The largest K_r-free set of vertices in a random graph

Tom Bohman[*] Marcus Michelen[†] Dhruv Mubayi[‡]

Abstract

For $r \geq 2$ and a graph G, let $\alpha_r(G)$ be the maximum number of vertices in a K_r-free subgraph of G. We investigate the value $\alpha_r(G)$ when G is the random graph $G \sim G_{n,1/2}$ and discover the following phenomenon: with high probability, $\alpha_r(G)$ lies in an interval of constant length that varies in a non-monotonic fashion from 1 to $\lfloor r/2 \rfloor + 1$ depending on the value of n. The special case $r = 2$ corresponds to the independence number of random graphs which is well-known to have two-point concentration; our results therefore extend and generalize this basic fact in random graph theory, showing more complicated behavior when $r > 2$. We also prove similar results where K_r is replaced by any color critical graph like C_5.

1 Introduction

Consider the binomial random graph $G_{n,1/2}$ and let $\alpha(G_{n,1/2})$ be its *independence number*, i.e. the size of the largest independent set of G. In the 1970's, classical works of Bollobás and Erdős [6] and Matula [15] independently showed that $\alpha(G_{n,1/2})$ is concentrated on at most two values meaning that there is a deterministic function $f(n)$ so that

$$\lim_{n\to\infty} \mathbb{P}(\alpha(G_{n,1/2}) \in \{f(n), f(n)+1\}) = 1\,.$$

In fact, the works of Bollobás-Erdős and Matula [6, 15] imply that for a typical integer n, $\alpha(G_{n,1/2})$ is concentrated on a single value, meaning that for most n we have $\alpha(G_{n,1/2}) = f(n)$ with high probability. Similar results are also known for $G \sim G_{n,p}$ as long as p is not sufficiently small in terms of n [11, 8, 5, 4]. One may interpret an independent set as simply a subset of $G_{n,1/2}$ that avoids the presence of an edge, i.e. a clique on two vertices. With this in mind, for a graph G, define $\alpha_r(G)$ to be the maximum number of vertices in an induced subgraph of G that does not contain a copy of K_r, the clique on r vertices. We note in passing that instead of vertices one can

[*]Department of Mathematical Science, Carnegie Mellon University. Email: tbohman@math.cmu.edu. Research partially supported by NSF Award DMS-2246907

[†]Department of Mathematics, Northwestern University. Email: michelen@northwestern.edu. Research partially supported by NSF awards DMS-2336788 and DMS-2246624

[‡]Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL 60607. Email: mubayi@uic.edu. Research partially supported by NSF Award DMS-2153576.

also consider the maximum number of edges in a subgraph of $G_{n,p}$ that contains no copy of K_r; this problem, posed by Babai-Simonovits-Spencer [2] in 1990, has a long and storied history. Some of the influential results here are due to Conlon-Gowers [7], Schacht [18], and DeMarco-Kahn [9]. In analogy with the independence number, we study the problem of concentration of the random quantity $\alpha_r(G_{n,1/2})$. In particular, does two-point concentration still hold for $\alpha_r(G_{n,1/2})$?

Perhaps surprisingly, one consequence of our work is that two-point concentration holds for $r = 2, 3$ and fails for $r \geq 4$. We show that for each fixed[1] $r \geq 2$ we have that $\alpha_r(G_{n,1/2})$ lies in an interval of constant length that varies in a non-monotonic fashion from 1 to $\lfloor r/2 \rfloor + 1$ depending on the specific value of n. This phenomenon is closely tied to the fact that as n varies, the number of independent sets of size $\alpha(G_{n,1/2})$ can be as small as $O(1)$ or as large as $n^{1+o(1)}$ depending on n. This non-monotonicity is also what drives the changing exponent of the concentration window for the chromatic number of $G_{n,1/2}$ in the work of Heckel [12] (see also Heckel-Riordan [13] for a precise conjecture).

