Two-regular subgraphs of hypergraphs

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Abstract

We prove that the maximum number of edges in a k-uniform hypergraph on n vertices containing no 2-regular subhypergraph is $\binom{n-1}{k-1}$ if $k \geq 4$ is even and n is sufficiently large. Equality holds only if all edges contain a specific vertex v. For odd k we conjecture that this maximum is $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$, with equality only for the hypergraph described above plus a maximum matching omitting v.

1 Introduction

One of the most basic facts in combinatorics is that an acyclic graph on n vertices has at most $n-1$ edges, with equality only for trees. A natural generalization to hypergraphs (see Berge [3] for more details) is obtained by defining a circuit to be a hypergraph consisting of distinct vertices $v_1, v_2, \ldots, v_k$ and distinct edges $e_1, \ldots, e_k$ such that $v_i \in e_i$ for $i = 1, 2 \ldots, k$, $v_{i+1} \in e_i$ for $i = 1, 2 \ldots, k-1$, and $v_1 \in e_k$. Then a hypergraph H with no circuit satisfies

$$\sum_{e \in H} (|e| - 1) \leq |V(H)| - 1.$$

In this paper, we consider a generalization to hypergraphs in a different direction. Since a cycle is a 2-regular graph, we may ask for the maximum number of edges that a hypergraph on n vertices can have without a 2-regular subgraph – i.e. a subhypergraph in which every vertex has degree two. Throughout the paper, hypergraphs where all edges have size k are called k-uniform hypergraphs or, simply, k-graphs. A star is a hypergraph in which there is a vertex v such that all possible edges containing v are present and there are no other edges. Our main result shows that stars are the extremal hypergraphs not containing a 2-regular subgraph when k is even:

Theorem 1. *For every even integer $k > 2$, there exists an integer n_k such that for $n \geq n_k$, if H is an n-vertex k-graph with no 2-regular subgraph, then $|H| \leq \binom{n-1}{k-1}$. Equality holds if and only if H is a star.*

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The non-uniform analog of this theorem, which is much simpler, is proved in Section 2. As one might expect, the proof of Theorem 1 needs completely new techniques than the graph case. The result is proved via the stability approach. Stability results were introduced in extremal graph theory by Erdős and Simonovits [15] in the 60's. The program of using stability to prove exact results has been recently used with great success in extremal set theory (see [5, 6, 7, 8, 9, 10, 11]). Perhaps the main difficulty in passing to an exact result when k is odd is that stars are not extremal when k is odd: it is possible to add to a star on n vertices a matching of size $\lfloor \frac{n-1}{k} \rfloor$, resulting in an n-vertex k-graph with no 2-regular subgraph with a few more edges than a star. We conjecture that this "star-plus-matching" construction is the unique extremal configuration when k is odd:

Conjecture 1. *For every odd integer $k \geq 3$, there exists an integer n_k such that for $n \geq n_k$, if H is an n-vertex k-graph with no 2-regular subgraph then $|H| \leq \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$. Equality holds if and only if H is a star with center v together with a maximal matching omitting v.*

Conjecture 1 is a weaker version of a conjecture due to Füredi, that for $k > 3$, a k-graph containing no two pairs of disjoint sets with the same union has at most $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$ edges. For odd $k > 3$, this implies Conjecture 1; in fact, a hypergraph consisting of two pairs of disjoint edges with the same union is the smallest possible 2-regular k-graph when k is odd. The question of determining the maximum number of edges $f_k(n)$ of a k-graph on n vertices containing no two pairs of disjoint edges with the same union was originally raised by Erdős (see [4]). This problem was studied by Frankl and Füredi [4], and the authors [12], who showed that $f_k(n) < 3\binom{n}{k-1}$. The best bounds are given in Pikhurko and the second author [13], where it is shown that $f_3(n) < \frac{13}{9}\binom{n}{2}$ and $f_k(n) < (1 + \frac{2}{\sqrt{k}})\binom{n}{k-1}$ for all k.

This paper is organized as follows. In the next section, we prove the non-uniform analog of Theorem 1, that a collection of subsets of an n-element set with no 2-regular subsystem has size at most 2^{n-1} with equality (for $n \geq 3$) only for a star. In fact, the same proof shows the nonuniform analogue of Füredi's conjecture, that the maximum size of a collection of nonempty subsets of $[n]$ containing no two pairs of disjoint sets with the same union is $2^{n-1} + 1$ (it is easy to see that there are many families achieving this bound, and hence there is no simple characterization of equality).

In Section 3, we present two lemmas used to prove Theorem 1. The proof of Theorem 1 is in Sections 4–6, and has three parts. First we shall show (see Section 4) that if H is an n-vertex k-graph with no 2-regular subgraph, then $|H| \lesssim \binom{n-1}{k-1}$. Using this result, we prove the stability result (see Section 5), which says that if $|H| \sim \binom{n-1}{k-1}$ then $\triangle(H) \sim \binom{n-1}{k-1}$. Finally, we use this stability theorem to prove Theorem 1 in Section 6. The final section mentions related open problems.

Terminology. A hypergraph is a family of subsets of a set of vertices, called edges. We denote by $|H|$ the number of edges in a hypergraph H. If H is a hypergraph, then $V(H)$ denotes the set of vertices. The degree of a vertex v, written $d(v)$, is the number of edges containing that vertex. A matching is a hypergraph in which every vertex has degree one – such a hypergraph M consists of pairwise disjoint edges $e_1, e_2, \ldots, e_m$ for some m and $V(M) = e_1 \cup e_2 \cup \cdots \cup e_m$. A k-graph is a hypergraph where all sets have size k, and a hypergraph is r-regular if all its vertices have degree r. We write $\binom{X}{k}$ for the collection of all k-sets of X. A star is a hypergraph on a vertex set X consisting of all possible edges containing a fixed vertex of X. In the context of k-graphs, a star consists of all possible k-sets containing a fixed vertex of X. For a hypergraph H, denote by $\triangle(H)$ its maximum degree. For $v \in V(H)$, let $H - \{v\} = \{e \in H : v \notin e\}$ and $H_v = \{e \setminus \{v\} : v \in e \in H\}$. If $f, g : \mathbb{N} \to \mathbb{R}$ are two functions then we write $f(n) \gtrsim g(n)$ to denote that $f(n) \geq g(n)h(n)$ for some function $h(n)$ such that $\liminf_{n\to\infty} h(n) = 1$. This is an equivalent but more convenient way to write $f(n) \geq (1 + o(1))g(n)$. In the case $f(n) = (1 + o(1))g(n)$ we write $f(n) \sim g(n)$. If there is a constant $c > 0$ such that $f(n) \geq cg(n)$ for all n, then we write $f(n) \gg g(n)$. Throughout this paper, k is always fixed relative to n.

