Two-regular subgraphs of hypergraphs

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June 23, 2008

Abstract

We prove that the maximum number of edges in a $k$-uniform hypergraph on $n$ vertices containing no 2-regular subhypergraph is $\binom{n-1}{k-1}$ if $k \geq 4$ is even and $n$ is sufficiently large. Equality holds only if all edges contain a specific vertex $v$. For odd $k$ we conjecture that this maximum is $\binom{n-1}{k-1} + \lfloor \frac{n}{k} \rfloor$, with equality only for the hypergraph described above plus a maximum matching omitting $v$.

1 Introduction

One of the most basic facts in combinatorics is that an acyclic graph on $n$ vertices has at most $n - 1$ edges, with equality only for trees. A natural generalization to hypergraphs (see Berge [3] for more details) is obtained by defining a circuit to be a hypergraph consisting of distinct vertices $v_1, v_2, \ldots, v_k$ and distinct edges $e_1, \ldots, e_k$ such that $v_i \in e_i$ for $i = 1, 2, \ldots, k$, $v_{i+1} \in e_i$ for $i = 1, 2, \ldots, k-1$, and $v_1 \in e_k$. Then a hypergraph $H$ with no circuit satisfies

$$\sum_{e \in H} (|e| - 1) \leq |V(H)| - 1.$$ 

In this paper, we consider a generalization to hypergraphs in a different direction. Since a cycle is a 2-regular graph, we may ask for the maximum number of edges that a hypergraph on $n$ vertices can have without a 2-regular subgraph – i.e. a subhypergraph in which every vertex has degree two. Throughout the paper, hypergraphs where all edges have size $k$ are called $k$-uniform hypergraphs or, simply, $k$-graphs. A star is a hypergraph in which there is a vertex $v$ such that all possible edges containing $v$ are present and there are no other edges. Our main result shows that stars are the extremal hypergraphs not containing a 2-regular subgraph when $k$ is even:

**Theorem 1.** For every even integer $k > 2$, there exists an integer $n_k$ such that for $n \geq n_k$, if $H$ is an $n$-vertex $k$-graph with no 2-regular subgraph, then $|H| \leq \binom{n-1}{k-1}$. Equality holds if and only if $H$ is a star.

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The non-uniform analog of this theorem, which is much simpler, is proved in Section 2. As one might expect, the proof of Theorem 1 needs completely new techniques than the graph case. The result is proved via the stability approach. Stability results were introduced in extremal graph theory by Erdős and Simonovits [15] in the 60’s. The program of using stability to prove exact results has been recently used with great success in extremal set theory (see [5, 6, 7, 8, 9, 10, 11]). Perhaps the main difficulty in passing to an exact result when $k$ is odd is that stars are not extremal when $k$ is odd: it is possible to add to a star on $n$ vertices a matching of size $\left\lfloor \frac{n-1}{k} \right\rfloor$, resulting in an $n$-vertex $k$-graph with no 2-regular subgraph with a few more edges than a star. We conjecture that this “star-plus-matching” construction is the unique extremal configuration when $k$ is odd.

**Conjecture 1.** For every odd integer $k \geq 3$, there exists an integer $n_k$ such that for $n \geq n_k$, if $H$ is an $n$-vertex $k$-graph with no 2-regular subgraph then $|H| \leq \binom{n-1}{k-1} + \left\lfloor \frac{n-1}{k} \right\rfloor$. Equality holds if and only if $H$ is a star with center $v$ together with a maximal matching omitting $v$.

Conjecture 1 is a weaker version of a conjecture due to Füredi, that for $k > 3$, a $k$-graph containing no two pairs of disjoint sets with the same union has at most $\binom{n-1}{k-1} + \left\lfloor \frac{n-1}{k} \right\rfloor$ edges. For odd $k > 3$, this implies Conjecture 1; in fact, a hypergraph consisting of two pairs of disjoint edges with the same union is the smallest possible 2-regular $k$-graph when $k$ is odd.

The question of determining the maximum number of edges $f_k(n)$ of a $k$-graph on $n$ vertices containing no two pairs of disjoint edges with the same union was originally raised by Erdős (see [4]). This problem was studied by Frankl and Füredi [4], and the authors [12], who showed that $f_k(n) < 3\binom{n}{k-1}$. The best bounds are given in Pikhurko and the second author [13], where it is shown that $f_3(n) < \frac{13}{9} \binom{n}{2}$ and $f_k(n) < (1 + \frac{2}{\sqrt{k}}) \binom{n}{k-1}$ for all $k$.

This paper is organized as follows. In the next section, we prove the nonuniform analog of Theorem 1, that a collection of subsets of an $n$-element set with no 2-regular subsystem has size at most $2^{n-1}$ with equality (for $n \geq 3$) only for a star. In Section 3, we present two lemmas used to prove Theorem 1. The proof of Theorem 1 is in Sections 4–6, and has three parts. First we shall show (see Section 4) that if $H$ is an $n$-vertex $k$-graph with no 2-regular subgraph, then $|H| \leq \binom{n-1}{k-1}$. Using this result, we prove the stability result (see Section 5), which says that if $|H| \sim \binom{n-1}{k-1}$ then $\Delta(H) \sim \binom{n-1}{k-1}$. Finally, we use this stability theorem to prove Theorem 1 in Section 6. The final section mentions related open problems.

**Terminology.** A hypergraph is a family of subsets of a set of vertices, called edges. We denote by $|H|$ the number of edges in a hypergraph $H$. If $H$ is a hypergraph, then $V(H)$ denotes this set of vertices. The degree of a vertex $v$, written $d(v)$, is the number of edges containing that vertex. A matching is a hypergraph in which every vertex has degree one — such a hypergraph $M$ consists of pairwise disjoint edges $e_1, e_2, \ldots, e_m$ for some $m$ and $V(M) = e_1 \cup e_2 \cup \cdots \cup e_m$. A $k$-graph is a hypergraph where all sets have size $k$, and a hypergraph is $r$-regular if all its vertices have degree $r$. We write $\binom{X}{k}$ for the collection of all $k$-sets of $X$. A star is a hypergraph on a vertex set $X$ consisting of all possible edges containing a fixed vertex of $X$. In the context of $k$-graphs, a star consists of all possible $k$-sets containing a fixed vertex of $X$. For a hypergraph $H$, denote by $\Delta(H)$ its maximum degree. For $v \in V(H)$, let $H - \{v\} = \{e \in H : v \not\in e\}$ and $H_v = \{e \setminus \{v\} : v \in e \in H\}$. If $f, g : \mathbb{N} \to \mathbb{R}$
are two functions then we write \( f(n) \geq g(n) \) to denote that \( f(n) \geq g(n)h(n) \) for some function \( h(n) \) such that \( \liminf_{n \to \infty} h(n) = 1 \). This is an equivalent but more convenient way to write \( f(n) \geq (1 + o(1))g(n) \). In the case \( f(n) = (1 + o(1))g(n) \) we write \( f(n) \sim g(n) \). If there is a constant \( c > 0 \) such that \( f(n) \geq cg(n) \) for all \( n \), then we write \( f(n) \gg g(n) \). Throughout this paper, \( k \) is always fixed relative to \( n \).

