

Solutions to HW 4

Section 6.7:

2. Let $(4, 6, 7, 8) = (a_1, a_2, a_3, a_4)$, and let A_i be the number of positive integers less than 10,000 divisible by a_i . Then we seek \overline{N} as defined in class. Clearly $|A_i| = \lfloor 10000/a_i \rfloor$, $|A_i \cap A_j| = \lfloor 10000/lcm(a_i a_j) \rfloor$ etc. Careful computation gives 5429 as the answer.

5. We seek the number of solutions to $x_1 + x_2 + x_3 + x_4 = 10$, where $x_1 \geq 0$, $0 \leq x_2 \leq 4$, $0 \leq x_3 \leq 5$, and $0 \leq x_4 \leq 7$. For $i = 2, 3, 4$, let c_i be the condition that $x_i \geq b_i$ and $x_j \geq 0$ for all $j \neq i$, where $(b_2, b_3, b_4) = (5, 6, 8)$. Then $N(c_2) = \binom{8}{3}$, $N(c_3) = \binom{7}{3}$, $N(c_4) = \binom{5}{3}$, $N(c_i c_j) = 0$ for all $i \neq j$. Therefore we obtain $\overline{N} = \binom{13}{3} - (\binom{8}{3} + \binom{7}{3} + \binom{5}{3})$.

6. This is the same as the number of integer solutions to $x_1 + x_2 + x_3 = 12$, where $0 \leq x_1 \leq 6, 0 \leq x_2 \leq 6, 0 \leq x_3 \leq 3$. Computing as before, the answer is 10.

7. No two x_i can exceed 8. If exactly one x_i exceeds 8, then the number of solutions is $\binom{9}{3}$. Hence the answer is $\binom{17}{3} - 4\binom{8}{3}$.

10. The fact that there is at least one r -combination of S , means that $r \leq |S| = \sum_i n_i$. On the other hand, A_i is the number of r -combinations where we use at least $n_i + 1$ objects of type i . Therefore $\cap_i A_i$ is the number of r -combinations where we use at least $n_i + 1$ objects of type i for each i . Thus the total number of objects chosen is at least $\sum_i (n_i + 1) = \sum_i n_i + k > \sum_i n_i \geq r$, hence the intersection is empty.

13. The number of permutations where a fixed set of i distinct odd integers are in their natural positions is $(9 - i)!$. The number of ways to choose i odd integers is $\binom{5}{i}$. Hence the answer is $\sum_{i=1}^5 (-1)^{i-1} \binom{5}{i} (9 - i)!$.

19.

$$\begin{aligned}
 (n-1)(D_{n-2} + D_{n-1}) &= (n-1) \left[(n-2)! \sum_{i=0}^{n-2} \frac{(-1)^i}{i!} + (n-1)! \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \right] \\
 &= (n-1)! \left[\sum_{i=0}^{n-2} \frac{(-1)^i}{i!} + (n-1) \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \right] \\
 &= (n-1)! \left[\sum_{i=0}^{n-2} \frac{(-1)^i}{i!} + n \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} - \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \right] \\
 &= (n-1)! \left[\frac{-(-1)^{n-1}}{(n-1)!} + n \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \right] = n! \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} + (-1)^n \\
 &= n! \sum_{i=0}^n \frac{(-1)^i}{i!} = D_n.
 \end{aligned}$$

22. This is trivial.

23. The identity is trivial by writing $(n - k)/k! = n/k! - k/k!$. Replacing each term in the expression in the previous problem with the two terms obtained in this problem yields the result:

$$q_n = (n - 1)! \sum_{k=0}^n (-1)^k \frac{n - k}{k!} = (n - 1)! \left[\sum_{k=0}^n n \frac{(-1)^k}{k!} + \frac{(-1)^{k-1}}{(k - 1)!} \right] = D_n + D_{n-1}.$$

26. Consider a 6 by 6 board containing two disjoint shaded subboards C_1 and C_2 , where C_1 consists of squares $(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)$, and C_2 consists of squares $(5, 5), (5, 6), (6, 5), (6, 6)$. Then $r(C_1, x) = 1 + 5x + 4x^2$ and $r(C_2, x) = 1 + 4x + 2x^2$, so for $C = C_1 \cup C_2$, we have $r(C, x) = r(C_1, x)r(C_2, x) = 1 + 9x + 26x^2 + 26x^3 + 8x^4$. These coefficients are used in computing the coefficient of x^6 in the rook polynomial of the unshaded board. By inclusion/exclusion, this coefficient, which is the number we seek, is

$$6! - 9 \cdot 5! + 26 \cdot 4! - 26 \cdot 3! + 8 \cdot 2!.$$