2 Non-uniform hypergraphs

In this section, we prove the non-uniform analog of Theorem 1. We stipulate that edges of a hypergraph are non-empty sets. A star on n vertices is a hypergraph consisting of all 2^{n-1} sets containing a fixed vertex.

Theorem 2. *Let $n \geq 1$ and let H be a hypergraph on n vertices containing no 2-regular subgraph. Then $|H| \leq 2^{n-1}$. If $n \geq 3$ and equality holds, then H is a star.*

Proof. We remark that it is easy to obtain an upper bound 2^{n-1}: if H has no 2-regular subgraph, then H contains at most one complementary pair – a complementary pair consists of the edge e and the edge $V(H)\backslash e$. This shows $|H| \leq 2^{n-1} + 1$, but if H contains both edges of some complementary pair, then $V(H)$ cannot be an edge of H, showing $|H| \leq 2^{n-1}$. For the characterization of equality, we proceed by induction on n for $n \geq 3$.

It is straightforward to check the case $n = 3$; we omit the details. Now we proceed to the induction step. Let us assume that $n \geq 4$ and H has size 2^{n-1} and no 2-regular subgraph. We will show that H is a star, which proves Theorem 2. First we show that every vertex of H, apart from at most one vertex, has degree exactly 2^{n-2}. If there is a vertex $v \in V(H)$ with $d(v) < 2^{n-2}$, then $H - \{v\}$ has a 2-regular subgraph, by induction. So every vertex of H has degree at least 2^{n-2}. Pick a vertex $x \in V(H)$. If x is contained in every set in H, then H is a star with center x and all other vertices have degree 2^{n-2}. We may therefore assume that there exists an $e \in H$ missing x. Assume that $|e| = k$ where $1 \leq k \leq n$. For each subset $f \subset V(H)\backslash(e \cup \{x\})$, the number of edges in H containing x whose intersection with $V(H)\backslash(e \cup \{x\})$ is f is at most 2^{k-1}, for otherwise two of these edges have complementary intersections in e and these together with e give a 2-regular subgraph, a contradiction. Hence the number of edges containing x is at most $2^{n-k-1}2^{k-1} = 2^{n-2}$. So x has degree exactly 2^{n-2}, in which case $|H - \{x\}| = 2^{n-2}$. By induction, $H - \{x\}$ is a star with center at some vertex w. Suppose, for a contradiction, that there exist distinct edges e, f containing x but not w. Then the edges

$$e, \quad f, \quad \{w\} \cup (e\backslash f), \quad \{w\} \cup (f\backslash e)$$

form a 2-regular subgraph of H, a contradiction. So at most one edge containing x does not contain w. If such an edge e exists, then pick an edge f containing x and w – this is possible since x has degree $2^{n-2} \geq 2$. It follows that

$$e, \quad f, \quad e \triangle f$$

is a 2-regular subgraph of H. So we have shown that all edges containing x must also contain w. Therefore H is a star with center w. □

3 Preliminary Lemmas

In this section, we present two lemmas which will be used in proving Theorem 1. The first lemma involves matchings. If M_1 and M_2 are distinct matchings and $V(M_1) = V(M_2)$, then $M_1 \triangle M_2$ is a hypergraph whose vertices all have degree two. This observation is the key point of the following lemma.

Lemma 1. *Let H be a k-graph on n vertices containing no 2-regular subgraph. Then $|H| \leq 6n\Delta^{(k-1)/k}$ or $|H| < 2k\triangle$.*

Proof. Let $d = k|H|/n$ and suppose $|H| \geq 2k\triangle$. Then it is enough to prove that $\triangle \geq (1/k)(d/6)^{k/(k-1)}$ to prove the lemma, for this implies the second inequality in

$$|H| \leq 6n \left(\frac{1}{k} \left(\frac{d}{6} \right)^{k/(k-1)} \right)^{(k-1)/k} \leq 6n\Delta^{(k-1)/k}.$$

Suppose, for a contradiction, that this is not true. We count matchings in H of size $m = \lfloor |H|/k\triangle \rfloor$ to show that H contains a 2-regular subgraph. Note that $m \geq 2$ since $|H| \geq 2k\triangle$. For a lower bound on the number of matchings of size m, we may greedily pick disjoint edges $f_1, f_2, \ldots, f_m$ where at each step we exclude all edges that intersect previously chosen edges. Since at each step we exclude at most $k\Delta$ new edges, the number of matchings of size m in H is at least

$$\frac{1}{m!} \prod_{i=0}^{m-1} (|H| - k\Delta i) = \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} \left(1 - \frac{k\triangle i}{|H|}\right) \geq \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} \left(1 - \frac{i}{m}\right) \geq (k\triangle)^m.$$

To complete the proof, we show that there exist distinct matchings M_1, M_2 of H such that $\bigcup_{f \in M_1} f = \bigcup_{f \in M_2} f$. This suffices, since the edges in $M_1 \triangle M_2$ form a 2-regular subgraph, contradicting the fact that H has no 2-regular subgraph. First note that

$$\binom{n}{mk} < \left(\frac{3n}{mk}\right)^{mk} \leq \left(\frac{6k\triangle}{d}\right)^{km} < (k\triangle)^m.$$

Here we used $m \geq dn/2k^2\triangle$ and then the assumed upper bound on $\triangle$. Since $\binom{n}{mk}$ is the number of sets of mk vertices of H, and there are more than $(k\triangle)^m$ matchings of size m in H, we find the two required distinct matchings M_1, M_2. □