## 2 Non-uniform hypergraphs

In this section, we prove the nonuniform analog of Theorem 1. We stipulate that edges of a hypergraph are non-empty sets. A star on \( n \) vertices is a hypergraph consisting of all \( 2^{n-1} \) sets containing a fixed vertex.

**Theorem 2.** Let \( n \geq 1 \) and let \( H \) be a hypergraph on \( n \) vertices containing no 2-regular subgraph. Then \( |H| \leq 2^{n-1} \). If \( n \geq 3 \) and equality holds, then \( H \) is a star.

**Proof.** We remark that it is easy to obtain an upper bound \( 2^{n-1} \): if \( H \) has no 2-regular subgraph, then \( H \) contains at most one complementary pair – a complementary pair consists of the edge \( e \) and the edge \( V(H) \setminus e \). This shows \( |H| \leq 2^{n-1} + 1 \), but if \( H \) contains both edges of some complementary pair, then \( V(H) \) cannot be an edge of \( H \), showing \( |H| \leq 2^{n-1} \). For the characterization of equality, we proceed by induction on \( n \) for \( n \geq 3 \).

It is straightforward to check the case \( n = 3 \); we omit the details. Now we proceed to the induction step. Let us assume that \( n \geq 4 \) and \( H \) has size \( 2^{n-1} \) and no 2-regular subgraph. We will show that \( H \) is a star, which proves Theorem 2. First we show that every vertex of \( H \), apart from at most one vertex, has degree exactly \( 2^{n-2} \). If there is a vertex \( v \in V(H) \) with \( d(v) < 2^{n-2} \), then \( H - \{v\} \) has a 2-regular subgraph, by induction. So every vertex of \( H \) has degree at least \( 2^{n-2} \). Pick a vertex \( x \in V(H) \). If \( x \) is contained in every set in \( H \), then \( H \) is a star with center \( x \) and all other vertices have degree \( 2^{n-2} \). We may therefore assume that there exists an \( e \in H \) missing \( x \). Assume that \( |e| = k \) where \( 1 \leq k \leq n \). For each subset \( f \subset V(H) \setminus (e \cup \{x\}) \), the number of edges in \( H \) containing \( x \) whose intersection with \( V(H) \setminus (e \cup \{x\}) \) is \( f \) is at most \( 2^{k-1} \), for otherwise two of these edges have complementary intersections in \( e \) and these together with \( e \) give a 2-regular subgraph, a contradiction. Hence the number of edges containing \( x \) is at most \( 2^{n-k-1}2^{k-1} = 2^{n-2} \). So \( x \) has degree exactly \( 2^{n-2} \), in which case \( |H - \{x\}| = 2^{n-2} \). By induction, \( H - \{x\} \) is a star with center at some vertex \( w \). Suppose, for a contradiction, that there exist distinct edges \( e, f \) containing \( x \) but not \( w \). Then the edges

\[
e, \quad f, \quad \{w\} \cup (e \setminus f), \quad \{w\} \cup (f \setminus e)
\]

form a 2-regular subgraph of \( H \), a contradiction. So at most one edge containing \( x \) does not contain \( w \). If such an edge \( e \) exists, then pick an edge \( f \) containing \( x \) and \( w \) – this is possible since \( x \) has degree \( 2^{n-2} \geq 2 \). It follows that

\[
e, \quad f, \quad e \triangle f
\]

is a 2-regular subgraph of \( H \). So we have shown that all edges containing \( x \) must also contain \( w \). Therefore \( H \) is a star with center \( w \).
3 Preliminary Lemmas

In this section, we present two lemmas which will be used in proving Theorem 1. The first lemma involves matchings. If $M_1$ and $M_2$ are distinct matchings and $V(M_1) = V(M_2)$, then $M_1 \triangle M_2$ is a hypergraph whose vertices all have degree two. This observation is the key point of the following lemma.

**Lemma 1.** Let $H$ be a $k$-graph on $n$ vertices containing no 2-regular subgraph. Then $|H| \leq 6n\Delta^{(k-1)/k}$ or $|H| < 2k\Delta$.

**Proof.** Let $d = k|H|/n$ and suppose $|H| \geq 2k\Delta$. Then it is enough to prove that $\Delta \geq (1/k)(d/6)^{k/(k-1)}$ to prove the lemma, for this implies the second inequality in

$$|H| \leq 6n\left(\frac{1}{k}\left(\frac{d}{6}\right)^{k/(k-1)}\right)^{(k-1)/k} \leq 6n\Delta^{(k-1)/k}.$$ 

Suppose, for a contradiction, that this is not true. We count matchings in $H$ of size $m = \lfloor|H|/k\Delta\rfloor$ to show that $H$ contains a 2-regular subgraph. Note that $m \geq 2$ since $|H| \geq 2k\Delta$. For a lower bound on the number of matchings of size $m$, we may greedily pick disjoint edges $f_1, f_2, \ldots, f_m$ where at each step we exclude all edges that intersect previously chosen edges. Since at each step we exclude at most $k\Delta$ new edges, the number of matchings of size $m$ in $H$ is at least

$$\frac{1}{m!} \prod_{i=0}^{m-1} (|H| - k\Delta i) = \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} \left(1 - \frac{k\Delta i}{|H|}\right) \geq \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} \left(1 - \frac{i}{m}\right) \geq (k\Delta)^m.$$ 

To complete the proof, we show that there exist distinct matchings $M_1, M_2$ of $H$ such that $\bigcup_{f \in M_1} f = \bigcup_{f \in M_2} f$. This suffices, since the edges in $M_1 \triangle M_2$ form a 2-regular subgraph, contradicting the fact that $H$ has no 2-regular subgraph. First note that

$$\binom{n}{mk} < \left(\frac{3n}{mk}\right)^{mk} \leq \left(\frac{6k\Delta}{d}\right)^{km} < (k\Delta)^m.$$ 