As is the case with the independence number, for a typical value of n, $\alpha_r(G_{n,1/2})$ is concentrated on a single value; when not concentrated on a single value, the size of the interval can take on multiple values for a given r that depend on divisibility properties of r. For instance, in the case of, say, $r = 1001$, there will be values of n for which $\alpha_r(G_{n,1/2})$ is concentrated on exactly 2 values, some where it is concentrated on exactly 501 values, some where it is concentrated on exactly 84 values. In fact, we will show that in the case of $r = 1001$, as n varies $\alpha_r(G_{n,1/2})$ is concentrated on intervals of length $\{1, 2, 3, 4, 5, 6, 8, 11, 12, 15, 18, 25, 35, 51, 84, 168, 501\}$. The pattern of what intervals of concentration occurs is non-monotone, with always going to 1 before a larger value. In particular, we first get 1 point concentration, then 2 point concentration; it alternates between 1-point and 2-point concentration 34 times before seeing 3-point. Further, the interval can shrink even aside from going back to 1 point. In the case of $r = 1001$ we have a stretch where we see 3-point then 1-point, 2-point, 1-point, 3-point, 1-point, 2-point. In general the pattern is complicated, but is fully described by Theorem 2.

Our most general main theorem is Theorem 1 which describes a criterion for witnessing both upper and lower bounds of $\alpha_r(G_{n,1/2})$ along with a statement identifying Poissonian behavior for these witnesses; we use Theorem 1 to deduce a complete description of the intervals of concentration in Theorem 2. Before jumping in to describe this general case, we describe the story for the case of $r = 3$, i.e. the largest triangle-free subgraph of $G_{n,1/2}$. We describe the case of triangles along with a heuristic description of what happens for larger cliques in Section 1.1, state our more general theorem in Section 1.2 and describe the intervals of concentration in Section 1.3. Finally, we describe some slightly coarser results for when avoiding *color critical* graphs other than K_r in Section 1.4.

[1] Our main result Theorem 1 will in fact apply for $r = o(\log\log n/\log\log\log n)$.

1.1 The case of triangles and a heuristic description for larger cliques

The expected number of triangle-free subgraphs of size k in $G_{n,1/2}$ is precisely given by $N_3(k)\binom{n}{k}2^{-\binom{k}{2}}$ where $N_3(k)$ is the number of triangle-free graphs on vertex set $[k]$. A classical theorem of Erdős-Kleitman-Rothschild [10] states that almost all triangle-free graphs are bipartite graphs. With this in mind, one might hope that rather than searching for a large triangle-free graph in $G_{n,1/2}$, we might be able to simply search for a large bipartite graph. Interestingly, it turns out that this is not precisely the case.

Suppose we are in the case where for some value of n, the independence number is k with high probability. As n increases, the independence number will eventually be $k+1$ with high probability. As this transition occurs, before seeing genuine independent sets of size $k+1$, we will begin to see some number of sets of $k+1$ containing only 1 edge. We call such an edge a *defect*. If one considers an independent set of size k and a $(k+1)$-set with a defect, it will not be too common that the union of these two sets induces a triangle free graph—indeed one can compute that the probability a given k-set and $(k+1)$-set with a defect combine to give a triangle-free graph is $(3/4)^{-k}$—but despite this probability being so small, it turns out that by the time there is a $(k+1)$-set with a defect there are enough independent sets of size k that in fact it is likely that a triangle-free graph of size $2k+1$ occurs.

In particular, if we define Y_k to be the number of independent sets of size k and $Z_{k,1}$ to be the number of k-sets of vertices that contain exactly 1 edge, then Theorem 1 implies that

$$\begin{aligned}\mathbb{P}(\alpha_3(G_{n,1/2}) \geq 2k+1) &= \mathbb{P}(Z_{k+1,1} \geq 1) + o(1)\\ \mathbb{P}(\alpha_3(G_{n,1/2}) \geq 2k) &= \mathbb{P}(Y_k \geq 2) + o(1)\end{aligned}$$

for each k chosen so that $n^{-O(1)} \leq \mathbb{E}[Y_k] \leq n^{O(1)}$. Further, each of $Z_{k,1}$ and Y_k behave like Poisson random variables provided their mean is, say, no more than $n^{1/4}$. To describe the intervals of concentration, for a small parameter $\varepsilon = o(1)$ we may define

$$\begin{aligned}a_k &= \min\{n : \mathbb{E}[Y_k] \geq \varepsilon^{-1}\}\\ b_{k,1} &= \min\{n : \mathbb{E}[Z_{k+1,1}] \geq \varepsilon\}, \qquad c_{k,1} = \min\left\{n : \mathbb{E}[Z_{k+1,1}] \geq \varepsilon^{-1}\right\},\\ b_{k,2} &= \min\{n : \mathbb{E}[Y_{k+1}] \geq \varepsilon\}, \qquad c_{k,2} = \min\left\{n : \mathbb{E}[Y_{k+1}] \geq \varepsilon^{-1}\right\}.\end{aligned}$$

