Solutions to Homework #2:

4) Find an infinite counterexample to the statement of the marriage theorem
Solution: Let $X = Z^+ \cup \{a\}$ and $Y = Z^+$ and join $x \in X$ to $y \in Y$ iff $x = y$ or $x = a$. Then Halls conditions clearly holds. On the other hand, a matching saturating X must saturate $Z^+ \subset X$, and since these vertices have degree 1, it cannot saturate a. Hence there is no matching saturating X.

5) Let k be an integer. Show that any two partitions of a finite set into k-sets admit a common choice of representatives.
Solution: Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be the two partitions. Form a bipartite graph with parts $\{A_i\}$ and $\{B_i\}$, and join A_i to B_j if they have an element in common. For any given collection of t A_i's, the number of elements in their union is tk, so the number of B_j''s covering these tk elements is at least t. Hence the number of neighbors of these t A_is in the bipartite graph is at least t, so Halls condition holds. By Halls theorem, we have a perfect matching. Each edge of this matching corresponds to an element of the ground set, and no two of these elements are the same since the A_is form a partition. Hence this matching gives a CSDR.

6) Let A be a finite set with subsets $A_1, \ldots, A_n$ and let $d_1, \ldots, d_n \in N$. Show that there are disjoint subsets $D_k \subset A_k$ with $D_k| = d_k$ for all $k \leq n$, if and only if $|\cup_{i \in I} A_i| \geq \sum_{i \in I} d_i$ for all $I \subset [n]$.
Solution: The condition is clearly necessary. Form a bipartite graph B with parts X, A, where $X = a_i^j$, for $i \in [n]$ and $j \in [d_i]$. Join $a_i^j \in X$ to $s \in A$ if $s \in A_i$. Then the given condition implies Halls condition in B, so B has a matching saturating all of X. The construction of B implies that we obtains the sets required in the problem.

13) Show that a graph G contains k independent edges if and only if $q(G - S) \leq |S| + |V(G)| - 2k$ for all sets $S \subset V(G)$.
Solution: Let $n = |V(G)|$ and form the graph G by adding $n - 2k$ new vertices each adjacent to all vertices of G. Then the condition of the problem corresponds to Tutte's condition on G', so by Tutte's theorem, G' has a perfect matching M. The number vertices of $V(G)$ that are matched to some other vertex of $V(G)$ is at least $n - (n - 2k) = 2k$, so we have at least k edges of M that lie entirely in G.

17) Does there exist a function $g(k)$ so that every multigraph with minimum degree at least 3 and at least $g(k)$ vertices contains k disjoint cycles?
Solution: No, let W_n be the graph obtained from C_n by adding a new vertex adjacent to all vertices of C_n – this is sometimes called the wheel. Then W_n has $n + 1$ vertices, minimum degree 3 and no two disjoint cycles.

18) Prove that the vertices of a graph G can be covered by at most $\alpha(G)$ disjoint subgraphs each isomorphic to a cycle, K_2, or K_1.
Solution: We can proceed by induction on $\alpha(G)$. The claim clearly holds for $\alpha(G) = 1$. Take a longest path P in G with endpoints u, v. If P has no edges, then the result holds trivially, so assume there is at least one edge in P. All edges incident with u are on P. Hence either $d_G(u) = 1$ or there is a cycle C such that $u \in V(C) \subset V(P)$ and u has no

edges to $G - C$ (by picking the furthest neighbor of u on P). Let $C' = C$ or the edge incident to u if $d_G(u) = 1$. Any independent set of $G - C'$ can be augmented by adding u to obtain an independent set in G, hence $\alpha(G - C') < \alpha(G)$. By induction, we can cover $G - C'$ and then cover G by adding C'.

21) Derive Hall's theorem from the Gallai-Milgram theorem.
Solution Suppose we are given the bipartite graph $B = X, Y$ with $|N(S)| \geq |S|$ for all $S \subset X$. Form a directed graph D by directing all edges of B from X to Y. Pick an independent set $I \subset V(D)$. Then $N_B(I \cap X) \subset Y - I$ so by Halls condition, $|Y - I| \geq |I \cap X|$. Then

$$|I| = |I \cap X| + |I \cap Y| \leq |Y - I| + |I \cap Y| = |Y|.$$

So by Gallai-Milgram, D can be covered by at most $|Y|$ directed paths. Each of these paths must have an endpoint in Y, so X is covered by paths of length at least 1. These paths provide a matching saturating all of X.