Here we used $m \geq dn/2k^2\Delta$ and then the assumed upper bound on $\Delta$. Since $\binom{n}{mk}$ is the number of sets of $mk$ vertices of $H$, and there are more than $(k\Delta)^m$ matchings of size $m$ in $H$, we find the two required distinct matchings $M_1, M_2$. \hfill $\square$

Our second lemma involves circuits in hypergraphs. A circuit is a hypergraph consisting of distinct vertices $v_1, v_2, \ldots, v_k$ and distinct edges $e_1, \ldots, e_k$ such that $v_i \in e_i$ for $i = 1, 2, \ldots, k$, $v_{i+1} \in e_i$ for $i = 1, 2, \ldots, k-1$, and $v_1 \in e_k$. We require the following lemma on 2-regular subgraphs arising from circuits in hypergraphs of a certain bipartite structure:

**Lemma 2.** Let $G$ be a $k$-graph and $V(G) = A \cup B$, where $A \cap B = \emptyset$, all edges $e \in G$ have $|e \cap A| = k-1$, and every $(k-1)$-set in $A$ lies in at least two edges of $G$. If $G$ has no 2-regular subgraph, then

$$|G| < 2 |B| \binom{|A| + k - 3}{k - 2}.$$
Proof. It is enough to show $|G| < 2|B| \binom{|A| - 1}{k - 2}$ when $k - 1$ divides $|A|$, since we may always add at most $k - 2$ points to $A$ so that $k - 1$ divides $|A|$. Baranyai’s Theorem [2] states that if $s$ divides $n$, then the complete $s$-graph on $n$ vertices can be partitioned into $\binom{n-1}{s-1}$ perfect matchings. Using this theorem with $s = k - 1$, we write

$$\left(\begin{array}{c} A \\ k-1 \end{array}\right) = M^1 \cup M^2 \cup \cdots \cup M^{\binom{|A|-1}{k-2}}$$

where each $M^i$ is a matching and the matchings are edge-disjoint. For each matching $M^i = \{e^i_1, \ldots, e^i_a\}$, let $G^i$ be the set of edges in $G$ whose intersection with $A$ is $e^i_j$ for some $j$. Let $f^i_j$ be the set of vertices $v \in B$ such that $e^i_j \cup \{v\} \in G^i$. Consider the hypergraph $H^i_j$ with edges $f^i_j$, $j = 1, \ldots, a$. If $H^i_j$ contains a circuit with vertices $v_1, v_2, \ldots, v_p$, then $G$ contains the 2-regular subgraph with edges

$$e^i_1 \cup \{v_1\} \ e^i_2 \cup \{v_2\} \ e^i_2 \cup \{v_2\} \ e^i_p \cup \{v_p\} \ e^i_p \cup \{v_1\}$$

which contradicts that $G$ has no 2-regular subgraph. Consequently, $H^i_j$ has no circuit. It is well-known that a hypergraph $H$ with no circuit satisfies

$$\sum_{e \in H} (|e| - 1) \leq (|V(H)| - 1). \quad (1)$$

Since every $(k - 1)$-set in $A$ lies in at least two edges of $G$, $|f^i_j| \geq 2$ for all $i, j$. Applying (1) to $H^i_j$, we therefore obtain

$$\sum_j |f^i_j| \leq \sum_j 2(|f^i_j| - 1) < 2|B|. \quad (2)$$

Adding (2) over different $i, j$, we obtain

$$|G| = \sum_i \sum_j |f^i_j| \leq 2|B| \binom{|A|-1}{k-2}. \quad \Box$$

4 The Asymptotic Result

**Theorem 3.** Let $k \geq 3$ and let $H$ be an $n$-vertex $k$-graph with no 2-regular subgraph. Then

$$|H| - \binom{n-1}{k-1} \ll n^{k-1-1/11}.$$ 

**Proof.** We prove the following more precise statement: for all $n > k^{100}$,

$$|H| < \left(1 + cn^{-\gamma}\right) \binom{n-1}{k-1}$$

where $c = 4(k+1)!$ and $\gamma = 1/11$. Define $\alpha = (k+1)/(3k-1)$ for $k > 3$ and $\alpha = 7/11$ for $k = 3$. Suppose, for a contradiction, that $|H|$ is at least this upper bound for some $H$. By deleting some edges, we may assume that $|H|$ is equal to the stated upper bound. Let $T$ denote the
set of vertices of \( H \) of degree at least \( D = n^{k-1-\alpha} \), and set \( t = |T| \). Then \( tD \leq k|H| \) and, since \( n > k^{100} \),

\[
t < D^{-1}k(1 + cn^{-\gamma}) \left( \frac{n - 1}{k - 1} \right) < kn^\alpha.
\]

(3)

Let \( H_i = \{ e \in H : |e \cap T| = i \} \) for \( i \leq k \), and define \( G = \{ e \in H_1 : \#f \in H_1 : e \setminus T = f \setminus T \} \). In particular, it is clear that \( |G| \leq \binom{n-1}{k-1} \).

**Claim 1.** \(|H_i| < \begin{cases} 6n^1(k-1)(k-1-\alpha)/k & \text{for } i = 0 \\ |G| + 2kn^{k-2+\alpha} & \text{for } i = 1 \end{cases} \)

**Proof.** Since \( \Delta(H_0) < D \), by definition of \( T \), the first bound follows from Lemma 1. For the second bound, we apply Lemma 2 to \( H_i \\ G \) with \( A = V(H) \setminus T \) and \( B = T \) to obtain \(|H_1 \setminus G| < 2|T| \frac{n^{k-3}}{k-2} < 2tn^{k-2} \). The bound on \(|H_1| \) now follows from (3).

**Claim 2.** \(|H \setminus (H_0 \cup H_1)| < \begin{cases} k^2n^{k-2+2\alpha} & \text{for } k > 3 \\ 6(n^{1+\alpha} + n^{3\alpha}) & \text{for } k = 3 \end{cases} \)

**Proof.** For \( k > 3 \), by definition, every edge in \( H \setminus (H_0 \cup H_1) \) contains two vertices of \( T \) and \( k-2 \) vertices of \( V(H) \), so certainly \(|H \setminus (H_0 \cup H_1)| \leq \binom{|T|}{3}n^{k-2} \). Now apply (3). For \( k = 3 \), observe that \(|H_3| < \binom{|T|}{3} \). Furthermore, by Lemma 2, with \( A = T \) and \( B = V(H) \setminus T \), \(|H_2| < 2|T|(n - |T|) + \binom{|T|}{2} < 2tn \). Here we note that there could be \( \binom{|T|}{2} \) pairs in \( T \) contained in only one triple of \( H_2 \). Those contained in two triples or more are the ones to which Lemma 2 applies, giving the bound \( 2|T|(n - |T|) \) for those triples. Using (3) gives the claim.

