MCS 421 – Combinatorics, Test 3/Final Exam, Spring 2020

Friday, May 1

Time: 50 minutes

There are 5 problems worth 10 points each. You will not get credit unless you completely justify each solution. You are not allowed to quote (without proof) results that we did not cover in class. But you are free to use any theorem we proved in class (you must state the theorem properly).

You may use class notes and the textbook, but you MAY NOT use the internet, other humans, or calculators. Good Luck!

1. Let a_n be the number of n digit ternary sequences with an even number of 0's, an even number of 1's and any number of 2's. For example, when $n = 3$ the sequences are $002, 020, 200, 112, 121, 211, 222$ so $a_3 = 7$. Use exponential generating functions to find a formula for a_n. (10 pts)

2. (a) Determine the (ordinary) generating function for the constant sequence $1, 1, 1, \ldots$, i.e., the sequence $a_n = 1$ for all $n \geq 0$. (5 pts)

(b) Using part (a) and derivatives, evaluate

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{5^{n-2}}.$$

(5pts)

3. Using difference sequences, find a closed form expression for the sum of the first n cubes, i.e., evaluate $\sum_{i=1}^{n} i^3$. You must use difference sequences to get any credit for this problem. (10pts)

4. (a) Draw the Ferrers diagram for the partition $11 = 5 + 3 + 2 + 1$. (3pts)

(b) Write the generating function for the number of partitions of n into parts of size at most 2. (4pts)

(c) Find a formula for the number of partitions of n into parts of size at most 2. Hint: Don't use part (b), do it from scratch. (3pts)

5. (a) Find a formula for the number of lattice paths from $(0, 0)$ to $(2n, 0)$ using steps of the form $(1, 1)$ or $(1, -1)$ such that the path is never below the x-axis (but it is allowed to touch the x-axis, and of course it does so at the beginning and at the end). So we are allowed to move a distance of $\sqrt{2}$ in the northeast or southeast direction in each step. For example, when $n = 2$, there are 2 paths: they are $(1, 1)(1, -1)(1, 1)(1, -1)$ and $(1, 1)(1, 1)(1, -1)(1, -1)$ when listed by the moves at each step. (6pts)

(b) Calculate $S(100, 99)$ and $s(100, 1)$ where $S(p, k)$ is the Stirling number of the second kind and $s(p, k)$ is the Stirling number of the first kind. (4pts)