Our second lemma involves circuits in hypergraphs. A circuit is a hypergraph consisting of distinct vertices $v_1, v_2, \ldots, v_k$ and distinct edges $e_1, \ldots, e_k$ such that $v_i \in e_i$ for $i = 1, 2 \ldots, k$,

$v_{i+1} \in e_i$ for $i = 1, 2 \ldots, k-1$, and $v_1 \in e_k$. We require the following lemma on 2-regular subgraphs arising from circuits in hypergraphs of a certain bipartite structure:

Lemma 2. *Let G be a k-graph and $V(G) = A \cup B$, where $A \cap B = \emptyset$, all edges $e \in G$ have $|e \cap A| = k-1$, and every $(k-1)$-set in A lies in at least two edges of G. If G has no 2-regular subgraph, then*

$$|G| < 2|B|\binom{|A|+k-3}{k-2}.$$

Proof. It is enough to show $|G| < 2|B|\binom{|A|-1}{k-2}$ when $k-1$ divides $|A|$, since we may always add at most $k-2$ points to A so that $k-1$ divides $|A|$. Baranyai's Theorem [2] states that if s divides n, then the complete s-graph on n vertices can be partitioned into $\binom{n-1}{s-1}$ perfect matchings. Using this theorem with $s = k-1$, we write

$$\binom{A}{k-1} = M^1 \cup M^2 \cup \cdots \cup M^{\binom{|A|-1}{k-2}}$$

where each M^i is a matching and the matchings are edge-disjoint. For each matching $M^i = \{e_1^i, \ldots, e_a^i\}$, let G^i be the set of edges in G whose intersection with A is e_j^i for some j. Let f_j^i be the set of vertices $v \in B$ such that $e_j^i \cup \{v\} \in G^i$. Consider the hypergraph H_j^i with edges f_j^i, $j = 1, \ldots, a$. If H_j^i contains a circuit with vertices $v_1, v_2, \ldots, v_p$, then G contains the 2-regular subgraph with edges

$$e_1^i \cup \{v_1\} \quad e_1^i \cup \{v_2\} \quad e_2^i \cup \{v_2\} \quad e_2^i \cup \{v_3\} \quad e_p^i \cup \{v_p\} \quad e_p^i \cup \{v_1\}$$

which contradicts that G has no 2-regular subgraph. Consequently, H_j^i has no circuit. It is well-known that a hypergraph H with no circuit satisfies

$$\sum_{e \in H} (|e| - 1) \leq (|V(H)| - 1). \tag{1}$$

Since every $(k-1)$-set in A lies in at least two edges of G, $|f_j^i| \geq 2$ for all i, j. Applying (1) to H_j^i, we therefore obtain

$$\sum_j |f_j^i| \leq \sum_j 2(|f_j^i| - 1) < 2|B|. \tag{2}$$

Adding (2) over different i, j, we obtain

$$|G| = \sum_i \sum_j |f_j^i| \leq 2|B|\binom{|A|-1}{k-2}. \quad \square$$

4 The Asymptotic Result

Theorem 3. *Let $k \geq 3$ and let H be an n-vertex k-graph with no 2-regular subgraph. Then*

$$|H| - \binom{n-1}{k-1} \ll n^{k-1-1/11}.$$

Proof. We prove the following more precise statement: for all $n > k^{100}$,

$$|H| < \left\lfloor (1+cn^{-\gamma})\binom{n-1}{k-1} \right\rfloor$$

where $c = 4(k+1)!$ and $\gamma = \frac{1}{11}$. Define $\alpha = (k+1)/(3k-1)$ for $k > 3$ and $\alpha = 7/11$ for $k = 3$. Suppose, for a contradiction, that $|H|$ is at least this upper bound for some H. By deleting some edges, we may assume that $|H|$ is equal to the stated upper bound. Let T denote the set of vertices of H of degree at least $D = n^{k-1-\alpha}$, and set $t = |T|$. Then $tD \leq k|H|$ and, since $n > k^{100}$,

$$t < D^{-1}k(1+cn^{-\gamma})\binom{n-1}{k-1} < kn^{\alpha}. \qquad (3)$$

Let $H_i = \{e \in H : |e \cap T| = i\}$ for $i \leq k$, and define $G = \{e \in H_1 : \nexists f \in H_1 : e \setminus T = f \setminus T\}$. In particular, it is clear that $|G| \leq \binom{n-1}{k-1}$.

Claim 1. $|H_i| < \begin{cases} 6n^{1+(k-1)(k-1-\alpha)/k} & \text{for } i = 0 \\ |G| + 2kn^{k-2+\alpha} & \text{for } i = 1 \end{cases}$

Proof. Since $\triangle(H_0) < D$, by definition of T, the first bound follows from Lemma 1. For the second bound, we apply Lemma 2 to $H_1 \backslash G$ with $A = V(H)\backslash T$ and $B = T$ to obtain $|H_1 \backslash G| < 2|T|\binom{n+k-3}{k-2} < 2tn^{k-2}$. The bound on $|H_1|$ now follows from (3).

Claim 2. $|H\backslash(H_0 \cup H_1)| < \begin{cases} k^2 n^{k-2+2\alpha} & \text{for } k > 3 \\ 6(n^{1+\alpha} + n^{3\alpha}) & \text{for } k = 3 \end{cases}$

Proof. For $k > 3$, by definition, every edge in $H\backslash(H_0 \cup H_1)$ contains two vertices of T and $k-2$ vertices of $V(H)$, so certainly $|H\backslash(H_0 \cup H_1)| \leq \binom{|T|}{2}n^{k-2}$. Now apply (3). For $k = 3$, observe that $|H_3| < \binom{|T|}{3}$. Furthermore, by Lemma 2, with $A = T$ and $B = V(H)\backslash T$, $|H_2| < 2|T|(n-|T|) + \binom{|T|}{2} < 2tn$. Here we note that there could be $\binom{|T|}{2}$ pairs in T contained in only one triple of H_2. Those contained in two triples or more are the ones to which Lemma 2 applies, giving the bound $2|T|(n-|T|)$ for those triples. Using (3) gives the claim.