Now we complete the proof. By definition of \( \alpha \), the bounds in Claims 1 and 2 are all of order at most \( n^{k-1-\gamma} \) (the case \( i = 0 \) in Claim 1 needs a somewhat tedious calculation). Specifically,

\[
|H \setminus G| = |H_0| + |H_1 \setminus G| + |H \setminus (H_0 \cup H_1)| < (6 + k^2 + 2k)n^{k-1-\gamma} < 4k^2n^{k-1-\gamma}.
\]

(4)

Using the bound \(|G| \leq \binom{n-1}{k-1} \) in (4), we obtain

\[
|H| = |G| + |H \setminus G| < \binom{n-1}{k-1} + 4k^2n^{k-1-\gamma} < \left( 1 + 4(k + 1)!n^{-\gamma} \right) \binom{n-1}{k-1}.
\]

The constant \( c = 4(k+1)! \) appears here: we used the fact that \( 4k^2n^{k-1-\gamma} < 4(k+1)!n^{-\gamma} \binom{n-1}{k-1} \) for \( n > k^{100} \). This contradiction completes the proof. \( \square \)

## 5 Stability

**Theorem 4.** Let \( k \geq 3 \) and let \( H_n \) be an \( n \)-vertex \( k \)-graph with no 2-regular subgraph. If \(|H_n| \sim \binom{n-1}{k-1} \), then \( \Delta(H_n) \sim \binom{n-1}{k-1} \).
Proof. For simplicity of notation, we let $H = H_n$ and omit the subscript $n$ when dealing with hypergraphs constructed from $H$. As in the proof of Theorem 3, let $T$ denote the set of vertices in $H$ of degree at least $n^{k-1-\alpha}$, $H_1 = \{ e \in H : |e \cap T| = 1 \}$ and $G = \{ e \in H_1 : \exists f \in H_1 : e \setminus T = f \setminus T \}$. Define

$$G' = \{ e \setminus T : e \in G \}.$$ 

For each $x \in T$, let $G_x = \{ e \in G' : e \cup \{ x \} \in G \}$. Let $v$ be a vertex such that $|G_v| = \max_{x \in T} |G_x|$. Note that all sets in $G$ have size $k$, and all sets in $G'$ or any $G_x$ have size $k-1$.

By (4), $|G'| = |G| \sim |H| \sim \binom{n}{k-1}$, so it suffices to prove that $|G_v| \sim |G'|$. Suppose, for a contradiction, that for some positive $\varepsilon < \frac{1}{2}$,

$$|G_v| \lesssim (1 - \varepsilon)|G'|.$$  

(5)

The strategy is to use (5) to derive a contradiction by finding edges $e, e' \in G_x$ and $f, f' \in G_y$, for some $x \neq y$, such that $|e \cap f| = 1 = |e' \cap f'|$, $e \Delta f = e' \Delta f'$ and $e \cap f \neq e' \cap f'$ (sometimes the latter condition will be guaranteed by $e \cap e' = \emptyset = f \cap f'$). For in this case, the edges

$$e \cup \{ x \}, e' \cup \{ x \}, f \cup \{ y \}, f' \cup \{ y \}$$

(6)

form a 2-regular subgraph of $H$.

For any hypergraph $F$, define $P(F) = \{ \{ e, f \} \subset F : |e \cap f| = 1 \}$. Define $P_1(G') \subset P(G')$ to be the set of pairs $\{ e, f \} \in P(G')$ such that $e \in G_x$ for some $x$, and $P_2(G') = P(G') \setminus P_1(G')$.

**Claim 1.** $|P_2(G')| \leq \frac{1}{2} \binom{k}{2} \binom{2k-4}{k-2} \binom{n-1}{2k-4}$.

**Proof.** Fix distinct vertices $x, y \in T$. We show that the number of $\{ e, f \} \in P_2(G')$ such that $e \in G_x$ and $f \in G_y$ is at most $\frac{1}{2} \binom{2k-4}{k-2} \binom{n-1}{2k-4}$. This completes the proof, since there are $\binom{k}{2}$ choices for $x$ and $y$.

Given a set $S$ of size $2k - 4$, let us count the number of pairs $\{ e, f \} \in P_2(G')$ with $e \Delta f = S$ that satisfy $e \in G_x$ and $f \in G_y$. Suppose that we have at least one such pair $\{ e, f \}$ with $e \cap f = \{ z \}$. Any other such pair $\{ e', f' \}$ must also satisfy $e' \cap f' = \{ z \}$, otherwise the four edges $e \cup \{ x \}, e' \cup \{ x \}, f \cup \{ y \}, f' \cup \{ y \}$ form a 2-regular subgraph. Hence the number of such pairs is at most the number of (unordered) partitions of $S$ into two sets of size $k - 2$, which is $(1/2)^{k-2} \binom{2k-4}{k-2}$. The number of ways to choose $S$ is at most $\binom{n-1}{2k-4}$. Putting this all together we obtain the required bound in the claim.

For the rest of the proof, let $\psi(\varepsilon) = ((1 - \varepsilon)^2 + \varepsilon^2)^{1/2}$. For $i \in \{1, 2\}$, let $Q_i(G')$ denote the set of pairs $\{ \{ e, f \}, \{ e', f' \} \}$ such that $\{ e, f \}, \{ e', f' \} \in P_i(G')$, $e \cap e' = \emptyset = f \cap f'$ and $e \Delta f = e' \Delta f'$. These are called type $i$ quadrilaterals of $G'$. For $x \in T$, define $Q_1(G_x)$ to be the collection of pairs $\{ \{ e, f \}, \{ e', f' \} \} \subset Q_1(G')$ such that $\{ e, f, e', f' \} \subset G_x$. These are type 1 quadrilaterals of $G_x$. Let $K$ be the complete $(k - 1)$-graph on $V(G')$. Recall that $P(K)$ is the number of pairs $\{ e, f \} \subset K$ such that $|e \cap f| = 1$. So in the case that $k = 2$, when $K$ is the complete graph, this is just the number of paths of length two. More generally, we have

$$|P(K)| \sim \frac{1}{2} \binom{k-1}{n-1} \binom{n-1}{k-1} \binom{n-1}{k-2}.$$  

(7)
Claim 2. \(|P_1(G')| \lesssim \psi(\varepsilon) \cdot |P(K)|.\)

