Solutions to Homework #3:

7) Show without Menger's theorem that every two vertices in a 2-connected graph lie on a common cycle.
Solution: It suffices to show that for any two vertices x, y, of G there are two internally vertex disjoint $x - y$ paths. Let us show this by induction on $d = dist(u, v)$. If $d = 1$, then since G is 2-edge connected, this holds, so assume that $d \geq 2$. Let x be the vertex on a shortest $u - v$ path preceding v. By induction, there are two disjoint $u - x$ paths P, P'. If v is on one of them, we are done, so assume this is not the case. Since G is 2-connected, there is a $u - v$ path Q in $G - x$. Let w be the last vertex of Q on $P \cup P'$, say its on P. Let R be the path $u - P - w - Q - v$. Then P' and R are the two required paths.

11) Show that every cubic 3-edge-connected graph is 3-connected.
Solution: Let S be a vertex cut of size at most 2. Let G_1, G_2 be two components of $G - S$. Then each vertex of S has an edge to each G_i (by minimality of S). Since G is cubic, for each $v \in S$ there is an i for which v has exactly edge to G_i. If $|S| = 1$, then this edge is an edge-cut, contradiction, so assume that $S = \{u, v\}$. If both u and v have exactly one edge to the same G_i, then these two edges form an edge cut of size 2. Otherwise, uv is not an edge, u has exactly one edge to G_1, and v has exactly one edge to G_2. Then these two edges form an edge cut, contradiction.

16) Show that every k connected graph ($k \geq 2$) with at least $2k$ vertices contains a cycle of length at least $2k$.
Solution: Consider a longest cycle C. If no $v \notin C$ exists than $|C| \geq 2k$ and we're done, so assume that $v \notin C$. By Menger, there are k disjoint $v - C$ paths. If $|C| < 2k$, then the endpoints of two of these paths must be adjacent on C, by pigeonhole. Detouring along these paths then yields a longer cycle, contradiction.

17) Show that in a k connected graph ($k \geq 2$) any k vertices lie on a common cycle.
Solution: Let S be a given set of k vertices and consider a cycle C with the maximum number of vertices from S. Suppose that some $v \in S - C$. Then by Menger, there are k $v - C$ paths. Partition C into at most $k - 1$ paths P_j, where the ith path begins from the ith vertex of S on C (in clockwise order say), and ends just before the $i + 1$st vertex. Since $|S| = k$, by pigeonhole, two of the $v - C$ paths have their endpoints in the same P_i. Detouring along these two paths yields a cycle with more vertices of S, contradiction.

25) Show, using the Thomas-Wollan result, that average degree cr^2 for some constant c implies the existence of a K_r subdivision
Solution: By moving to a subgraph (using Mader's result), we may assume that our graph is $c'r^2$-connected, and then by Thomas-Wollan, r^2-linked. Now pick r vertices $v_1, \ldots, v_r$ and sets S_i such that $|S_i| = r - 1$, $S_i \subset N(v_i)$, and $(S_i \cup \{v_i\}) \cap (S_j \cup \{v_j\}) = \emptyset$ for $i \neq j$. We can choose such sets since the minimum degree is at least r^2. Now, for each $i \neq j$, pick a vertex of S_i and pair it with a vertex in S_j, as long as these vertices haven't already been paired. By r^2-linkage, we can find disjoint paths between pairs and avoiding the v_i's as well. These paths together yield a subdivision of K_r with branch vertices v_i.