1) Find gcd(30, 102). Express this number in the form 30x + 102y, where x and y are (not necessarily positive) integers.
Sol: Using the division algorithm we have 102 = 3 · 30 + 12 and then 30 = 2 · 12 + 6, and then 12 = 2 · 6 + 0, hence the gcd is 6. Working backwards, we have 6 = 30 − 2 · 12 = 30 − 2(102 − 3 · 30) = 7 · 30 − 2 · 102. Hence x = 7 and y = −2.

2) In how many ways can the letters of the English alphabet be arranged so that the letters a and b are not adjacent?
Sol: The total number of arrangements of the letters is 26!. The number where a and b are adjacent is 2 · 25!, where the 2 comes from the order that a and b appear in, and 25! is the number of arrangements, where a and b are thought of as a single letter. Hence the answer is 26! − 2 · 25! = 24 · 25!.

3) Solve the recurrence \(a_n = 5a_{n-1} + 2^n\), where \(a_1 = 30\).
Sol: The characteristic equation to the homogeneous part is \(x − 5 = 0\), so the general solution is \(A5^n\). For the particular solution, we try \(B2^n\) which gives \(B2^n = 5B2^{n-1} + 2^n\) or \(2B = 5B + 2\) or \(B = −2/3\). Hence the general solution is \(a_n = A5^n − (2/3)2^n\). Plugging in \(a_1 = 30\) yields \(A = 94/15\).

4) Suppose that five points are located in an equilateral triangle of side length 2. Show that some two of these points are within distance 1 of each other.
Sol: Partition the triangle into four equilateral triangles each of side length 1 (there is only 1 way of doing this, and if you did it wrong, you get essentially no credit). By pigeonhole, some two of the points must lie in the same small triangle, and these two points are within distance at most 1 of each other.

5) Find and prove a formula, by induction, for \(1 + 3 + 3^2 + ⋅⋅⋅ + 3^n\).
Sol: The formula is \((3^{n+1} − 1)/2\). To prove it by induction, first observe that the base case gives \((3^2 − 1)/2 = 4 = 1 + 3\) which is correct. For the induction step, suppose that \(1 + ⋅⋅⋅ + 3^k = (3^{k+1} − 1)/2\). Then \(1 + ⋅⋅⋅ + 3^k + 3^{k+1} = (3^{k+1} − 1)/2 + 3^{k+1} = (3^{k+1} − 1 + 2 · 3^{k+1})/2 = (3^{k+2} − 1)/2\) as desired.