Proof. Let \(\{\{e, f\}, \{e', f'\}\} \in Q_1(G').\) If \(e, f \in G_x\) and \(e', f' \in G_y\) with \(x \neq y,\) then we obtain a 2-regular subgraph similar to that in (6). We conclude that if \(e, f \in G_x,\) then also \(e', f' \in G_x.\) It follows that

\[
|Q_1(G')| = \sum_{x \in T} |Q_1(G_x)|. \tag{8}
\]

For a pair \(\{g, h\}\) of disjoint sets of size \(k - 2\) in \(V(G'),\) let \(p(g, h)\) denote the number of pairs \(\{e, f\} \in P_1(G')\) with \(e \setminus f = g\) and \(f \setminus e = h.\) The number of such pairs \(\{g, h\}\) is at most

\[
\binom{(n-1)}{2} := N.
\]

Note also that the sum of \(p(g, h)\) over all \(\{g, h\} \subset V(G')\) is exactly \(|P_1(G')|.| By convexity of binomial coefficients,

\[
|Q_1(G')| = \sum_{\{g, h\}} \binom{p(g, h)}{2} \geq \left(\frac{|P_1(G')|/N}{2}\right) \cdot N \sim \frac{|P_1(G')|^2}{(\binom{n-1}{2})^2}. \tag{9}
\]

The first equality is the hypergraph analog of the fact that the number of quadrilaterals in a graph \(F\) is exactly \(\sum_{u, v \in V(F)} \binom{p(u, v)}{2}\) where \(p(u, v)\) is the number of paths of length two from \(u\) to \(v\) in \(F.\) On the other hand, we observe that \(|Q_1(G_x)| \leq \frac{1}{2}(k - 1)^2|G_x|^2\), since if we fix two disjoint edges, say \(e, e' \in G_x,\) then the number of type 1 quadrilaterals of the form \(\{\{e, f\}, \{e', f'\}\}\) is at most \((k - 1)^2.\) The same type 1 quadrilaterals are counted if we had fixed the two disjoint edges \(f, f' \in G_x\) instead of \(e, e',\) and this gives the additional factor of 2 in the observation. Therefore, by (8),

\[
|Q_1(G')| \leq \frac{1}{2}(k - 1)^2 \sum_{x \in T} \binom{|G_x|^2}{2}.
\]

By convexity, this sum is a maximum when \(|G_v| \sim (1 - \varepsilon)|G'|\) and \(|G_w| \sim \varepsilon|G'|\) for some \(w \neq v,\) and the rest of the \(|G_x|s\) are zero. Therefore

\[
|Q_1(G')| \lesssim \frac{1}{4}(k - 1)^2 \psi(\varepsilon)^2|G'|^2 \tag{10}.
\]

Combining (9), (10), \(|G'| \sim \binom{n-1}{k-1},\) and (7) we obtain

\[
|P_1(G')| \lesssim \psi(\varepsilon) \cdot \frac{1}{2}(k - 1)|G'|\binom{n-1}{k-2} \lesssim \psi(\varepsilon)|P(K)|.
\]

This proves Claim 2.

The next claim is intuitively obvious since \(|G'| \sim |K| \sim \binom{n-1}{k-1}.\) We present a formal proof below.

Claim 3. \(|P(G')| \sim |P(K)|.\)
Proof. We note that $|P(K)| = n\binom{n-1}{k-\frac{3}{2}}$, since we may choose any vertex and two disjoint $(k-1)$-sets containing it. Let $d_x$ be the number of sets in $K\backslash G'$ which contain $x\in V(G')$. Then

$$\sum_{x\in V(G')} d_x = (k-1)|K\backslash G'|.$$ 

Using this we obtain

$$|P(K)\backslash P(G')| \leq \sum_{x\in V(G')} \left(\frac{d_x}{2}\right) + \sum_{x\in V(G')} d_x \left(\binom{n-2}{k-2} - d_x\right)$$

$$= (k-1)|K\backslash G'|\binom{n-2}{k-2} - \frac{1}{2} \sum_{x\in V(G')} d_x^2 - \frac{1}{2} (k-1)|K\backslash G'|. \quad (11)$$

Now since $|P(K)|$ is of order $n^{2k-3}$, and $|G'| \sim |K|$, we see that all terms in (11) are negligible relative to $|P(K)|$, except possibly the sum of $d_x^2$. We wish to find

$$\max \sum_{x\in V(G')} d_x^2 \text{ if } \sum_{x\in V(G')} d_x = |K\backslash G'|.$$

The maximum possible value of $d_x$ is $\binom{n-2}{k-2}$. For a maximum of the sum of squares, we let

$$\frac{(k-1)|K\backslash G'|}{\binom{n-2}{k-2}}$$

of the $d_x$ take the value $\binom{n-2}{k-2}$, and the rest are zero (note that for a maximum, it is not necessary that there exist a hypergraph $K\backslash G'$ realizing these values of $d_x$). Therefore

$$\max \sum_{x\in V(G')} d_x^2 \leq (k-1)|K\backslash G'|\binom{n-2}{k-2}$$

and again this is negligible relative to $|P(K)|$ since $|K| \sim |G'|$ and $|P(K)|$ has order $n^{2k-3}$. This proves the claim.

We complete the proof of Theorem 4 for $k > 3$. By (3), $t \leq kn^\alpha$ where $\alpha < \frac{1}{2}$ (this relies on $k > 3$). Therefore Claims 1, 2, and 3 imply that

$$|P(K)| \sim |P(G')| = |P_1(G')| + |P_2(G')|$$

$$\lesssim \psi(\varepsilon)|P(K)| + \binom{t}{2} \binom{2k-4}{k-2} \binom{n}{2k-4}$$

$$\sim \psi(\varepsilon)|P(K)|. \quad (12)$$

However, $\psi(\varepsilon) = ((1-\varepsilon)^2 + \varepsilon^2)^{1/2}$ is bounded away from 1, so the above inequality is a contradiction.

