Homework #2 Solutions:

1) Define the multicolor Ramsey number $R_k(3)$ to be the minimum n such that no matter how the edges of K_n are colored with k colors, there is a monochromatic copy of K_3. Prove that

$$R_k(3) - 1 \leq 1 + k(R_{k-1}(3) - 1).$$

Use this to prove the upper bound $R_k(3) < 1 + ek!$, where e is the (usual) base of the natural logarithm, i.e., $e = 2.718..$.

Sol: Let $N = 2+k(R_{k-1}(3)-1)$, and suppose we have a k-coloring of $E(K_n)$. Pick a vertex v. By the pigeonhole principle, there is a color i and a set S of size $R_{k-1}(3)$ such that all v, S edges have color i. If color i appears anywhere within S, then we have a triangle in color i. Otherwise the edges within S are $(k-1)$-colored, and since $|S| \geq R_{k-1}(3)$, we conclude that S contains a monochromatic triangle in some other color.

We show by induction on k that $R_k(3) \leq 1 + k!(\sum_{i=0}^{k} 1/i!) < 1 + k!e$. The base case follows by $R(3,3) \leq 6$. For the induction step,

$$R_k(3) \leq 2 + k(R_{k-1}(3) - 1) \leq 2 + k\left((k-1)!\sum_{i=0}^{k-1} 1/i!\right) = 1 + k!\sum_{i=0}^{k} 1/i!.$$

2) Let $k = 50$, $L = \{0, 26, 27\}$, and $\mathcal{F} \subseteq 2^{[n]}$ be an L-intersecting k-uniform family. Prove, using the uniform RW Theorem, that $m = |\mathcal{F}| \leq \binom{n}{2}$.

Hint: For $A, B \in \mathcal{F}$, let $A \sim B$ if $A \cap B \neq \emptyset$. Prove that this is an equivalence relation. You may also use the easy fact that if $\sum_i n_i = n$, then $\sum_i \binom{n_i}{2} \leq \binom{n}{2}$.

Sol: If $A \sim B$ and $B \sim C$, then both $|A \cap B|$ and $|B \cap C|$ are at least $26 > |B|/2$, so $|A \cap C| > 0$. This implies that $A \sim C$. This relation divides the sets in $\mathcal{F}$ into equivalence classes. Each equivalence class is a subfamily $\mathcal{F}_i$ that is L'-intersecting, where $L' = \{26, 27\}$. By the uniform RW-inequality, $|\mathcal{F}_i| \leq \binom{n_i}{2}$, where $n_i = |\cup_{A \in \mathcal{F}_i} A|$. By definition of $\sim$, $(\cup_{A \in \mathcal{F}_i} A) \cap (\cup_{B \in \mathcal{F}_j} B) = \emptyset$, so $|\mathcal{F}| \leq \sum_i \binom{n_i}{2} \leq \binom{n}{2}$.

3) Prove that in Nagy's coloring given in class, if $t \equiv 2$ or $3 \pmod 4$, then there is no blue K_r for $r > t - 2$.

Recall that in the coloring, the vertex set of K_n is $\binom{[t]}{3}$, and an edge is blue iff the endpoints intersect in a set of size zero or two.

Sol: Let $C_1, \ldots, C_r$ be the vertex of a blue K_r. Put $C_i \sim C_j$ if $|C_i \cap C_j| = 2$. Then it is easy to see that this defines an equivalence relation, since no two of these sets intersect in exactly one element. Each equivalence class is a subfamily $\mathcal{F}_i$ that is 2-intersecting. Let $A_1, \ldots, A_{m_i}$ be the sets in $\mathcal{F}_i$. We will show that either $m_i = 1$, $A_j \cap A_k$ is the same set (of size two) for all j, k (this is called a sunflower), or $\mathcal{F}_i$ consists of (at most four) 3-subsets of a four element set.

Suppose that $m_i > 1$ and that $\mathcal{F}_i$ is not a sunflower. Let $A, B, C \in \mathcal{F}_i$ with $|A \cap B| = 2$, and $C \cap A \neq A \cap B$. We may assume by symmetry that $A = 123$ and $B = 234$ and $C = 124$. Then it is easy to see that the only choice for a fourth set $D \in \mathcal{F}_i$ is $D = 134$.

Let $m_i = |\mathcal{F}_i|$, and $X_i = \cup_{A \in \mathcal{F}_i} A$ with $t_i = |X_i|$. By definition, $X_i \cap X_j = \emptyset$ for $i \neq j$. Clearly $m_i \leq t_i - 2$ if $\mathcal{F}_i$ is a sunflower, and $m_i \leq t_i$ otherwise. By the choice of t, there are at least two points that are in an X_i for which $\mathcal{F}_i$ is a sunflower. This gives $m = \sum m_i \leq (\sum t_i) - 2 = t - 2$.

