Homework Set 6

1) A vertex cover in a hypergraph is a vertex subset that intersects each edge. Let $\mathcal{H}$ be a hypergraph and $\tau(\mathcal{H})$ be the minimum size of a vertex cover of $\mathcal{H}$. Call $\mathcal{H}$ *critical* if $\tau(\mathcal{H} - A) < \tau(\mathcal{H})$ for every $A \in E(\mathcal{H})$. Prove that if $\mathcal{H}$ is a critical r-uniform hypergraph, and $\tau(\mathcal{H}) = s + 1$, then

$$\mathcal{H} \leq \binom{r+s}{r}.$$

2) Let $\mathcal{F} \subseteq 2^{[n]}$ be a maximal family with the property that if $A, B \in \mathcal{F}$, then $A \cup B \neq [n]$. Prove that $\mathcal{F}$ is an ideal and that $|\mathcal{F}| = 2^{n-1}$. Recall that an ideal is a family $\mathcal{I}$ with the property that whenever $A \in \mathcal{I}$ and $B \subseteq A$, then also $B \in \mathcal{I}$.

3) Prove the following extension of the Erdős-Ko-Rado Theorem: If $1 \leq r < n/2$ and $\mathcal{F}$ is an intersecting sperner family of subsets of $\binom{[n]}{\leq r}$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$.

4) Prove the following extension of the Erdős-Ko-Rado Theorem: Families $\mathcal{F}$ and $\mathcal{G}$ are *cross intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$. Let k, l, n be positive integers with $k + l \leq n$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and $\mathcal{G} \subseteq \binom{[n]}{l}$ be cross intersecting. Then either $|\mathcal{F}| < \binom{n-1}{k-1}$ or $|\mathcal{G}| \leq \binom{n-1}{l-1}$.
Hint: Consider the complements of one of the families and use the Kruskal Katona theorem like we did to prove EKR in class.