Solutions to HW 5

Chapter 7:

1. (a) It equals f_{2n}. It is true for $n = 1$. Now the induction step:

$$f_1 + f_3 + \cdots + f_{2n+1} = (f_1 + f_3 + \cdots + f_{2n-1}) + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2} = f_{2(n+1)}.$$

(b) It equals $f_{2n+1} - 1$. It is true for $n = 0$. Now the induction step:

$$f_0 + f_2 + \cdots + f_{2n+2} = (f_0 + f_2 + \cdots + f_{2n}) + f_{2n+2} = f_{2n+1} - 1 + f_{2n+2} = f_{2n+3} - 1.$$

(c) When n is odd the sum is $-(f_{n-1} + 1)$ and when n is even the sum is $f_{n-1} - 1$. The cases $n = 0$, and $n = 1$ hold. Now first suppose that $n \geq 2$ is even. Then $n - 1$ is odd so by induction

$$f_0 - f_1 + \cdots + f_n = (f_0 - f_1 + \cdots - f_{n-1}) + f_n = -f_{n-2} - 1 + f_n = f_{n-1} - 1.$$

On the other hand, if $n \geq 3$ is odd, then $n - 1$ is even and

$$f_0 - f_1 + \cdots - f_n = (f_0 - f_1 + \cdots + f_{n-1}) - f_n = f_{n-2} - 1 - f_n = -f_{n-1} - 1.$$

(d) The sum of the squares of the first n fibonacci numbers is $f_n f_{n+1}$. The induction step follows since

$$f_0^2 + \cdots + f_n^2 = (f_0^2 + \cdots f_{n-1}^2) + f_n^2 = f_{n-1}f_n + f_n^2 = f_n(f_{n-1} + f_n) = f_n f_{n+1}.$$

5. Using the fibonacci recurrence repeatedly, one obtains $f_{n+8} = 21 f_{n+2} + 13 f_n$. Thus if 7 divides f_n, then 7 also divides f_{n+8}. Since 7 divides f_0, we conclude that 7 divides f_n when n is a multiple of 8. On the other hand, if 7 divides f_{n+8}, then 7 divides $13 f_n$, but $(7, 13) = 1$, therefore 7 divides f_n. Clearly 7 does not divide f_n for $1 \leq n \leq 7$, therefore we conclude that 7 divides f_n if and only if n is a multiple of 8.

8. If the last square is red, then the second last last square must be blue, and there are h_{n-2} ways to color the first $n - 2$ squares. If the last square is blue, then there is no restriction on the second last square, and there are h_{n-1} ways to color the remaining squares. Thus $h_n = h_{n-1} + h_{n-2}$. Clearly $h_1 = 2$ and $h_2 = 3$, from which we conclude that $h_0 = 1$. Thus $h_0 = f_2$, and $h_n = f_{n+2}$, where f_i is given in (7.11) page 198.

9. If the last square is red, then the second last last square must be blue or white, and there are h_{n-2} ways to color the first $n-2$ squares for each of these two possibilities. If the last square is blue (or white), then there is no restriction on the second last square, and there are h_{n-1} ways to color the remaining squares. Thus $h_n = 2h_{n-1} + 2h_{n-2}$. Clearly $h_1 = 3$ and $h_2 = 8$, from which we conclude that $h_0 = 1$. The characteristic equation for the recurrence is $x^2 - 2x + 2$, from which we get $h_n = A(1 + \sqrt{3})^n + B(1 - \sqrt{3})^n$. The initial values yield $A = (\sqrt{3} + 2)/2\sqrt{3}$ and $B = (\sqrt{3} - 2)/2\sqrt{3}$.

16. How many positive integer solutions to $a+b+c+d=n$ where $a \in \{0,1,2\}, b \in \{0,2,4,6\}, c \in \{0,2,4,6,..\}, d \in \{1,2,3,..\}$. The answer is the coefficient of x^n in the generating function given

17. $g(x) = (1+x^2+\cdots)(1+x+x^2)(1+x^3+x^6+\cdots)(1+x)$. This simplifies to $(1-x)^{-2} = \sum(n+1)x^n$. Hence the answer is $n+1$.

21. Let C be a convex $(n+2)$-gon with vertices $x_1, \ldots, x_{n+2}$. Removing x_1 yields an $(n+1)$-gon C' with h_{n-1} regions. Putting x_1 back yields n new regions formed by diagonals containing x_1 and the diagonal x_2x_{n+2}. However, the diagonals from x_1 cut certain old regions into two parts, thus creating new regions. Thus $h_n = h_{n-1} + n + R$, where R is the number of regions in C' that are cut by a diagonal containing x_1. There are two ways to compute R. We may bijectively map each region in R to a set of three vertices in C' as follows: Let the diagonal x_1x_i first encounter the region in the diagonal x_jx_k. Then we associate the three points $\{x_i, x_j, x_k\}$. It is easy to check that this mapping is bijective, so $R = \binom{n+1}{3}$. On the other hand, for fixed i, the regions in R cut by x_1x_i correspond to the number of diagonals x_jx_k, where $j < i < k$, so $R = \sum_{i=3}^{n+1}(i-2)(n+2-i) = \binom{n+1}{3}$. Using generating functions, we obtain $g(x) = x^2/(1-x)^5 + x/(1-x)^3$. This gives $h_n = \binom{n+2}{4} + \binom{n+1}{2}$ for $n \geq 2$. in class: $(1+x)^n$

26. The exponential generating function is

$$([e^x + e^{-x}]/2)^2 e^{2x} = (e^{4x} + 2e^{2x} + 1)/4.$$

The coefficient of $x^n/n!$ is $4^{n-1} + 2^{n-1}$.

33. The characteristic equation is $x^3 - x^2 - 9x + 9$ which has roots $1, \pm 3$. Thus $h_n = A + B3^n + C(-3)^n$. The initial values give the equations $A+B+C=0, A+3B-3C=1, A+9B+9C=2$, and we obtain $A=-1/4, B=1/3, C=-1/12$.

34. The characteristic equation is $x^4 - 8x + 16$, so the general solution is $(A+Bn)4^n$. The initial values give $A=-1, B=1$.

42. The particular solution is Bn, since 4 is a root of the characteristic equation of multiplicity 1. Running it through the equation gives $B=1$. Thus $h_n = (A+n)4^n$. The initial value gives $A=3$.

47. The particular solution is $C+Dn$, and running it through the equation yields $C=13, D=3$. The general solution to the nonhomogeneous part is $(A+Bn)2^n$. Thus $h_n = (A+Bn)2^n + 13 + 3n$. Using the initial values gives $A=-12, B=5$.