Solutions to HW 7

Chapter 10:

42. For each $r = 1, 2, 3, 4$, let L^r be defined by $L^r_{i,j} = ri + j$ for each $i, j \in \{0, \ldots, 5\}$, where everything is taken modulo 5. Then we proved in class that these are MOLS.

Chapter 14 :

7. Label the corners $1, 2, 3, 4$. Think of one corner being on top, and the other three forming a triangle as a base. Then for each choice of the corner on top, we can rotate the other three points about the line through the top point by 0, 60 or 120 degrees. So we obtain all 12 elements of the group in this way.

$$\{id, (234), (243)\}, \{(132), (12)(34), (142)\}, \{(124), (134), (14)(23)\}, \{(123), (13)(24), (143)\}$$

Note that it is impossible to just transpose two elements and leave the other two elements fixed, so the group cannot be all of S_4, so its order is certainly less than 24. Since the order of a subgroup divides the order of a group (not covered in class, but easy fact about groups), once we have 12 elements, we know that this must be all of them, since we do not have 24 elements.

8. This is similar to the previous problem. Just keep track of where the edges get mapped under the symmetries.

12. This is exactly the same group as the vertex symmetry group of the square.

13d. First observe that $(gf)^{-1} = (345)$. So $(gf) * c$ when applied to each element yields the color vector (R, B, R, R, B, R). similarly, $(fg)^{-1} = (152)$, so $(fg) * c$ yields (R, R, B, R, B, R).

15. There is only one equivalence class in each case: RRRR, BBBB, RBBB, BRRR, RRBB. Thus there are 5 inequivalent colorings. Now suppose we have three colors, R, B, and W. There are three colorings where only one color is used. There are nine colorings if some two colors are used (three for every pair of colors). If all three colors are used, then one color is used twice, and the other two are used once. Then there is only one way to color, so in total there are three colorings of this type. Thus there are 15 colorings when we have 3 colors at our disposal.

19. The corner symmetry group is S_3, of order 6. The identity permutation fixes all p^3 colorings, the two 3-cycles fix p colorings (when only one color is used), and the three 2-cycles fix p^2 colorings. So we have $(p^3 + 2p + 3p^2)/6$ inequivalent colorings.

26. The dihedral group D_7 has 14 elements. There are $\binom{7}{3} = 35$ colorings total; the identity fixes all of them. The rotation through the point i fixes three colorings, which are obtained by coloring i with B, and coloring one point on each side of i also with B, where the shortest distance between the two points is 2. The remaining symmetries fix no colorings. So we obtain $(35 + 7 \cdot 3 + 0)/14 = 4$ colorings in total.

35. In the Dihedral group, the six rotations yield

$$i_d, (123456), (135)(246), (14)(25)(36), (153)(264), (165432),$$

and the six reflections yield

$$(26)(35),(31)(46),(42)(51),(12)(36)(45),(23)(14)(56),(16)(25)(34)$$

We get that $\#(f) = 4$ for three different f, $\#(f) = 3$ for four different f, $\#(f) = 2$ for two different f, and $\#(f) = 1$ for one f. Together with the identity, we obtain that $(k^6 + 3k^4 + 4k^3 + 2k^2 + 2k)/12$ is the number of colorings.

48. If you think of the window as stationary, then the problem is somewhat easier (you'll get credit). More interesting is if you think of the window as a piece of glass that we can move in space. I have solved that problem below. Label the squares using 1 through 9 row by row, from left to right. There are four rotations, by 0, 90, 180, 270 degrees:

$$i_d, (1397)(2684)(5), (19)(28)(37)(46)(5), (1793)(2486)(5)$$

There are four reflections, about the middle vertical line, the middle horizontal line, the main diagonal, the other diagonal:

$$(13)(46)(79)(2)(5)(8), (17)(28)(39)(4)(5)(6), (24)(37)(68)(1)(5)(9), (26)(19)(48)(3)(5)(7).$$

A composition of these two types of permutations yields another of these types, so this is the symmetry group. Calculating the types of each of these permutations, we obtain

$$P_G(z_1, \ldots, z_9) = (z_1^9 + 2z_1z_4^2 + z_1z_2^4 + 4z_1^3z_2^3)/8.$$

Plugging in $k = 2$ for each z_i we obtain $2^6 + 2 + 2^2 + 2^5 = 102$ inequivalent 2-colorings.