Homework Set 4

1) Let F be a family of subsets of $N = \{1, 2, \ldots, n\}$, and suppose that there are no two distinct $A, B \in F$ with $A \subset B$. Let $\sigma \in S_n$ be a random permutation of the elements of N and consider the random variable

$$X = |\{i : \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} \in F\}|.$$

By considering the expectation of X prove that $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$.

2) Let $v_i = (x_i, y_i), i = 1, \ldots, n$ be n two-dimensional vectors, where each x_i and each y_i is an integer whose absolute value does not exceed $2^{n/2}/(100\sqrt{n})$. Show that there are two disjoint sets $I, J \subset [n]$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

Hint: Chebyshev's inequality

3) Let $G = (V, E)$ be a bipartite graph with n vertices and a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

4) Let X be a random variable taking integral nonnegative values, let $E(X^2)$ denote the expectation of its square, and let $Var(X)$ denote its variance. Prove that

$$P(X = 0) \leq \frac{Var(X)}{E(X^2)}.$$

You may use the Cauchy-Schwarz inequality which states that $E(AB)^2 \leq E(A^2)E(B^2)$. Note that in class we proved this result with $E(X)^2$ in the denominator so this result is stronger (since $E(X)^2 \leq E(X^2)$).