Homework Set 4

1) Let H be a graph, and let $n > |V(H)|$ be an integer. Suppose that there is a graph on n vertices and t edges containing no copy of H, and suppose that $tk > n^2 \log_e n$. Show that there is a coloring of the edges of the complete graph on n vertices by k colors with no monochromatic copy of H.

2) Let F be a family of subsets of $N = \{1, 2, \ldots, n\}$, and suppose that there are no two distinct $A, B \in F$ with $A \subset B$. Let $\sigma \in S_n$ be a random permutation of the elements of N and consider the random variable

$$X = |\{i : \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} \in F\}|.$$

By considering the expectation of X prove that $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$.

3) Let $G = (V, E)$ be a bipartite graph with n vertices and a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

4) Let X be a random variable taking integral nonnegative values, let $E(X^2)$ denote the expectation of its square, and let $Var(X)$ denote its variance. Prove that

$$P(X = 0) \leq \frac{Var(X)}{E(X^2)}.$$