Then $\alpha_3(G_{n,1/2}) \in I_n$ with high probability where I_n is the interval

$$I_n = \begin{cases}\{2k\} & \text{if } a_k \leq n < b_{k,1}\\ \{2k, 2k+1\} & \text{if } b_{k,1} \leq n < c_{k,1}\\ \{2k+1\} & \text{if } c_{k,1} \leq n < b_{k,2}\\ \{2k+1, 2k+2\} & \text{if } b_{k,2} \leq n < c_{k,2} = a_{k+1}.\end{cases}$$

For larger r, we recall that classical works of Erdős-Kleitman-Rothschild [10] and Kolaitis-Prömel-Rothschild [14] show that almost every K_{r+1}-free graph is r-partite. As n varies, we will see that for most values of n the subgraphs that realize $\alpha_{r+1}(G_{n,1/2})$ are in fact r-partite graphs. As n increases and we transition from having independence number k to independence number $k+1$, there will be vertex sets of size $k+1$ that have a small number of edges. As in the case of triangles, we refer to these edges as *defects* in potential independent sets. It turns out that we can incorporate these $(k+1)$-sets with few defects into K_{r+1}-free subgraphs. We show that this can be done by "covering" each individual defect with an independent set of size k, where a k-set A *covers* a defect e if there is no copy of K_3 that consists of e and a vertex in A. The point of this definition is that any copy of K_{r+1} that contains a defect from a $(k+1)$-set cannot have any vertex in a k-set that covers the defect; this allows us to include both these sets in a K_{r+1}-free subgraph. It turns out that for n in this regime we can form a maximum K_{r+1}-free subgraph of $G_{n,1/2}$ by identifying disjoint sets $A_1, \ldots, A_r$ where each set A_i is either an independent set of size k or a set of size $k+1$ with a small number of defects with the property that each defect is covered by an independent set of size k in the collection. It will also be the case that each k-set covers only one defect. We show that in this regime X is equal to the maximum number of vertices in such a structure. We do not show that all K_{r+1}-free subgraphs of maximum size have this form; indeed, in some cases different—but closely related—maximum structures are also possible.

1.2 Concentration via Poissonian statistics

We now prepare to describe our main result. As a typical K_r-free graph is $(r-1)$-partite it will be more notationally convenient to work with K_{r+1}-free subgraphs. For $r = o(\log\log n/\log\log\log n)$, set $X = \alpha_{r+1}(G_{n,1/2})$. In analogy with the case of triangles, define

$$Y_k = \text{number of independent sets of size } k$$
$$Z_{k,i} = \text{number of } k\text{-sets of vertices that contain exactly } i \text{ edges.}$$

For each $j \in \{1, \ldots, r\}$ define

$$\mu_j = \left\lfloor \frac{r-j}{j} \right\rfloor = \left\lfloor \frac{r}{j} \right\rfloor - 1 \quad \text{and} \quad \xi_j = j(\mu_j + 2) - r.$$

Note that we have $1 \le \xi_j \le j$. Just as we were occasionally able to form a triangle-free graph by combining an independent set of size k with a $k+1$-set with a single defect, we will see that there are values for n in the regime where $\mathbb{E}[Y_{k+1}]$ goes to zero sufficiently slowly (to be precise, n in the regime defined by $\mathbb{E}[Y_{k+1}] \approx (\log n)^{-2\mu_j}$) for which we can form a K_{r+1}-free subgraph in $G_{n,1/2}$ with $kr+j$ vertices by using ξ_j many $(k+1)$-sets that have μ_j defects each and $j-\xi_j$ many $(k+1)$-sets with μ_j+1 defects each. Each defect in this structure is covered by an independent set of size k, and it follows that the total number of k-sets and $(k+1)$-sets involved in the structure is

$$\xi_j(\mu_j+1) + (j-\xi_j)(\mu_j+2) = j(\mu_j+2) - \xi_j = r.$$

Note that we cannot form such a structure with fewer than ξ_j many $(k+1)$-sets with μ_j defects each as this would require too many k-sets to cover all of the defects. Of course, in order for such a structure to appear n must be large enough for $(k+1)$-sets with μ_j defects to appear in $G_{n,1/2}$. Note that as n increases the smallest number of defects in a $(k+1)$-set decreases. Our main result is that structures of the type discussed above govern the evolution of X; the appearance of these structures is in turn governed by the number of $(k+1)$-sets with few defects; and the number of $(k+1)$-sets with a particular number of defects has the Poissonian statistics that we would anticipate.