4) We gave superpolynomial lower bounds in class for the Ramsey number $R(t,t)$ for infinitely many t. Prove the same lower bound for all t, namely, for any fixed $\epsilon > 0$, there is a t_0 such that for $t > t_0$ we have $R(t,t) > t^{(1-\epsilon)\omega(t)}$, where $\omega(t) = \ln t/(4 \ln \ln t)$.

Hint: As in class, let $n = p^3$. Now let p be the largest prime such that $2\binom{n}{p-1} < t$. You may use the following consequence of the Prime Number Theorem: for any $\delta > 0$, there is a q_0 such that, if $q > q_0$ is a prime, then the next largest prime $q' > q$ has the property that $q' < (1+\delta)q$. Use this to prove that for any $\delta' > 0$,

$$\frac{(1-\delta')\ln t}{2\ln\ln t} < p < \frac{(1+\delta')\ln t}{2\ln\ln t}$$

for sufficiently large t. Then use the estimates for binomial coefficients we have proved to complete the proof.

Sol: Let q be the next largest prime after p. Then

$$2\binom{p^3}{p-1} < t < 2\binom{q^3}{q-1}.$$

Using standard estimates for binomial coefficients (see the similar calculations given in class), and the Prime Number Theorem, this yields

$$(2 - o(1))p\log p < \log t < (2 + o(1))q \log q < (2 + o(1))p \log p.$$

This implies that $\log t \sim 2p \log p$ and also that $\log\log t \sim \log p$. This yields

$$p = (1 + o(1))(\log t)/(2\log p) = (1 + o(1))(\log t)/(2\log\log t).$$

To prove the lower bound on $R(t,t)$, we construct a graph on $n = \binom{p^3}{p^2-1}$ vertices, with no clique or independent set of size $2\binom{p^3}{p-1} < t$. This was done in class. We only need to show that $n > t^{(1-\epsilon)\omega(t)}$. Lower bounding n and taking logs, this amounts to showing that $p^2 \log p > (1-\epsilon)\log^2 t/(4\log\log t)$. The bounds for p in terms of t derived above imply precisely this.

5) Let $K = \{k_1, k_2\}$ and $L = \{l_1, \ldots, l_s\}$ be two sets of nonnegative integers with $k_i > s-2$ for $i = 1, 2$. Let $\mathcal{F} \subseteq 2^{[n]}$ be an L-intersecting family with $|S| \in K$ for each $S \in \mathcal{F}$. Prove that

$$m = |\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1}.$$

Hint: Proceed as in the proof of the uniform RW Theorem presented in class. Instead of the function $(\sum_i x_i - k)$, use the function $(\sum_i x_i - k_1)(\sum_i x_i - k_2)$, and instead of letting $|I| \le s-1$, let $|I| \le s-2$.

Remark: This can be easily generalized to $K = \{k_1, \ldots, k_r\}$ (no need to do it), and then it provides a common proof of both the uniform and nonuniform RW Theorems (Alon-Babai-Suzuki 1991).

Sol: We prove the more general version in the Remark. Recall the following Lemma proved in class

Lemma: Let $f : \Omega \to \mathbf{R}$. Assume that $f(I) \neq 0$ for any $|I| \le r$. Then the set of functions $\{x_I f : |I| \le r\}$ is linearly independent.

Proof: Order the subsets of $[n]$ such that $I < J$ implies that $|I| \le |J|$. Then for $I, J \subseteq [n]$ with $|I|, |J| \le r$ we have $x_I(J)f(J) \neq 0$ if $I = J$ and $= 0$ if $J < I$. By the triangular criterion proved in class, we conclude that these functions are linearly independent.

Let $\mathcal{F} = \{A_1, \ldots, A_m\}$, where $|A_i| \le |A_{i+1}|$ for all i, and define the polynomials

$$f_i(x) = \prod_{k:\ l_k < |A_i|} (v_i \cdot x - l_k), \qquad (x \in \mathbf{R}^n),$$

where v_i is the incidence vector of A_i. Set

$$f = \prod_{i=1}^{r} \left(\sum_{j=1}^{n} x_j - k_i \right).$$

Then the Lemma implies that the set of functions $\{x_I f : |I| \le r\}$ is linearly independent. We next show that this set of functions together with the set $\{f_i : 1 \le i \le m\}$ is linearly independent. To prove this, suppose that

$$\sum_{i=1}^{m} \lambda_i f_i + \sum_{|I| \le s-r} \mu_I x_I f = 0.$$

We first argue that each $\lambda_i = 0$. If not, suppose that i_0 is the smallest i for which $\lambda_{i_0} \neq 0$. Substituting A_{i_0} above yields the contradiction $\lambda_{i_0} = 0$. Now it follows the all the μ_I are zero by the Lemma.

We may assume that all these function are multilinear. Thus we have $m + \sum_{i=0}^{s-r} \binom{n}{i}$ linearly independent functions, each of which can be represented by polynomials of degree at most s. Consequently $m \le \sum_{i=s-r+1}^{s} \binom{n}{i}$.