Homework #4 Solutions:

0) Continuation of 29.5 from previous homework. Show that
\[ n + \lfloor \log n \rfloor \leq \chi_2(f) \leq n + 2\lfloor \log n \rfloor. \]

Sol: The problem is equivalent to showing that \( \lfloor \log n \rfloor \leq \chi(G) \leq 2\lfloor \log n \rfloor \), where \( G \) is the graph with vertex set \( V = ([n] \times [n]) - \{(i,i) : i \in [n]\} \) and vertex \((x,y)\) is adjacent to \((a,b)\) iff \( a = y \) or \( b = x \). For the upper bound, suppose first that \( n = 2^k \). Divide the square \( V \) into four equal parts in the natural way. Color the bottom left and top right subsquares inductively with the same set of colors. Give a single new color to the bottom right subsquare, and another one to the top left subsquare. This is a proper coloring and it uses 2 more colors than are needed for \( n = 2^k - 1 \). Thus we can color in total by \( 2^k \) colors. In general, \( 2^{k-1} < n \leq 2^k \), so we can color the larger square with side length \( 2^k \) and restrict to \( V \).

For the upper bound, just focus on the portion of \( V \) above the diagonal. Let \( C_i \) be the \( i \)th column, and let \( S_i \) be the set of colors received by \( C_i \). Then \( S_i \neq S_j \) if \( i < j \), since the vertex \((i,j)\) is adjacent to all of \( S_j \). If we have a proper \( k \)-coloring, this immediately gives \( 2^k \geq n \), since different rows receive distinct subsets of colors. In other words, \( k \geq \lfloor \log n \rfloor \).

1) 7.1

Sol: The projective plane of order \( s - 1 \) is a weak delta system with \( s^2 - s + 1 \) sets. It is not a sunflower since the number of sets containing a fixed point is \( s < s^2 - s + 1 \).

2) 7.2

Sol: Suppose \( A_1, \ldots, A_k \) is a sunflower. There must be a \( V_i \) where the \( A_i \)'s are pairwise disjoint, but \( |V_i| < k \).

3) 7.5

Sol: Copy the proof of Lemma 7.1 verbatim. The only difference is to do induction on \( k \) starting with \( k = 2 \). The base case is trivial, since any two sets have the required property.

4) 7.8

Sol: Suppose \( G^s \) has a clique of size larger than \( s!(k-1)^s \). By the sunflower lemma, we have a sunflower with \( k \) petals. In \( G \), the union of these \( k \) sets is at most \( sk \), and since they form a clique in \( G^s \), there is an edge of \( G \) between every two petals. Consequently, there is a subset of \( G \) of size at most \( ks \) that spans at least \( \binom{k}{2} \) edges, contradiction.

5) Suppose \( \mathcal{F} \subset \binom{[n]}{s} \) is an intersecting family. Show that \( \mathcal{F} \) can be augmented to an intersecting family of size \( 2^n - 1 \).

Sol: We claim that for every set \( A \), we can add \( A \) or \([n] - A \) to \( \mathcal{F} \). For if \( A \) couldn't be added, then there is \( B \in \mathcal{F} \) such that \( A \cap B = \emptyset \). Similarly, there exists \( C \in \mathcal{F} \) such that \( ([n] - A) \cap C = \emptyset \). But this means that \( B \cap C = \emptyset \), contradiction. Hence, we may keep adding sets to \( \mathcal{F} \) as long as a set and its complement remain. This means that we can augment \( |\mathcal{F}| \) to \( 2^n - 1 \).