Homework Set 5

1) Let F be a finite collection of binary strings of finite lengths and assume no member of F is a prefix of another one. Let N_i denote the number of strings of length i in F. Prove that $\sum_i N_i/2^i \leq 1$.

2) Let $v_i = (x_i, y_i), i = 1, \ldots, n$ be n two-dimensional vectors, where each x_i and each y_i is an integer whose absolute value does not exceed $2^{n/2}/(100\sqrt{n})$. Show that there are two disjoint sets $I, J \subset [n]$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

Hint: Chebyshev's inequality

3) Prove that for every integer $d > 1$, there is a finite $c(d)$ such that the edges of any bipartite graph with maximum degree d in which every cycle has at least $c(d)$ edges can be colored by $d+1$ colors so that there are no two adjacent edges with the same color and there is no two-colored cycle. Hint: Use König's theorem, that the edges can be partitioned into d matchings.

4) Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that $c \in S(v)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.