Homework #2:

1) Define the multicolor Ramsey number $R_k(3)$ to be the minimum $n$ such that no matter how the edges of $K_n$ are colored with $k$ colors, there is a monochromatic copy of $K_3$. Prove that

$$R_k(3) - 1 \leq 1 + k(R_{k-1}(3) - 1).$$

Use this to prove the upper bound $R_k(3) < 1 + ek!$, where $e$ is the (usual) base of the natural logarithm, i.e., $e \approx 2.718$. .

2) Let $k = 50$, $L = \{0, 26, 27\}$, and $\mathcal{F} \subseteq 2^{[n]}$ be an $L$-intersecting $k$-uniform family. Prove, using the uniform RW Theorem, that $m = |\mathcal{F}| \leq \binom{n}{2}$. 

Hint: For $A, B \in \mathcal{F}$, let $A \sim B$ if $A \cap B \neq \emptyset$. Prove that this is an equivalence relation. You may also use the easy fact that if $\sum_i n_i = n$, then $\sum_i \binom{n_i}{2} \leq \binom{n}{2}$.

3) Prove that in Nagy’s coloring given in class, if $t \equiv 2$ or $3$ (mod 4), then there is no blue $K_r$ for $r > t - 2$.

Recall that in the coloring, the vertex set of $K_n$ is $\binom{[t]}{3}$, and an edge is blue iff the endpoints intersect in a set of size zero or two.

4) We gave superpolynomial lower bounds in class for the Ramsey number $R(t, t)$ for infinitely many $t$. Prove the same lower bound for all $t$, namely, for any fixed $\epsilon > 0$, there is a $t_0$ such that for $t > t_0$ we have $R(t, t) > t^{\left(1 - \epsilon\right)\omega(t)}$, where $\omega(t) = \ln t / (4 \ln \ln t)$.

Hint: As in class, let $n = p^3$. Now let $p$ be the largest prime such that $2^{\binom{n}{2} - 1} < t$. You may use the following consequence of the Prime Number Theorem: for any $\delta > 0$, there is a $q_0$ such that, if $q > q_0$ is a prime, then the next largest prime $q' > q$ has the property that $q' < (1 + \delta)q$. Use this to prove that for any $\delta' > 0$,

$$\frac{(1 - \delta') \ln t}{2 \ln \ln t} < p < \frac{(1 + \delta') \ln t}{2 \ln \ln t}$$

for sufficiently large $t$. Then use the estimates for binomial coefficients we have proved to complete the proof.

5) Let $K = \{k_1, k_2\}$ and $L = \{l_1, \ldots, l_s\}$ be two sets of nonnegative integers with $k_i > s - 2$ for $i = 1, 2$. Let $\mathcal{F} \subseteq 2^{[n]}$ be an $L$-intersecting family with $|S| \in K$ for each $S \in \mathcal{F}$. Prove that

$$m = |\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1}.$$

Hint: Proceed as in the proof of the uniform RW Theorem presented in class. Instead of the function $\left(\sum_i x_i - k_1\right)$, use the function $\left(\sum_i x_i - k_1\right)\left(\sum_i x_i - k_2\right)$, and instead of letting $|I| \leq s - 1$, let $|I| \leq s - 2$.

Remark: This can be easily generalized to $K = \{k_1, \ldots, k_r\}$ (no need to do it), and then it provides a common proof of both the uniform and nonuniform RW Theorems (Alon-Babai-Suzuki 1991).