Now we complete the proof. By definition of α, the bounds in Claims 1 and 2 are all of order at most $n^{k-1-\gamma}$ (the case $i = 0$ in Claim 1 needs a somewhat tedious calculation). Specifically,

$$|H\backslash G| = |H_0| + |H_1\backslash G| + |H\backslash(H_0 \cup H_1)| < (6 + k^2 + 2k)n^{k-1-\gamma} < 4k^2 n^{k-1-\gamma}. \qquad (4)$$

Using the bound $|G| \leq \binom{n-1}{k-1}$ in (4), we obtain

$$|H| = |G| + |H\backslash G| < \binom{n-1}{k-1} + 4k^2 n^{k-1-\gamma} < \left\lfloor (1 + 4(k+1)!n^{-\gamma})\binom{n-1}{k-1} \right\rfloor.$$

The constant $c = 4(k+1)!$ appears here: we used the fact that $4k^2 n^{k-1-\gamma} < 4(k+1)!n^{-\gamma}\binom{n-1}{k-1}$ for $n > k^{100}$. This contradiction completes the proof. □

5 Stability

Theorem 4. *Let $k \geq 3$ and let H_n be an n-vertex k-graph with no 2-regular subgraph. If $|H_n| \sim \binom{n-1}{k-1}$, then $\triangle(H_n) \sim \binom{n-1}{k-1}$.*

Proof. For simplicity of notation, we let $H = H_n$ and omit the subscript n when dealing with hypergraphs constructed from H. As in the proof of Theorem 3, let T denote the set of vertices in H of degree at least $n^{k-1-\alpha}$, $H_1 = \{e \in H : |e \cap T| = 1\}$ and $G = \{e \in H_1 : \nexists f \in H_1 : e \setminus T = f \setminus T\}$. Define

$$G' = \{e \setminus T : e \in G\}.$$

For each $x \in T$, let $G_x = \{e \in G' : e \cup \{x\} \in G\}$. Let v be a vertex such that $|G_v| = \max_{x \in T} |G_x|$. Note that all sets in G have size k, and all sets in G' or any G_x have size $k-1$. By (4), $|G'| = |G| \sim |H| \sim \binom{n-1}{k-1}$, so it suffices to prove that $|G_v| \sim |G'|$. Suppose, for a contradiction, that for some positive $\varepsilon < \frac{1}{2}$,

$$|G_v| \lesssim (1-\varepsilon)|G'|. \tag{5}$$

The strategy is to use (5) to derive a contradiction by finding edges $e, e' \in G_x$ and $f, f' \in G_y$, for some $x \neq y$, such that $|e \cap f| = 1 = |e' \cap f'|$, $e \triangle f = e' \triangle f'$ and $e \cap f \neq e' \cap f'$ (sometimes the latter condition will be guaranteed by $e \cap e' = \emptyset = f \cap f'$). For in this case, the edges

$$e \cup \{x\} \quad e' \cup \{x\} \quad f \cup \{y\} \quad f' \cup \{y\} \tag{6}$$

form a 2-regular subgraph of H.

For any hypergraph F, define $P(F) = \{\{e, f\} \subset F : |e \cap f| = 1\}$. Define $P_1(G') \subset P(G')$ to be the set of pairs $\{e, f\} \in P(G')$ such that $e, f \in G_x$ for some x, and $P_2(G') = P(G') \backslash P_1(G')$.

Claim 1. $|P_2(G')| \leq \frac{1}{2}\binom{t}{2}\binom{2k-4}{k-2}\binom{n-1}{2k-4}$.

Proof. Fix distinct vertices $x, y \in T$. We show that the number of $\{e, f\} \in P_2(G')$ such that $e \in G_x$ and $f \in G_y$ is at most $\frac{1}{2}\binom{2k-4}{k-2}\binom{n-1}{2k-4}$. This completes the proof, since there are $\binom{t}{2}$ choices for x and y.

Given a set S of size $2k-4$, let us count the number of pairs $\{e, f\} \in P_2(G')$ with $e \triangle f = S$ that satisfy $e \in G_x$ and $f \in G_y$. Suppose that we have at least one such pair $\{e, f\}$ with $e \cap f = \{z\}$. Any other such pair $\{e', f'\}$ must also satisfy $e' \cap f' = \{z\}$, otherwise the four edges $e \cup \{x\}, e' \cup \{x\}, f \cup \{y\}, f' \cup \{y\}$ form a 2-regular subgraph. Hence the number of such pairs is at most the number of (unordered) partitions of S into two sets of size $k-2$, which is $(1/2)\binom{2k-4}{k-2}$. The number of ways to choose S is at most $\binom{n-1}{2k-4}$. Putting this all together we obtain the required bound in the claim.

For the rest of the proof, let $\psi(\varepsilon) = ((1-\varepsilon)^2 + \varepsilon^2)^{1/2}$. For $i \in \{1, 2\}$, let $Q_i(G')$ denote the set of pairs $\{\{e, f\}, \{e', f'\}\}$ such that $\{e, f\}, \{e', f'\} \in P_i(G')$, $e \cap e' = \emptyset = f \cap f'$ and $e \triangle f = e' \triangle f'$. These are called type i quadrilaterals of G'. For $x \in T$, define $Q_1(G_x)$ to be the collection of pairs $\{\{e,f\},\{e',f'\}\} \in Q_1(G')$ such that $\{e,f,e',f'\} \subset G_x$. These are type 1 quadrilaterals of G_x. Let K be the complete $(k-1)$-graph on $V(G')$. Recall that $P(K)$ is the number of pairs $\{e,f\} \subset K$ such that $|e \cap f| = 1$. So in the case that $k = 2$, when K is the complete graph, this is just the number of paths of length two. More generally, we have

$$|P(K)| \sim \frac{1}{2}(k-1)\binom{n-1}{k-1}\binom{n-1}{k-2}. \tag{7}$$

Claim 2. $|P_1(G')| \lesssim \psi(\varepsilon) \cdot |P(K)|$.

