Some Solutions to HW 8

5.3.4 The characteristic polynomial is \( x^2 - 7x + 10 = (x - 5)(x - 2). \)
Therefore the solution is \( a_n = c_1 5^n + c_2 2^n. \) Solve
\[
5c_1 + 2c_2 = 10 \\
25c_1 + 4c_2 = 29
\]
to get \( c_1 = 3/5, \ c_2 = 7/2, \) so \( a_n = 3 \cdot 5^{n-1} + 7 \cdot 2^{n-1}. \)

5.3.14

a) The characteristic polynomial is \( x^2 - 5x + 6 = (x - 2)(x - 3). \)
Therefore the solution is \( a_n = c_1 2^n + c_2 3^n. \) Solve
\[
c_1 + c_2 = 2 \\
2c_1 + 3c_2 = 11
\]
to get \( c_1 = -5, \ c_2 = 7, \) so \( a_n = -5 \cdot 2^n + 7 \cdot 3^n. \)

b) (We use the solution to Problem 21 for guidance). To get a particular
solution \( a + bn \) let
\[
a + bn = 5(a + b(n - 1)) - 6(a + b(n - 2)) + 3n
\]
so
\[
a = -a + 7b \quad \text{constant coefficient} \\
b = -b + 3 \quad \text{coefficient of } n
\]
which has solution \( a = 21/4, \ b = 3/2 \) (we are ignoring the initial condi-
tions), so the general solution is
\[
a_n = c_1 2^n + c_2 3^n + \frac{21}{4} + \frac{3}{2} n.
\]
The equations \( a_0 = 2, \ a_1 = 14 \) give
\[
c_1 + c_2 = -\frac{13}{4} \\
2c_1 + 3c_2 = \frac{29}{4}
\]
so \( c_1 = -17, \ c_2 = \frac{55}{4} \)

c) \( n = 0: \ -17 \cdot 2^0 + \frac{55}{4} 3^0 + \frac{21}{4} + \frac{3}{2} 0 = 2 \)
\( n = 1: \ -17 \cdot 2^1 + \frac{55}{4} 3^1 + \frac{21}{4} + \frac{3}{2} 1 = 14 \)
\( n = 2: \ -17 \cdot 2^2 + \frac{55}{4} 3^2 + \frac{21}{4} + \frac{3}{2} 2 = 64 = 5 \cdot 14 - 6 \cdot 2 + 3 \cdot 2 \)
\( n = 3: \ -17 \cdot 2^3 + \frac{55}{4} 3^3 + \frac{21}{4} + \frac{3}{2} 3 = 245 = 5 \cdot 64 - 6 \cdot 14 + 3 \cdot 3 \)
\( n = 4: \ -17 \cdot 2^4 + \frac{55}{4} 3^4 + \frac{21}{4} + \frac{3}{2} 4 = 853 = 5 \cdot 245 - 6 \cdot 64 + 3 \cdot 4 \)
\( n = 5: \quad -17 \cdot 2^5 + \frac{55}{4} 3^5 + \frac{21}{4} + \frac{3}{2} 25 = 2810 = 5 \cdot 853 - 6 \cdot 245 + 3 \cdot 5 \\
\( n = 6: \quad -17 \cdot 2^6 + \frac{55}{4} 3^6 + \frac{21}{4} + \frac{3}{2} 26 = 8950 = 5 \cdot 2810 - 6 \cdot 853 + 3 \cdot 6 \)

A more correct (and less boring) way to check this would be with mathematical induction, to show that the sequence has as solution

\[ a_n = -17 \cdot 2^n + \frac{55}{4} 3^n + \frac{21}{4} + \frac{3}{2} n. \]

We’ve computed above that

\[ a_0 = -17 \cdot 2^0 + \frac{55}{4} 3^0 + \frac{21}{4} + \frac{3}{2} 0, \]
\[ a_1 = -17 \cdot 2^1 + \frac{55}{4} 3^1 + \frac{21}{4} + \frac{3}{2} 1. \]

Suppose that \( n \geq 2 \) and that

\[ a_k = -17 \cdot 2^k + \frac{55}{4} 3^k + \frac{21}{4} + \frac{3}{2} k \]

for all \( k < n \). Then

\[ a_n = 5a_{n-1} - 6a_{n-2} + 3n \]

\[ = 5(-17)2^{n-1} + \frac{55}{4} 3^{n-1} + \frac{21}{4} + \frac{3}{2} (n - 1) \]

\[ - 6 \cdot 5(-17)2^{n-2} - \frac{55}{4} 3^{n-2} - \frac{21}{4} - \frac{3}{2} (n - 2) + 3n \]

\[ = -17 \cdot 2^n + \frac{55}{4} 3^n + \frac{21}{4} + \frac{3}{2} n. \]

5.4.4 Let \( G(x) \) be the generating function. Since \( a_n = 3a_{n-1} + 1, \ n \geq 1, \) and \( a_0 = 1, \) we have that \( G(x) = 3xG(x) + (1 + x + \cdots) \) or \( G(x)(1 - 3x) = 1 + x + \cdots \) or

\[ G(x) = \frac{1}{(1 - 3x)(1 - x)}. \]

Using the method of partial fractions to solve

\[ \frac{1}{(1 - 3x)(1 - x)} = \frac{\alpha}{1 - 3x} + \frac{\beta}{1 - x} \]

we get \( \alpha(1 - x) + \beta(1 - 3x) = 1 \) or

\[ \alpha + \beta = 1 \]
\[ -\alpha - 3\beta = 0 \]

so that \( \beta = -\frac{1}{2}, \ \alpha = \frac{3}{2}. \) Therefore

\[ G(x) = -\frac{1}{2} \frac{1}{1 - x} + \frac{3}{2} \frac{1}{1 - 3x} \]

\[ = 1 + \frac{3^2 - 1}{2} x + \frac{3^3 - 1}{2} x^2 + \cdots \]
that is,

\[ a_n = \frac{1}{2} (3^{n+1} - 1) \]

How does this compare with Problem 10 on p. 164? It’s basically the same answer except that in that problem the numbering starts at 1 rather than at 0, so the exponents are shifted by 1.