Theorem 1. *Let $k = 2\log_2 n + O(\log\log n)$, $r = o(\log\log n/\log\log\log n)$. Then, for n sufficiently large and $j \in \{1,\ldots,r\}$ we have*

$$\left|\mathbb{P}(X \geq kr + j) - \mathbb{P}(Z_{k+1,\mu_j} \geq \xi_j)\right| \leq (\log n)^{-1/2}\,.$$

For $\lambda = \mathbb{E}[Z_{k+1,\mu_j}] \leq n^{1/4}$ we also have

$$\left|\mathbb{P}(\mathrm{Poiss}(\lambda) \geq \xi_j) - \mathbb{P}(Z_{k+1,\mu_j} \geq \xi_j)\right| \leq n^{-1/4}\,.$$

See Figure 1 for a depiction of some of the structures that govern the evolution of $\alpha_{r+1}(G_{n,1/2})$ in the case $r = 11$.

1.3 The interval of concentration for K_{r+1}

We now deduce the extent of concentration of $\alpha_{r+1}(G_{n,1/2})$ for fixed r directly from Theorem 1. Implicit in Theorem 1 is that for each $j \in \{1,2,\ldots,r\}$ such that $\mu_j \neq \mu_{j+1}$ there is an interval in n in which $\alpha_{r+1}(G_{n,1/2})$ is likely to be equal to $rk+j$. In order to make this precise, we define a sequence of values of n, much like we did for the $r=3$ case in Section 1.1. We begin by defining the point at which it is likely that there is a K_{r+1}-free set of vertices in $G_{n,1/2}$ that consists of r independent sets of size k. Note that $\mathbb{E}[Y_k] = \binom{n}{k}2^{-\binom{k}{2}}$. Set[2] $\varepsilon = \varepsilon(k) = 1/\log k$ and define

$$a_k = \min\left\{n : \mathbb{E}[Y_k] \geq \varepsilon^{-1}\right\}.$$

First, let us observe that the sequence a_k is exponential in k. We begin by noting that the bound $\binom{n}{k} \leq (ne/k)^k$ implies $a_k > (k/e)2^{(k-1)/2}$. Next observe that if $n > 2^{(k-1)/2}$, then

$$\mathbb{E}[Y_k] = \left(\frac{ne}{k}2^{-\frac{k-1}{2}}\right)^k e^{O(k^2/2^{k-1})+O(\log k)}.$$

This implies

$$a_k = (1+o_k(1))\frac{k}{e}2^{\frac{k-1}{2}} \quad \text{and} \quad a_{k+1} = (\sqrt{2}+o_k(1))a_k.$$

We will see that for the majority of the values for n in the interval $[a_k, a_{k+1})$ we have $X = rk$ with probability at least $1 - O(\varepsilon)$. Then, as n approaches a_{k+1}, the structure of a maximum K_{r+1}-free subgraph goes through a series of transitions that interpolate between a subgraph of a Turán graph

[2] Note the choice of this particular function is made for convenience. Any $\varepsilon = o_k(1)$ suffices.