Proof. Let $\{\{e,f\},\{e',f'\}\} \in Q_1(G')$. If $e,f \in G_x$ and $e',f' \in G_y$ with $x \neq y$, then we obtain a 2-regular subgraph similar to that in (6). We conclude that if $e,f \in G_x$, then also $e',f' \in G_x$. It follows that

$$|Q_1(G')| = \sum_{x \in T} |Q_1(G_x)|. \tag{8}$$

For a pair $\{g,h\}$ of disjoint sets of size $k-2$ in $V(G')$, let $p(g,h)$ denote the number of pairs $\{e,f\} \in P_1(G')$ with $e \backslash f = g$ and $f \backslash e = h$. The number of such pairs $\{g,h\}$ is at most

$$\binom{\binom{n-1}{k-2}}{2} := N.$$

Note also that the sum of $p(g,h)$ over all $\{g,h\} \subset V(G')$ is exactly $|P_1(G')|$. By convexity of binomial coefficients,

$$|Q_1(G')| \;=\; \sum_{\{g,h\}} \binom{p(g,h)}{2} \;\gtrsim\; \binom{|P_1(G')|/N}{2} \cdot N \;\sim\; \frac{|P_1(G')|^2}{\binom{n-1}{k-2}^2}. \tag{9}$$

The first equality is the hypergraph analog of the fact that the number of quadrilaterals in a graph F is exactly $\sum_{u,v \in V(F)} \binom{p(u,v)}{2}$ where $p(u,v)$ is the number of paths of length two from u to v in F. On the other hand, we observe that $|Q_1(G_x)| \leq \frac{1}{2}(k-1)^2\binom{|G_x|}{2}$, since if we fix two disjoint edges, say $e,e' \in G_x$, then the number of type 1 quadrilaterals of the form $\{\{e,f\},\{e',f'\}\}$ is at most $(k-1)^2$. The same type 1 quadrilaterals are counted if we had fixed the two disjoint edges $f,f' \in G_x$ instead of e,e', and this gives the additional factor of 2 in the observation. Therefore, by (8),

$$|Q_1(G')| \leq \frac{1}{2}(k-1)^2 \sum_{x \in T} \binom{|G_x|}{2}.$$

By convexity, this sum is a maximum when $|G_v| \sim (1-\varepsilon)|G'|$ and $|G_w| \sim \varepsilon|G'|$ for some $w \neq v$, and the rest of the $|G_x|$s are zero. Therefore

$$|Q_1(G')| \lesssim \frac{1}{4}(k-1)^2\psi(\varepsilon)^2|G'|^2. \tag{10}$$

Combining (9), (10), $|G'| \sim \binom{n-1}{k-1}$, and (7) we obtain

$$|P_1(G')| \lesssim \psi(\varepsilon) \cdot \frac{1}{2}(k-1)|G'|\binom{n-1}{k-2} \lesssim \psi(\varepsilon)|P(K)|.$$

This proves Claim 2.

The next claim is intuitively obvious since $|G'| \sim |K| \sim \binom{n-1}{k-1}$. We present a formal proof below.

Claim 3. $|P(G')| \sim |P(K)|$.

Proof. We note that $|P(K)| = n\binom{\binom{n-1}{k-2}}{2}$, since we may choose any vertex and two disjoint $(k-1)$-sets containing it. Let d_x be the number of sets in $K \backslash G'$ which contain $x \in V(G')$. Then

$$\sum_{x \in V(G')} d_x = (k-1)|K \backslash G'|.$$

Using this we obtain

$$\begin{aligned} |P(K) \backslash P(G')| &\leq \sum_{x \in V(G')} \binom{d_x}{2} + \sum_{x \in V(G')} d_x \left(\binom{n-2}{k-2} - d_x \right) \\ &= (k-1)|K \backslash G'| \binom{n-2}{k-2} - \frac{1}{2} \sum_{x \in V(G')} d_x^2 - \frac{1}{2}(k-1)|K \backslash G'|. \end{aligned} \tag{11}$$

Now since $|P(K)|$ is of order n^{2k-3}, and $|G'| \sim |K|$, we see that all terms in (11) are negligible relative to $|P(K)|$, except possibly the sum of d_x^2. We wish to find

$$\max \sum_{x \in V(G')} d_x^2 \quad \text{if} \quad \sum_{x \in V(G')} d_x = |K \backslash G'|.$$

The maximum possible value of d_x is $\binom{n-2}{k-2}$. For a maximum of the sum of squares, we let

$$\frac{(k-1)|K \backslash G'|}{\binom{n-2}{k-2}}$$

of the d_x take the value $\binom{n-2}{k-2}$, and the rest are zero (note that for a maximum, it is not necessary that there exist a hypergraph $K \backslash G'$ realizing these values of d_x). Therefore

$$\max \sum_{x \in V(G')} d_x^2 \leq (k-1)|K \backslash G'| \binom{n-2}{k-2}$$

and again this is negligible relative to $|P(K)|$ since $|K| \sim |G'|$ and $|P(K)|$ has order n^{2k-3}. This proves the claim.

We complete the proof of Theorem 4 for $k > 3$. By (3), $t \leq kn^{\alpha}$ where $\alpha < \frac{1}{2}$ (this relies on $k > 3$). Therefore Claims 1,2, and 3 imply that

$$\begin{aligned} |P(K)| \sim |P(G')| &= |P_1(G')| + |P_2(G')| \\ &\lesssim \psi(\varepsilon)|P(K)| + \binom{t}{2}\binom{2k-4}{k-2}\binom{n}{2k-4} \\ &\sim \psi(\varepsilon)|P(K)|. \end{aligned} \tag{12}$$

However, $\psi(\varepsilon) = ((1-\varepsilon)^2 + \varepsilon^2)^{1/2}$ is bounded away from 1, so the above inequality is a contradiction.

