Homework 1

1) Let $d = d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ be a sequence of integers with sum $2(n-1)$. Prove that there is a tree with vertex set $\{v_1, \ldots, v_n\}$ and $d(v_i) = d_i$.

2) Prove the Erdős-Sós conjecture for P_3, the path with three edges. In other words, prove that an n vertex graph with no copy of P_3 has at most n edges. Can you prove a similar result for P_4 ($3n/2$ edges) and more generally for P_k ($(k-1)n/2$ edges)? (the general case of k is hard, extra credit)

3) Prove that every n vertex graph with e edges has at least $(e/3)(4e/n - n)$ triangles.

4) Use the Local Lemma to prove the following: there is a positive constant c such that for all n there is an edge-coloring of K_n with at most $c\sqrt{n}$ colors such that every copy of K_4 receives at least three colors (on its six edges).

5) Prove that every connected graph G contains a path of length at least $\min\{2\delta(G), n(G) - 1\}$ (here $\delta(G)$ is the minimum degree in G).

6) Let m be given. Show that if n is large enough, then every n by n $(0, 1)$-matrix has a principal submatrix of size m in which all elements below the diagonal are the same, and all elements above the diagonal are the same.

7) Let G be a graph such that every two odd cycles in G have a vertex in common. Prove that $\chi(G) \leq 5$.