For $k = 3$, $G'$ is a graph and $P(G')$ is the set of paths of length two in $G'$. The problem with the above arguments for $k = 3$ is that (3) only gives $t \leq 3n^{7/11}$, which is too large for (12) to
hold (since \( \binom{n}{2k-4} \) has order \( n^{3+3/11} \)). Therefore we go one step further, and count paths of length three in \( G' \) instead of paths of length two. Let \( P_3(G') \) be the number of paths of length three in \( G' \) with edges from three different \( G_x \)s. By Claims 2 and 3
\[
|P_2(G')| = |P(G')| - |P_1(G')| \geq (1 - \psi(\varepsilon))|P(K)| \gg n^3. \tag{13}
\]
As in Claim 1, if \( \{|e, f\}, \{e', f'\} \} \) is a type 2 quadrilateral of \( G \) and \( e, e' \in G_x \) and \( f, f' \in G_y \), then we obtain a 2-regular subgraph of \( H \). So each type 2 quadrilateral contains edges from at least three different \( G_x \)s, and these edges form a path of length three in \( G' \). Consequently, as in (9), the convexity of binomial coefficients and (13) give
\[
|P_3(G')| \geq \frac{1}{4} |Q_2(G')| \geq \frac{1}{4} \left( \frac{|P_2(G')|}{N} \right) N \gg n^4
\]
since \( N = \binom{n-2}{2} \). Let \( (A, B) \) be a random partition of \( V(G') \), defined by placing a vertex in \( A \) with probability \( \frac{1}{2} \) and in \( B \) with probability \( \frac{1}{2} \), independently for each vertex of \( V(G') \). Let \( G^* \) denote the graph consisting of all edges between \( A \) and \( B \). Then the expected value of \( |P_3(G^*)| \) is exactly \( \frac{1}{8} |P_3(G')| \), so there is a partition of \( G' \) for which
\[
|P_3(G^*)| \geq \frac{1}{8} |P_3(G')| \gg n^4. \tag{14}
\]
Let \( e_1e_2e_3 \) and \( f_1f_2f_3 \) be two paths in \( G^* \) with the same pair of endpoints. Suppose \( e_i \in G_j(i) \) and \( f_i \in G_h(i) \) where \( \{j(1), j(2), j(3)\} = \{h(1), h(2), h(3)\} \). Since \( G^* \) is bipartite, amongst these edges there is a cycle \( C \) of length four or six containing exactly zero or two edges from each \( G_j(i), i = 1, 2, 3 \). It is easily checked that the unique edges of \( H' \) which contain the edges of \( C \) form a 2-regular subgraph of \( H \), which is a contradiction. We conclude that at most \( \binom{3}{3} \) paths of length three in \( G^* \) with edges in different \( G_i \)s have the same pair of endpoints. It follows that
\[
|P_3(G^*)| \leq \binom{t}{3} \binom{n}{2} < n^{4-\frac{1}{11}}
\]
using (3). This contradicts (14), and completes the proof of Theorem 4. \( \square \)

6 The Exact Result

In this section we prove Theorem 1. Our main tools are the asymptotic and stability result. Let \( H \) be an \( n \)-vertex \( k \)-graph containing no 2-regular subgraph, where \( k \geq 4 \) is even, and suppose \( |H| = \binom{n-1}{k-1} \). Let \( \varepsilon = \frac{1}{100k^2} \). By Theorem 4, for large enough \( n \), there is a vertex \( v \in V(H) \) such that
\[
|H - \{v\}| \leq \varepsilon n^{k-1}. \tag{15}
\]
Let \( H^* = H - \{v\} \). To complete the proof, we show \( |H^*| = 0 \). Suppose, for a contradiction, that \( |H^*| > 0 \). For \( |e| = k - 2 \), let \( d_v(e) \) be the number of sets in \( H_v \) containing \( e \). Let
\[
s = n - k + 1 - \frac{2k|H^*|}{\binom{k-1}{k-2}}.
\]

Claim 1. There are pairwise disjoint \( (k-2) \)-sets \( e_1, e_2, \ldots, e_k \subset V(H) \setminus \{v\} \) such that \( d_v(e_i) \geq s \) for each \( i \in \{1, 2, \ldots, k\} \).

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Proof. Let $F$ be the family of $(k-2)$-sets in $V(H_v)$ whose degree is at least $s$, and let $F^c$ be the rest of the $(k-2)$-sets in $V(H_v)$. Then

$$(k-1)|H_v| = \sum_e d_v(e) \leq |F|(n-k+1) + |F^c|s,$$

where the sum is over $e \subset V(H_v)$ of size $k-2$. As $|F| + |F^c| = \binom{n-1}{k-2}$, this implies

$$\frac{2k|H^*||F|}{\binom{n-1}{k-2}} \geq (k-1)|H_v| - s \binom{n-1}{k-2} = 2k|H^*| - (k-1)|H^*|$$

since $|H^*| = \binom{n-1}{k-2} - |H_v|$. Hence $|F| \geq \left(1 - \frac{k-1}{2k}\right) \binom{n-1}{k-2} > \frac{1}{2} \binom{n-1}{k-2}$. Let $\{e_1, e_2, \ldots, e_l\}$ be a maximum matching in $F$. If $l < k$, then all other sets of $F$ have an element within $e_1 \cup e_2 \cup \cdots \cup e_l$, which implies (since we may take $n$ large enough) that

$$|F| \leq (k-1)(k-2) \binom{n-1}{k-3} < k^2 \binom{n-1}{k-3} < \frac{1}{2} \binom{n-1}{k-2}.$$ 

This contradiction shows that $l \geq k$ and the claim is proved.

Let $W = \{w \in V(H_v) \mid \exists e : e \cup \{v, w\} \notin H\}$. By Claim 1, $|W| < k(n-s)$. By adding points arbitrarily to $W$, we may assume that $|W| = [k(n-s)]$. Define, for each $i \in \{0, 1, \ldots, k\}$, $H_i = \{e \in H^* : |e \cap W| = i\}$ and let $G = H_0 \cup H_1 \cup \cdots \cup H_{k-2}$. Note that the $H_i$ partition $H^*$.

Claim 2. $|H_{k-1}| \leq \binom{|W|}{k-1}$.

Proof. Suppose there exists a $(k-1)$-set $e \subset W$ and elements $y, z \notin W$ such that $e \cup \{y\}, e \cup \{z\} \in H_{k-1}$. Since $|e| = k-1$, by Claim 1 and the definition of $W$ there exists $i$ such that $e_i \cap e = \emptyset$ and $e_i \cup \{v, y\}, e_i \cup \{e, z\} \in H$. Together with $e \cup \{y\}$ and $e \cup \{z\}$, this yields a 2-regular subgraph in $H$. This contradiction implies that we may count sets in $H_{k-1}$ by their intersection with $W$ to obtain $|H_{k-1}| \leq \binom{|W|}{k-1}$.

Claim 3. $|H^*| \geq \binom{n-k-1}{k/2-1}$.

Proof. Since $|H^*| \geq 1$, there exists $e \in H^*$. Let $e'$ be a $\frac{k}{2}$-subset of $e$. Now for each choice of a $(\frac{k}{2} - 1)$-set $f \subset V(H_v) \backslash e$, one of the sets $f \cup e \cup \{v\}$ or $f \cup (e \backslash e') \cup \{v\}$ must be missing from $H$, otherwise these two sets together with $e$ form a 2-regular subgraph of $H$. Consequently, $|H^*| \geq \binom{n-k-1}{k/2-1}$.