For $k = 3$, G' is a graph and $P(G')$ is the set of paths of length two in G'. The problem with the above arguments for $k = 3$ is that (3) only gives $t \leq 3n^{7/11}$, which is too large for (12) to hold (since $\binom{t}{2}\binom{n}{2k-4}$ has order $n^{3+3/11}$). Therefore we go one step further, and count paths of length three in G' instead of paths of length two. Let $P_3(G')$ be the number of paths of length three in G' with edges from three different G_xs. By Claims 2 and 3

$$|P_2(G')| = |P(G')| - |P_1(G')| \gtrsim (1 - \psi(\varepsilon))|P(K)| \gg n^3. \tag{13}$$

As in Claim 1, if $\{\{e, f\}, \{e', f'\}\}$ is a type 2 quadrilateral of G and $e, e' \in G_x$ and $f, f' \in G_y$, then we obtain a 2-regular subgraph of H. So each type 2 quadrilateral contains edges from at least three different G_xs, and these edges form a path of length three in G'. Consequently, as in (9), the convexity of binomial coefficients and (13) give

$$|P_3(G')| \geq \frac{1}{4}|Q_2(G')| \geq \frac{1}{4}\binom{|P_2(G')|/N}{2}N \gg n^4$$

since $N = \binom{n-1}{2}$. Let (A, B) be a random partition of $V(G')$, defined by placing a vertex in A with probability $\frac{1}{2}$ and in B with probability $\frac{1}{2}$, independently for each vertex of $V(G')$. Let G^* denote the graph consisting of all edges between A and B. Then the expected value of $|P_3(G^*)|$ is exactly $\frac{1}{8}|P_3(G')|$, so there is a partition of G' for which

$$|P_3(G^*)| \geq \frac{1}{8}|P_3(G')| \gg n^4. \tag{14}$$

Let $e_1e_2e_3$ and $f_1f_2f_3$ be two paths in G^* with the same pair of endpoints. Suppose $e_i \in G_{j(i)}$ and $f_i \in G_{h(i)}$ where $\{j(1), j(2), j(3)\} = \{h(1), h(2), h(3)\}$. Since G^* is bipartite, amongst these edges there is a cycle C of length four or six containing exactly zero or two edges from each $G_{j(i)}$, $i = 1, 2, 3$. It is easily checked that the unique edges of H' which contain the edges of C form a 2-regular subgraph of H, which is a contradiction. We conclude that at most $\binom{t}{3}$ paths of length three in G^* with edges in different G_is have the same pair of endpoints. It follows that

$$|P_3(G^*)| \leq \binom{t}{3}\binom{n}{2} \ll n^{4-\frac{1}{11}}$$

using (3). This contradicts (14), and completes the proof of Theorem 4. □

6 The Exact Result

In this section we prove Theorem 1. Our main tools are the asymptotic and stability result. Let H be an n-vertex k-graph containing no 2-regular subgraph, where $k \geq 4$ is even, and suppose $|H| = \binom{n-1}{k-1}$. Let $\varepsilon = \frac{1}{100k^{4k}}$. By Theorem 4, for large enough n, there is a vertex $v \in V(H)$ such that

$$|H - \{v\}| \leq \varepsilon n^{k-1}. \tag{15}$$

Let $H^* = H - \{v\}$. To complete the proof, we show $|H^*| = 0$. Suppose, for a contradiction, that $|H^*| > 0$. For $|e| = k-2$, let $d_v(e)$ be the number of sets in H_v containing e. Let

$$s = n - k + 1 - \frac{2k|H^*|}{\binom{n-1}{k-2}}.$$

Claim 1. There are pairwise disjoint $(k-2)$-sets $e_1, e_2, \dots, e_k \subset V(H)\backslash\{v\}$ such that $d_v(e_i) \geq s$ for each $i \in \{1, 2, \dots, k\}$.

Proof. Let F be the family of $(k-2)$-sets in $V(H_v)$ whose degree is at least s, and let F^c be the rest of the $(k-2)$-sets in $V(H_v)$. Then

$$(k-1)|H_v| = \sum_e d_v(e) \leq |F|(n-k+1) + |F^c|s.$$

where the sum is over $e \subset V(H_v)$ of size $k-2$. As $|F| + |F^c| = \binom{n-1}{k-2}$, this implies

$$\frac{2k|H^*||F|}{\binom{n-1}{k-2}} \geq (k-1)|H_v| - s\binom{n-1}{k-2} = 2k|H^*| - (k-1)|H^*|$$

since $|H^*| = \binom{n-1}{k-1} - |H_v|$. Hence $|F| \geq \left(1 - \frac{k-1}{2k}\right)\binom{n-1}{k-2} > \frac{1}{2}\binom{n-1}{k-2}$. Let $\{e_1, e_2, \dots, e_l\}$ be a maximum matching in F. If $l < k$, then all other sets of F have an element within $e_1 \cup e_2 \cup \dots \cup e_l$, which implies (since we may take n large enough) that

$$|F| \leq (k-1)(k-2)\binom{n-1}{k-3} < k^2\binom{n-1}{k-3} < \frac{1}{2}\binom{n-1}{k-2}.$$

This contradiction shows that $l \geq k$ and the claim is proved.

Let $W = \{w \in V(H_v) \mid \exists i : e_i \cup \{v, w\} \notin H\}$. By Claim 1, $|W| < k(n-s)$. By adding points arbitrarily to W, we may assume that $|W| = \lceil k(n-s) \rceil$. Define, for each $i \in \{0, 1, \dots, k\}$, $H_i = \{e \in H^* : |e \cap W| = i\}$ and let $G = H_0 \cup H_1 \cup \dots \cup H_{k-2}$. Note that the H_i partition H^*.

Claim 2. $|H_{k-1}| \leq \binom{|W|}{k-1}$.

Proof. Suppose there exists a $(k-1)$-set $e \subset W$ and elements $y, z \notin W$ such that $e \cup \{y\}, e \cup \{z\} \in H_{k-1}$. Since $|e| = k-1$, by Claim 1 and the definition of W there exists i such that $e_i \cap e = \emptyset$ and $e_i \cup \{v, y\}, e_i \cup \{v, z\} \in H$. Together with $e \cup \{y\}$ and $e \cup \{z\}$, this yields a 2-regular subgraph in H. This contradiction implies that we may count sets in H_{k-1} by their intersection with W to obtain $|H_{k-1}| \leq \binom{|W|}{k-1}$.

