Homework Set 5

1) Let $k = 50$, $L = \{0, 26, 27\}$, and $\mathcal{F} \subseteq 2^{[n]}$ be an L-intersecting k-uniform family. Prove, using the uniform RW Theorem, that $m = |\mathcal{F}| \leq \binom{n}{2}$.

Hint: For $A, B \in \mathcal{F}$, let $A \sim B$ if $A \cap B \neq \emptyset$. Prove that this is an equivalence relation. You may also use the easy fact that if $\sum_i n_i = n$, then $\sum_i \binom{n_i}{2} \leq \binom{n}{2}$.

2) Prove that in Nagy's coloring given in class, if $t \equiv 2$ or $3 \pmod 4$, then there is no blue K_r for $r > t - 2$. Recall that in the coloring, the vertex set of K_n is $\binom{[t]}{3}$, and an edge is blue iff the endpoints intersect in a set of size zero or two.

3) We gave superpolynomial lower bounds in class for the Ramsey number $R(t,t)$ for infinitely many t. Prove the same lower bound for all t, namely, for any fixed $\epsilon > 0$, there is a t_0 such that for $t > t_0$ we have $R(t,t) > t^{(1-\epsilon)\omega(t)}$, where $\omega(t) = \ln t/(4 \ln \ln t)$.

Hint: As in class, let $n = p^3$. Now let p be the largest prime such that $2\binom{n}{p-1} < t$. You may use the following consequence of the Prime Number Theorem: for any $\delta > 0$, there is a q_0 such that, if $q > q_0$ is a prime, then the next largest prime $q' > q$ has the property that $q' < (1+\delta)q$. Use this to prove that for any $\delta' > 0$,

$$\frac{(1-\delta')\ln t}{2\ln\ln t} < p < \frac{(1+\delta')\ln t}{2\ln\ln t}$$

for sufficiently large t. Then use the estimates for binomial coefficients we have proved to complete the proof.

4) Let $K = \{k_1, k_2\}$ and $L = \{l_1, \ldots, l_s\}$ be two sets of nonnegative integers with $k_i > s - 2$ for $i = 1, 2$. Let $\mathcal{F} \subseteq 2^{[n]}$ be an L-intersecting family with $|S| \in K$ for each $S \in \mathcal{F}$. Prove that

$$m = |\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1}.$$

Hint: Proceed as in the proof of the uniform RW Theorem presented in class. Instead of the function $(\sum_i x_i - k)$, use a similar function, and slightly change the condition $|I| \leq s - 1$,

Remark: This can be easily generalized to $K = \{k_1, \ldots, k_r\}$ (no need to do it), and then it provides a common proof of both the uniform and nonuniform RW Theorems (Alon-Babai-Suzuki 1991).