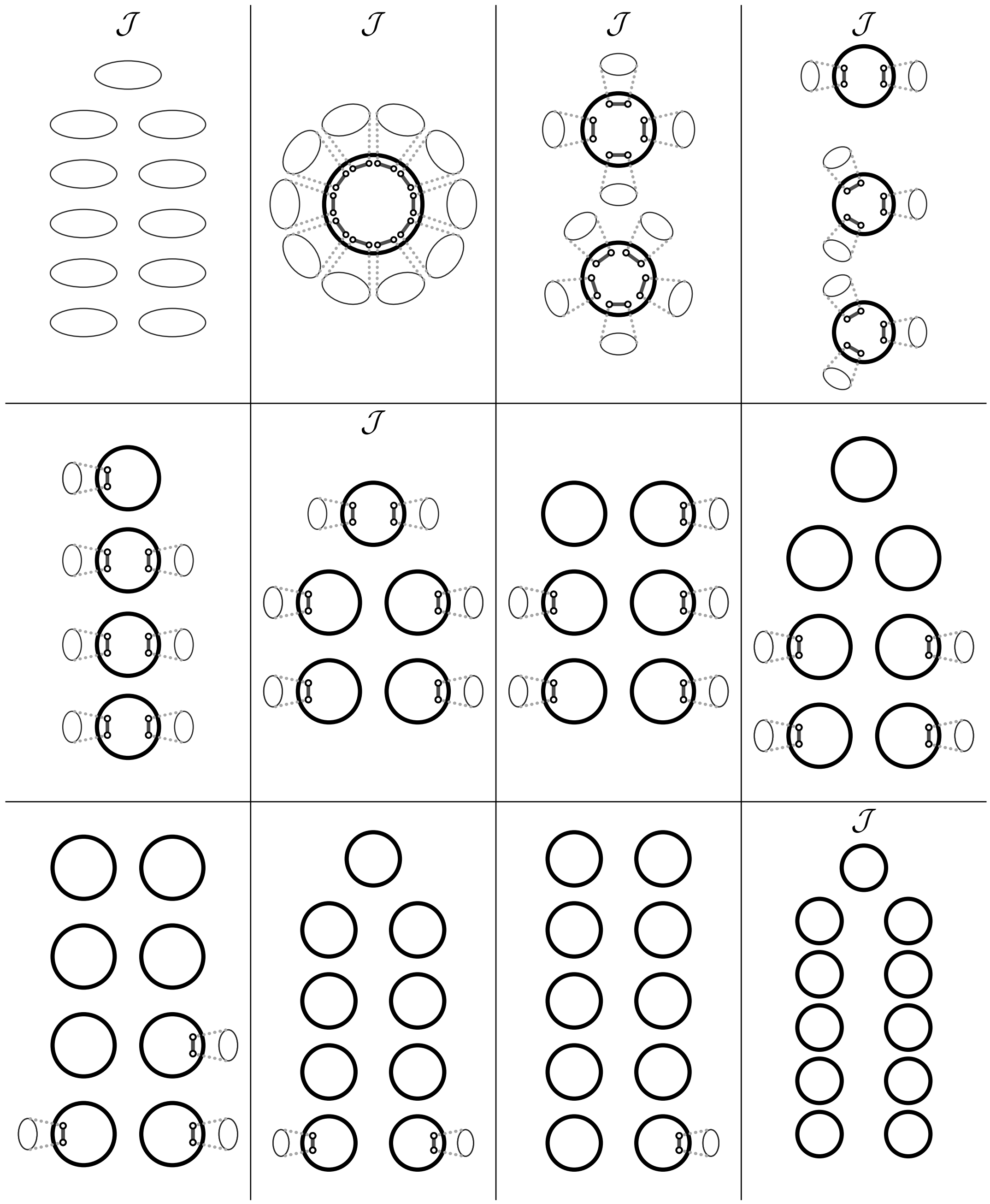

Figure 1: These images depict likely maximum induced subgraphs of $G_{n,1/2}$ not containing K_{12} at different points in the evolution of $G_{n,1/2}$ (where we view n as growing and the edge probability is fixed at 1/2). The black circles represent sets of $k+1$ vertices that have some number of defects, which are the gray edges. The gray ovals are independent sets of size k, and the dotted lines indicate which k-sets cover which defects. Note that we have $\mu_1 = 10, \xi_1 = 1; \mu_2 = 4, \xi_2 = 1; \mu_3 = 2, \xi_3 = 1; \mu_4 = 1, \xi_4 = 1; \mu_5 = 1, \xi_5 = 4$ and $\mathcal{J} = \{1, 2, 3, 5, 11\}$. We placed a $\mathcal{J}$ for sets whose size lies in $\mathcal{J} \mod 11$. These sizes form the endpoints of the intervals of concentration of X.