Claim 3. $|H^*| \geq \binom{n-k-1}{k/2-1}$.

Proof. Since $|H^*| \geq 1$, there exists $e \in H^*$. Let e' be a $\frac{k}{2}$-subset of e. Now for each choice of a $(\frac{k}{2}-1)$-set $f \subset V(H_v)\backslash e$, one of the sets $f \cup e' \cup \{v\}$ or $f \cup (e\backslash e') \cup \{v\}$ must be missing from H, otherwise these two sets together with e form a 2-regular subgraph of H. Consequently, $|H^*| \geq \binom{n-k-1}{k/2-1}$.

Claim 4. $|G| > \frac{99}{100}|H^*|$.

Proof. We show $|H_{k-1}|+|H_k| < \frac{1}{100}|H^*|$. By Theorem 3, there is a smallest integer $n_0 = n_0(k)$ such every k-graph on n vertices with no 2-regular subgraph and with $n > n_0$ has at most $2\binom{n_0-1}{k-1}$ edges. Assume also that $n_0 > 3k^2$. If $|W| < n_0$, then $|H_k| + |H_{k-1}| < |W|^k < n_0^k$. If n is large enough then, by Claim 3, this is less than $\frac{|H^*|}{100}$, as required. So we assume $|W| > n_0$. Since the k-graph H_k itself contains no 2-regular subgraph, $|H_k| \le 2\binom{|W|-1}{k-1}$. Recall that

$$|W| = \lceil k(n-s) \rceil = \left\lceil k\left(k-1+\frac{2k|H^*|}{\binom{n-1}{k-2}}\right)\right\rceil < k^2 + \frac{2k^2|H^*|}{\binom{n-1}{k-2}}.$$

Using this and $|W| > n_0 > 3k^2$, we obtain

$$|W| < \frac{3}{2}\frac{2k^2|H^*|}{\binom{n-1}{k-2}} = \frac{3k^2|H^*|}{\binom{n-1}{k-2}}.$$

Now suppose, for a contradiction, that $|H_k| + |H_{k-1}| > \frac{|H^*|}{100}$. By Claim 2,

$$\frac{|H^*|}{100} < |H_k| + |H_{k-1}| < 2\binom{|W|-1}{k-1} + \binom{|W|}{k-1} < \frac{3|W|^{k-1}}{(k-1)!} < \frac{k^{2k}|H^*|^{k-1}}{\binom{n-1}{k-2}^{k-1}}.$$

Simplifying,

$$|H^*|^{k-2} > \binom{n-1}{k-2}^{k-1}\frac{1}{100k^{2k}} > \left(\frac{n-1}{k-2}\right)^{(k-2)(k-1)}\frac{1}{100k^{2k}} > \frac{(n-1)^{(k-2)(k-1)}}{100k^{(k-2)(k-1)+2k}}.$$

This implies that $|H^*| > \frac{(n-1)^{k-1}}{100k^{k-1+2k/(k-2)}} > \varepsilon n^{k-1}$, which contradicts (15). This completes the proof of Claim 4.

Let p be the number of pairs (e, f) such that

(1) $v \notin e \in H$ and $|e \cap W| \le k-2$ (i.e. $e \in G = \cup_{i=0}^{k-2} H_i$)
(2) $v \in f \notin H$ and $|f| = k$ (so the number of such fs is $|H^*|$)
(3) $|e \cap f| = \frac{k}{2}$
(4) $e \cap f$ and $e \backslash f$ (which are both $\frac{k}{2}$-sets) have a point outside W.

Fix $e \in H$ as in (1) above. Since $|e \setminus W| \ge 2$, there is a $\frac{k}{2}$-subset $g \subset e$ such that neither g nor $e \backslash g$ lies within W. Let h be a $(\frac{k}{2}-1)$-subset of $V(H) \backslash (W \cup e \cup \{v\})$ and let $f = g \cup h \cup \{v\}$. Then the three sets $e, f, (e\backslash g) \cup h \cup \{v\}$ form a 2-regular subgraph. Consequently, either $g \cup h$ or $(e \backslash g) \cup h$ is not in H_v. The number of pairs $\{g, e\backslash g\}$ that we can take in this argument is at least

$$\frac{1}{2}\binom{k}{k/2} - \binom{|W \cap e|}{k/2} \ge \frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2}.$$

Therefore, counting p from the e's we have

$$p \ge (0.99)|H^*|\left(\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2}\right)\binom{n-|W|-k-1}{k/2-1}$$

$$> (0.98)|H^*|\left(\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2}\right)\binom{n}{k/2-1},$$

where the last inequality holds since $|W| < \varepsilon k^{4k} n$ and n is sufficiently large.

On the other hand, counting p from the fs we have $p \leq |H^*|\binom{k-1}{k/2}q$, where q is the number of different ways the $\frac{k}{2}$-sets $g \subset f\backslash\{v\}$ can extend to e, where $e \cap f = g$. Let F be the $\frac{k}{2}$-graph of these possible extensions of g to e. Let $F_0 \subset F$ be the $\frac{k}{2}$-graph whose edges have no points in W and $F_1 \subset F$ be the $\frac{k}{2}$-graph whose edges have at least one point in W.

Claim 5. $|F_0| < 2\binom{n}{k/2-1}/k$ and $|F_1| \leq \varepsilon k^{2k}\binom{n}{k/2-1}$.