Claim 4. $|G| > \frac{99}{100} |H^*|$.

Proof. We show $|H_{k-1}| + |H_k| < \frac{1}{100} |H^*|$. By Theorem 3, there is a smallest integer $n_0 = n_0(k)$ such every $k$-graph on $n$ vertices with no 2-regular subgraph and with $n > n_0$ has at most $2\binom{n-1}{k-1}$ edges. Assume also that $n_0 > 3k^2$. If $|W| < n_0$, then $|H_k| + |H_{k-1}| < |W|^k < n_0^k$. If $n$
is large enough then, by Claim 3, this is less than $\frac{|H^*|}{100}$, as required. So we assume $|W| > n_0$. Since the $k$-graph $H_k$ itself contains no 2-regular subgraph, $|H_k| \leq 2\left(\frac{|W|}{k-1}\right)$. Recall that

$$|W| = [k(n - s)] = \left[ k \left( k - 1 + \frac{2k|H^*|}{(n-1)(k-2)} \right) \right] < k^2 + \frac{2k^2|H^*|}{(n-1)(k-2)}.$$  

Using this and $|W| > n_0 > 3k^2$, we obtain

$$|W| < \frac{3}{2} \left( \frac{2k^2|H^*|}{(n-1)(k-2)} \right) = \frac{3k^2|H^*|}{(n-1)(k-2)}.$$  

Now suppose, for a contradiction, that $|H_k| + |H_{k-1}| > \frac{|H^*|}{100}$. By Claim 2,

$$\frac{|H^*|}{100} < |H_k| + |H_{k-1}| < 2\left(\frac{|W| - 1}{k-1} + \frac{|W|}{k-1}\right) < \frac{3|W|^{k-1}}{(k-1)!} < \frac{k^2|H^*|^{k-1}}{(k-2)(k-1)}.$$  

Simplifying,

$$|H^*|^{k-2} > \left( \frac{n-1}{k-2} \right)^{k-1} \frac{1}{100k2^k} > \left( \frac{n-1}{k-2} \right)^{(k-2)(k-1)} \frac{1}{100k2^k} > \frac{(n-1)(k-2)(k-1)}{100k(k-2)(k-1)+2k}.$$  

This implies that $|H^*| > \frac{(n-1)^{k-1}}{100k(k-2)(k-1)+2k} > \varepsilon n^{k-1}$, which contradicts (15). This completes the proof of Claim 4.

Let $p$ be the number of pairs $(e, f)$ such that

1. $v \notin e \in H$ and $|e \cap W| \leq k - 2$ (i.e. $e \in G = \cup_{i=0}^{k-2} H_i$)
2. $v \in f \notin H$ and $|f| = k$ (so the number of such $f$s is $|H^*|$)
3. $|e \cap f| = \frac{k}{2}$
4. $e \cap f$ and $e \setminus f$ (which are both $\frac{k}{2}$-sets) have a point outside $W$.

Fix $e \in H$ as in (1) above. Since $|e \setminus W| \geq 2$, there is a $\frac{k}{2}$-subset $g \subset e$ such that neither $g$ nor $e \setminus g$ lies within $W$. Let $h$ be a $(\frac{k}{2} - 1)$-subset of $V(H) \setminus (W \cup e \cup \{v\})$ and let $f = g \cup h \cup \{v\}$. Then the three sets $e, f, (e \setminus g) \cup h \cup \{v\}$ form a 2-regular subgraph. Consequently, either $g \cup h$ or $(e \setminus g) \cup h$ is not in $H_v$. The number of pairs $\{g, e \setminus g\}$ that we can take in this argument is at least

$$\frac{1}{2} \left( \frac{k}{k/2} \right) - \left( \frac{|W \cap e|}{k/2} \right) \geq \frac{1}{2} \left( \frac{k}{k/2} \right) - \left( \frac{k-2}{k/2} \right).$$  

Therefore, counting $p$ from the $e$’s we have

$$p \geq (0.99)|H^*| \left( \frac{1}{2} \left( \frac{k}{k/2} \right) - \left( \frac{k-2}{k/2} \right) \right) \left( n - |W| - k - 1 \right)$$

$$> (0.98)|H^*| \left( \frac{1}{2} \left( \frac{k}{k/2} \right) - \left( \frac{k-2}{k/2} \right) \right) \left( n - k/2 - 1 \right),$$

where the last inequality holds since $|W| < \varepsilon k^{4k}n$ and $n$ is sufficiently large.
On the other hand, counting $p$ from the $f$s we have $p \leq |H^*|\binom{k-1}{k/2}q$, where $q$ is the number of times the $\frac{k}{2}$-sets $g \subset f \setminus \{v\}$ can extend to $e$, where $e \cap f = g$. Let $F$ be the $\frac{k}{2}$-graph of these possible extensions of $g$ to $e$. Let $F_0 \subset F$ be the $\frac{k}{2}$-graph whose edges have no points in $W$ and $F_1 \subset F$ be the $\frac{k}{2}$-graph whose edges have at least one point in $W$.

**Claim 5.** $|F_0| < 2\binom{n}{k/2-1}/k$ and $|F_1| \leq \varepsilon k^{2k}\binom{n}{k/2-1}$.

**Proof.** We start with $F_1$: to each $\frac{k}{2}$-set $h \in F_1$ associate a $(\frac{k}{2} - 1)$-set $h' \subset h$ such that $h' \cap W \neq \emptyset$ and $W \cap h \subset h'$. Such an $h'$ exists by the definition of $F_1$. If there are distinct $h_1, h_2 \in F_1$ with $h'_1 = h'_2$, then there are distinct vertices $y, z \notin W$ such that $h_1 = h'_1 \cup \{y\}$ and $h_2 = h'_2 \cup \{z\}$. By Claim 1, there exists $i$ for which $e_i$ has no point of $W \cap (g \cup h_1 \cup h_2)$. Now the four sets $g \cup h_1, g \cup h_2, e_i \cup \{v, y\}, e_i \cup \{v, z\}$ form a 2-regular subgraph of $H$, contradicting that $H$ has no 2-regular subgraph. Consequently, $|F_1|$ is at most the number of $(\frac{k}{2} - 1)$-sets of $V(H)$ that contain at least one point of $W$. This is at most

$$|W|\binom{n}{k/2 - 2} < \frac{3k^2\varepsilon n^{k-1}}{(n-1)^2}\binom{n}{k/2 - 2} < \varepsilon k^{2k}\binom{n}{k/2 - 1}.$$ 

This gives the bound on $|F_1|$. If there are distinct $h_1, h_2 \in F_0$ with $|h_1 \cap h_2| = k/2 - 1$, then arguing as above we find a 2-regular subgraph of $H$. Consequently, $|F_0| < \binom{n}{k/2 - 1}/k^{k/2-1}$.

