Homework Set 1

1) Prove that in every 2-coloring of the edges of the complete graph on n vertices $(n \ge 3)$, there is a hamiltonian cycle which is either monochromatic, or consists of two monochromatic arcs.

2) Prove or disprove: In every 3-coloring of the points of the plane, there are two points at distance 1 with the same color.

3) Prove or disprove: In every 2-coloring of the points of the plane, there is a monochromatic regular triangle with side 1.

4) Let $k, r \geq 1$ be given. Prove that there is an N(k, r) such that if $n \geq N(k, r)$ and [n] is k-colored, then we can find natural numbers a, d_1, \ldots, d_r such that all sums

$$a + d_{i_1} + \dots + d_{i_{\nu}}$$
 $(1 \le i_1 < \dots < i_{\nu} \le r. \ 0 \le \nu \le r)$

have the same color (and, of course, $a + d_1 + \cdots + d_r \leq n$).

5) Prove the Hales Jewett theorem for combinatorial *m*-spaces. Hint: use induction on *m*. For the induction step from *m* to m + 1, consider words of length M + N, where M = HJ(1, r, t) and $N = HJ(m, r^{t^M}, t)$.

6) Let q be a prime power and $r \ge 1$, $k \ge 0$. Prove that if n is sufficiently large, then the following holds: Every r-coloring of the points of the affine n-dimensional space over GF(q) yields a monochromatic k-dimensional subspace.

7) Let $k, r \geq 1$. Construct, either using Ramsey's theorem or using the previous problem, a k-uniform hypergraph with chromatic number at least k + 1 and in which every two edges have at most one point in common. Recall that the chromatic number is the least number of vertex classes that partition the vertex set, where each class is an independent set.

8) Prove the following modular version of the sunflower lemma: Let \mathcal{F} be a family of s-sets of an n-set, and suppose that $|\mathcal{F}| > F(n, k, s)$, where

$$F(n,k,s) = (nk)^{s/2} \cdot \frac{1 \cdot 3 \cdots (s-1)}{2 \cdot 4 \cdots s}$$

if s is even and

$$F(n,k,s) = n^{(s-1)/2} k^{(s+1)/2} \cdot \frac{1 \cdot 3 \cdots s}{2 \cdot 4 \cdots (s-1)}$$

if s is odd. Then \mathcal{F} contains a sunflower with k+1 petals and a core Y such that $|Y| \equiv s + 1 \pmod{2}$.

9) Suppose that \mathcal{F} is a family of *s*-sets with $|\mathcal{F}| > 3s!$. Show that \mathcal{F} contains three sets whose pairwise intersection sizes are the same (the sunflower lemma gives this, and even more, as long as we have an additional factor of 2^s).