Proof. We start with F_1: to each $\frac{k}{2}$-set $h \in F_1$ associate a $(\frac{k}{2}-1)$-set $h' \subset h$ such that $h' \cap W \neq \emptyset$ and $W \cap h \subset h'$. Such an h' exists by the definition of F_1. If there are distinct $h_1, h_2 \in F_1$ with $h_1' = h_2'$, then there are distinct vertices $y, z \notin W$ such that $h_1 = h_1' \cup \{y\}$ and $h_2 = h_2' \cup \{z\}$. By Claim 1, there exists i for which e_i has no point of $W \cap (g \cup h_1 \cup h_2)$. Now the four sets $g \cup h_1, g \cup h_2, e_i \cup \{v, y\}, e_i \cup \{v, z\}$ form a 2-regular subgraph of H, contradicting that H has no 2-regular subgraph. Consequently, $|F_1|$ is at most the number of $(\frac{k}{2}-1)$-sets of $V(H)$ that contain at least one point of W. This is at most

$$|W|\binom{n}{k/2-2} < \frac{3k^2\varepsilon n^{k-1}}{\binom{n-1}{k-2}}\binom{n}{k/2-2} < \varepsilon k^{2k}\binom{n}{k/2-1}.$$

This gives the bound on $|F_1|$. If there are distinct $h_1, h_2 \in F_0$ with $|h_1 \cap h_2| = k/2-1$, then arguing as above we find a 2-regular subgraph of H. Consequently, $|F_0| < \binom{n}{k/2-1}/\binom{k/2}{k/2-1}$.

Putting these bounds together we have $q \leq \varepsilon k^{2k}\binom{n}{k/2-1} + 2\binom{n}{k/2-1}/k$, and this gives

$$p \leq (1+\varepsilon k^{4k})|H^*|\binom{k-1}{k/2}\frac{2}{k}\binom{n}{k/2-1}.$$

Comparing the upper and lower bounds for p and dividing by $|H^*|\binom{n}{k/2-1}$ yields

$$(0.98)\left(\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2}\right) < (1+\varepsilon k^{4k})\frac{2}{k}\binom{k-1}{k/2}.$$

Since $\varepsilon < k^{4k}/100$ this implies that

$$(0.97)\left(\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2}\right) < \frac{2}{k}\binom{k-1}{k/2}.$$

A short calculation shows that this is equivalent to $(0.97k-2)(k-1) < 0.97k(\frac{k}{2}-1)$, and it is easily verified that this is false for $k \geq 4$. This contradiction completes the proof of the theorem. $\square$

7 Concluding Remarks

A k-graph is r-regular if all its vertices have degree r. In contrast to Theorem 1, if the degrees in a k-graph are all the same, then a linear number of edges already forces a 2-regular subgraph.

Precisely, let ϕ_k denote the maximum number ϕ such that there exists a ϕ-regular k-graph containing no 2-regular subgraph. Then Lemma 1 immediately implies that $\phi_k \leq (6k)^k$. On the other hand, we have a lower bound of $\binom{3k/2-1}{k-1}$ when k is even and $\binom{2k-1}{k-1}$ when k is odd, by taking complete k-graphs of the appropriate size (these contain no 2-regular subgraphs because every 2-regular subgraph has at least $3k/2$ vertices when k is even and at least $2k$ vertices when k is odd). The lower bounds are of order c^k, so there is a substantial gap in the bounds for ϕ_k. We leave the open problem of determining ϕ_k and, in particular, ϕ_3. It is expected that if a k-graph is ϕ-regular and ϕ is a large enough constant depending on k and r, then every ϕ-regular k-graph has an r-regular subgraph (a subgraph in which every vertex has degree r). In fact, this should hold for multi-k-graphs – instead of a set of edges a multiset of edges is allowed. Therefore we make the following conjecture:

Conjecture 2. *Let $k, r \geq 2$. There exists an integer $\phi_k(r)$ such that for $\phi >> \phi_k(r)$, every ϕ-regular multi-k-graph contains an r-regular subgraph.*

This conjecture is wide open for $k, r \geq 3$. If r is a prime not dividing k and we superimpose $r-1$ copies of the complete k-graph on $k+1$ vertices, namely K_{k+1}^k, then we obtain a multi-k-graph $H_{r,k}$ containing no r-regular subgraph. To check this, let J be the all-one matrix, so that $J - I$ is the incidence matrix of K_{k+1}^k, with rows indexed by edges and columns by vertices. Then over $\mathbb{Z}_r$, the field of integers mod r, we have $(r-1)(J-I) = I - J$, and this matrix has full rank over $\mathbb{Z}_r$, since r does not divide k. Therefore no set of rows of $(r-1)(J-I)$ is linearly dependent over $\mathbb{Z}_r$, which means $H_{r,k}$ has no non-empty subgraph in which all vertices have degree zero modulo r. This simple construction shows that if $\phi_k(r)$ exists, then $\phi_k(r) \geq k(r-1)$. For $k = 2$, in other words, for multigraphs, Tâskinov [16] completely determined $\phi_k(r)$ using Tutte's f-Factor Theorem. Unfortunately, no analogous theorem for k-graphs is known when $k \geq 3$. The following positive evidence for Conjecture 2 follows immediately by extending the proof of Alon, Friedland, Kalai [1] and uses Chevalley's theorem:

Theorem 5. *Let H be an n-vertex multi-k-graph, such that H is $k(r-1)+1$-regular, where r is a prime number. Then H has a subgraph all of whose vertex degrees are elements of $\{r, 2r, \ldots, (k-1)r\}$.*

The multi-k-graph $H_{r,k}$ shows that Theorem 5 is tight. Further evidence for Conjecture 2 comes from Rödl's packing method [14]. A k-graph is linear if no two of its edges intersect in two or more points. If M is a matching in a k-graph H, let $\text{ex}(M)$ denote the number of vertices not covered by M. Rödl's Theorem [14] says that every linear n-vertex d-regular k-graph contains a matching M such that $\text{ex}(M) \leq d^{-\varepsilon}n$, for some constant $\varepsilon > 0$ depending only on k. In fact, the degrees of the vertices in the hypergraph are allowed to be between $(1-\delta)d$ and $(1+\delta)d$ for the same conclusion, provided $\delta > 0$ is a sufficiently small constant depending on ε. By repeatedly removing r such matchings from a linear d-regular k-graph, we see that for any fixed r, we obtain a subgraph in which all vertices have degree at most r, and at most $rd^{-\varepsilon}n$ vertices have degree less than r. In other words, we find an "almost r-regular" subgraph. On the other hand, we do not even have a verification that every large enough Steiner triple system has a three-regular subgraph.

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