Homework Set 6

1) Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability 1/2, independently, is connected (and spanning). Let Q denote the probability that in a random two-coloring of G, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$?

2) Let F_1, \ldots, F_k be k intersecting families of subsets of [n]. Prove, using Kleitman's inequalities, that

$$|\cup_{i=1}^{k} F_i| \le 2^n - 2^{n-k}.$$

Show an (easy) example that achieves the bound. Hint: use induction on k; you may also assume that each F_i is maximal, and hence monotone increasing (why?).

3) Show that the probability that in the random graph G(2k, 1/2) the maximum degree is at most k - 1 is at least $1/4^k$.

4) Suppose that $\Omega = A \cup B$ is a probability space, where $A \cap B = \emptyset$. Suppose that $f: \Omega \to R$ is a random variable. Let *a* be the expected value of *f* on *A*, *b* be the expected value of *f* on *B*, and *c* be the expected value of *f* on $A \cup B$ (so $a = (\sum_{x \in A} f(x)P(x))/P(A)$ etc.) Define Z(x) to be *a* if $x \in A$ and *b* if $x \in B$. Prove that E(Z) = c. Conclude that the edge exposure martingale X_0, \ldots, X_m for some graph function *f* satisfies $E(X_i|X_{i-1}, \ldots, X_0) = X_{i-1}$.