3ar - 2as = a(3r - 2s)$. Since $3r - 2s$ is an integer, $a|3b - 2c$.
(ii) Since $a|c$, Then there is an integer r such that $c = ar$. Then $bc = bar = a(br)$. Since br is an integer, $a|bc$.

2) Write down the truth tables for the following:
a) P and (not Q)
b) (not P) $\Rightarrow$ (Q or P)

3) Consider the following statement:
If I get a perfect score on this test, then Professor Mubayi is a great teacher.
Decide which of the following is logically equivalent to the statement above.
a) If Professor Mubayi is a great teacher, then I will get a perfect score on this test.
b) If Professor Mubayi is not a great teacher, then I will get a perfect score on this test.
c) If Professor Mubayi is not a great teacher, then I won't get a perfect score on this test.
d) Professor Mubayi being a great teacher is sufficient for my getting a perfect score on this test.
e) Professor Mubayi being a great teacher is necessary for my getting a perfect score on this test.

Solution: a) is the converse which is not equivalent. b) is also not equivalent. c) is the contrapositive so it is equivalent. d) is also the converse so it is not equivalent. e) is equivalent. So only c) and e) are equivalent.

4) Prove by induction on n that for every positive integer n,

$$\sum_{i=0}^{n} \frac{2}{3^i} = 3 - \frac{1}{3^n}.$$

Solution: The base case is $2+2/3 = 3-1/3$ which is true. For the induction step, suppose that $\sum_{i=0}^{k} \frac{2}{3^i} = 3 - \frac{1}{3^k}$. Then $\sum_{i=0}^{k+1} \frac{2}{3^i} = \sum_{i=0}^{k} \frac{2}{3^i} + \frac{2}{3^{k+1}} = 3 - \frac{1}{3^k} + \frac{2}{3^{k+1}} = 3 - \frac{1}{3^{k+1}}$.

5) Prove or disprove each of the following statements
(i) $\exists n \in \mathbf{Z}, \forall m \in \mathbf{Z}, m \leq n$.
(ii) $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}, \forall z \in \mathbf{R}, x + y + z = 0$.

Solution: (i) is false since for every $n \in \mathbf{Z}$, we can let $m = n + 1$. Then $m > n$. (ii) is false since we may let $x = 0$ and show that the ensuing claim is false. In other words, we must show that $\forall y \in \mathbf{R}, \exists z \in \mathbf{R}, x + y + z \neq 0$. So let us pick $y \in \mathbf{R}$. Now let $z = 1 - y$. Then $x + y + z = 0 + y + z = 1 \neq 0$ as required.