Homework 1

1) Let $d = d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ be a sequence of integers with sum $2(n-1)$. Prove that there is a tree with vertex set $\{v_1, \ldots, v_n\}$ and $d(v_i) = d_i$.

2) Prove that the Prüfer sequence construction from class gives a bijection between the number of labeled trees with n vertices and the number of sequences of length $n-2$ with entries from $[n] = \{1, \ldots, n\}$.

3) Prove the Erdős-Sós conjecture for P_3, the path with three edges. In other words, prove that an n vertex graph with no copy of P_3 has at most n edges. Can you prove a similar result for P_4 ($3n/2$ edges) and more generally for P_k ($(k-1)n/2$ edges)? (the general case of k is hard, extra credit)

4) Prove that every connected graph G contains a path of length at least $\min\{2\delta(G), n(G)-1\}$ (here $\delta(G)$ is the minimum degree in G).

5) Let m be given. Show that if n is large enough, then every n by n $(0,1)$-matrix has a principal submatrix of size m in which all elements below the diagonal are the same, and all elements above the diagonal are the same. Recall that a principal submatrix is obtained by deleting some set of rows and the same set of columns.

6) Let G be a graph such that every two odd cycles in G have a vertex in common. Prove that $\chi(G) \leq 5$.

7) Prove that in every 2-coloring of the edges of the complete graph on n vertices ($n \geq 3$), there is a hamiltonian cycle which is either monochromatic, or consists of two monochromatic arcs.