Homework Set 7

1) Let $d \geq 3$ be a constant and $\rho = p(n-1)$. Prove the following: If $\rho \ll n^{-1/d}$ then $G(n,p)$ does not have a vertex of degree d almost surely and if $\rho \gg n^{-1/d}$ then $G(n,p)$ has a vertex of degree d almost surely.

2) Let p be a prime congruent to 1 mod 4 and G_p the graph with vertex set $GF(p)$ and ij forming an edge iff $i-j$ is a quadratic residue mod p. Show that G_p is well-defined and the prove the following about G_p:
a) it is $(p-1)/2$-regular
b) any two adjacent vertices have $(p-5)/4$ common neighbors
c) any two nonadjacent vertices have $(p-1)/4$ common neighbors
d) For any two vertices a, b, there are precisely $(p-1)/4$ vertices $c \neq b$ joined to a and not joined to b.

3) Let B and C be disjoint sets of vertices in G_p defined above. Prove that

$$\left| e(B,C) - \frac{1}{2}|B||C| \right| \leq \frac{1}{2}|B|^{1/2}|C|^{1/2}p^{1/2}.$$

4) Let $G = (V,E)$ be an (n,d,λ)-graph and $k|n$. Suppose that c is a k-coloring of V so that each color appears precisely n/k times. Prove that there is a vertex of G which has a neighbor of each of the k colors, provided $k\lambda \leq d$.