Putting these bounds together we have $q \leq \varepsilon k^{2k}\binom{n}{k/2-1} + 2\binom{n}{k/2-1}/k$, and this gives

$$p \leq (1 + \varepsilon k^{4k})|H^*|\binom{k-1}{k/2}\frac{2}{k}\binom{n}{k/2 - 1}.$$ 

Comparing the upper and lower bounds for $p$ and dividing by $|H^*|\binom{n}{k/2-1}$ yields

$$\left(0.98\right)\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2} < (1 + \varepsilon k^{4k})\frac{2}{k}\binom{k-1}{k/2}.$$ 

Since $\varepsilon < k^{4k}/100$ this implies that

$$\left(0.97\right)\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2} < \frac{2}{k}\binom{k-1}{k/2}.$$ 

A short calculation shows that this is equivalent to $(0.97k - 2)(k - 1) < 0.97k(k/2 - 1)$, and it is easily verified that this is false for $k \geq 4$. This contradiction completes the proof of the theorem. \qed

## 7 Concluding Remarks

A $k$-graph is $r$-regular if all its vertices have degree $r$. In contrast to Theorem 1, if the degrees in a $k$-graph are all the same, then a linear number of edges already forces a 2-regular subgraph. Precisely, let $\phi_k$ denote the maximum number $\phi$ such that there exists a $\phi$-regular $k$-graph.
containing no 2-regular subgraph. Then Lemma 1 immediately implies that \( \phi_k \leq (6k)^k \). On the other hand, we have a lower bound of \( \left( \frac{3k}{2} - 1 \right) \) when \( k \) is even and \( \left( \frac{2k}{k-1} \right) \) when \( k \) is odd, by taking complete \( k \)-graphs of the appropriate size (these contain no 2-regular subgraphs because every 2-regular subgraph has at least \( 3k/2 \) vertices when \( k \) is even and at least \( 2k \) vertices when \( k \) is odd). The lower bounds are of order \( c^k \), so there is a substantial gap in the bounds for \( \phi_k \). We leave the open problem of determining \( \phi_k \) and, in particular, \( \phi_3 \). It is expected that if a \( k \)-graph is \( \phi \)-regular and \( \phi \) is a large enough constant depending on \( k \) and \( r \), then every \( \phi \)-regular \( k \)-graph has an \( r \)-regular subgraph (a subgraph in which every vertex has degree \( r \)). In fact, this should hold for multi-\( k \)-graphs – instead of a set of edges a multiset of edges is allowed. Therefore we make the following conjecture:

**Conjecture 2.** Let \( k, r \geq 2 \). There exists an integer \( \phi_k(r) \) such that for \( \phi \gg \phi_k(r) \), every \( \phi \)-regular multi-\( k \)-graph contains an \( r \)-regular subgraph.

This conjecture is wide open for \( k, r \geq 3 \). If \( r \) is a prime not dividing \( k \) and we superimpose \( r - 1 \) copies of the complete \( k \)-graph on \( k + 1 \) vertices, namely \( K_{k+1}^k \), then we obtain a multi-\( k \)-graph \( H_{r,k} \) containing no \( r \)-regular subgraph. To check this, let \( J \) be the all-one matrix, so that \( J - I \) is the incidence matrix of \( K_{k+1}^k \), with rows indexed by edges and columns by vertices. Then over \( \mathbb{Z}_r \), the field of integers mod \( r \), we have \( (r - 1)(J - I) = I - J \), and this matrix has full rank over \( \mathbb{Z}_r \), since \( r \) does not divide \( k \). Therefore no set of rows of \( (r - 1)(J - I) \) is linearly dependent over \( \mathbb{Z}_r \), which means \( H_{r,k} \) has no non-empty subgraph in which all vertices have degree zero modulo \( r \). This simple construction shows that if \( \phi_k(r) \) exists, then \( \phi_k(r) \geq k(r - 1) \). For \( k = 2 \), in other words, for multigraphs, Tăsăinov [16] completely determined \( \phi_k(r) \) using Tutte’s \( f \)-Factor Theorem. Unfortunately, no analogous theorem for \( k \)-graphs is known when \( k \geq 3 \). The following positive evidence for Conjecture 2 follows immediately by extending the proof of Alon, Friedland, Kalai [1] and uses Chevalley’s theorem:

**Theorem 5.** Let \( H \) be an \( n \)-vertex multi-\( k \)-graph, such that \( H \) is \( k(r - 1) + 1 \)-regular, where \( r \) is a prime number. Then \( H \) has a subgraph all of whose vertex degrees are elements of \( \{ r, 2r, \ldots, (k - 1)r \} \).

The multi-\( k \)-graph \( H_{r,k} \) shows that Theorem 5 is tight. Further evidence for Conjecture 2 comes from Rödl’s packing method [14]. A \( k \)-graph is linear if no two of its edges intersect in two or more points. If \( M \) is a matching in a \( k \)-graph \( H \), let \( \text{ex}(M) \) denote the number of vertices not covered by \( M \). Rödl’s Theorem [14] says that every linear \( n \)-vertex \( d \)-regular \( k \)-graph contains a matching \( M \) such that \( \text{ex}(M) \leq d^{-\varepsilon}n \), for some constant \( \varepsilon > 0 \) depending only on \( k \). In fact, the degrees of the vertices in the hypergraph are allowed to be between \( (1 - \delta)d \) and \( (1 + \delta)d \) for the same conclusion, provided \( \delta > 0 \) is a sufficiently small constant depending on \( \varepsilon \). By repeatedly removing \( r \) such matchings from a linear \( d \)-regular \( k \)-graph, we see that for any fixed \( r \), we obtain a subgraph in which all vertices have degree at most \( r \), and at most \( rd^{-\varepsilon}n \) vertices have degree less than \( r \). In other words, we find an “almost \( r \)-regular” subgraph. On the other hand, we do not even have a verification that every large enough Steiner triple system has a three-regular subgraph.
8 Acknowledgments

The authors are grateful to Laci Lovász for suggesting the idea of counting matchings to obtain a 2-regular subgraph. The authors also thank the referees for their careful reading which has helped to improve the presentation.

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