Homework Set 6

1) Suppose that $\Omega = A \cup B$ is a probability space, where $A \cap B = \emptyset$. Suppose that $f : \Omega \to R$ is a random variable. Let a be the expected value of f on A, b be the expected value of f on B, and c be the expected value of f on $A \cup B$ (so $a = (\sum_{x \in A} f(x)P(x))/P(A)$ etc.) Define $Z(x)$ to be a if $x \in A$ and b if $x \in B$. Prove that $E(Z) = c$. Conclude that the edge exposure martingale $X_0, \ldots, X_m$ for some graph function f satisfies $E(X_i | X_{i-1}, \ldots, X_0) = X_{i-1}$.

2) Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability $1/2$, independently, is connected (and spanning). Let Q denote the probability that in a random two-coloring of G, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$?

3) Let $F_1, \ldots, F_k$ be k intersecting families of subsets of $[n]$. Prove that

$$| \cup_{i=1}^k F_i | \leq 2^n - 2^{n-k}.$$

Hint: use induction on k; you may also assume that each F_i is maximal, and hence monotone increasing (why?).

4) Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k-1$ is at least $1/4^k$.