on kr vertices and a subgraph of a Turán graph on $(k+1)r$ vertices. In order to capture this behavior, we need some additional definitions. For $a_k \le n < a_{k+1}$ define

$$\mu = \mu(n) = \min\left\{i : \mathbb{E}[Z_{k+1,i}] > \varepsilon^{-1}\right\},$$

and note that $\mu(n+1) \le \mu(n)$. Recall that we define

$$\mu_j = \left\lfloor \frac{r-j}{j} \right\rfloor = \left\lfloor \frac{r}{j} \right\rfloor - 1,$$

and by Theorem 1 we expect to have K_{r+1}-free subgraphs with $kr + j$ vertices with some significant probability for those j such that $\mu \le \mu_j$. Note that the quantity $\mu(n)$ decreases from $\Theta(\log n / \log\log n)$ to 0 as n increases from a_k to a_{k+1} (see Fact 6 that appears later in the paper for the detailed calculations that imply this). Thus, for the vast majority of the interval there is no improvement relative to the bound $\alpha_{r+1}(G_{n,1/2}) \ge kr$, since sets of size $k+1$ will have too many defects to be included in our K_{r+1}-free subgraph. Then, as n approaches a_{k+1}, we see values of $\mu(n)$ that equal μ_j for $1 \le j \le r$, and it is here that we see increases in $\alpha_{r+1}(G_{n,1/2})$. There is a series of changes in this value as $\mu(n)$ decreases which we now describe.

Let $\mathcal{J}$ be the set of integers $j \in \{1, \ldots, r\}$ such that $\mu_j \ne \mu_{j+1}$ (where we set $\mu_{r+1} = -1$) and set $s = |\mathcal{J}|$. For example, if $r = 11$, then $\mathcal{J} = \{1, 2, 3, 5, 11\}$ and if $r = 23$, then $\mathcal{J} = \{1, 2, 3, 4, 5, 7, 11, 23\}$. Write

$$\mathcal{J} = \{j_i : i \in [s]\} = \{1 = j_1 < j_2 < \cdots < j_s = r\},$$

and for each $i \in [s]$ set

$$b_{k,i} = \min\left\{n : \mathbb{E}[Z_{k+1,\mu_{j_i}}] \ge \varepsilon\right\}, \qquad c_{k,i} = \min\left\{n : \mathbb{E}[Z_{k+1,\mu_{j_i}}] \ge \varepsilon^{-1}\right\},$$

and note that $c_{k,s} = a_{k+1}$. Observe that these definitions imply the following chain of inequalities. The calculations needed to justify these inequalities may be found in Fact 6.

$$a_k \le b_{k,1} \le c_{k,1} \le b_{k,2} \le c_{k,2} \le \cdots \le b_{k,s-1} \le c_{k,s-1} \le b_{k,s} \le c_{k,s} = a_{k+1}.$$

We are now ready to define the intervals on which X is concentrated. Set $j_0 = 0$ and $c_0 = a_k$. Define

$$I_n = \begin{cases} \{kr + j_{i-1}\} & \text{if } c_{k,i-1} \le n < b_{k,i} \text{ and } 1 \le i \le s \\ \{kr + j_{i-1}, \ldots, kr + j_i\} & \text{if } b_{k,i} \le n < c_{k,i} \text{ and } 1 \le i \le s \end{cases}$$

In other words, if $n \in [b_{k,j_i}, b_{k,j_{i+1}})$, then I_n is either $\{kr + j_{i-1}, \ldots, kr + j_i\}$ or $\{kr + j_i\}$ according to whether $n < c_{k,j_i}$ or $n \ge c_{k,j_i}$.

Example. If $r = 11$, then $\mathcal{J} = \{1, 2, 3, 5, 11\}$, $s = 5$, and $j_1 = 1, j_2 = 2, j_3 = 3, j_4 = 5, j_5 = 11$.

Consequently,

$$I_n = \begin{cases} \{11k\} & \text{if } a_k = c_{k,0} \le n < b_{k,1} \\ \{11k, 11k+1\} & \text{if } b_{k,1} \le n < c_{k,1} \\ \{11k+1\} & \text{if } c_{k,1} \le n < b_{k,2} \\ \{11k+1, 11k+2\} & \text{if } b_{k,2} \le n < c_{k,2} \\ \{11k+2\} & \text{if } c_{k,2} \le n < b_{k,3} \\ \{11k+2, 11k+3\} & \text{if } b_{k,3} \le n < c_{k,3} \\ \{11k+3\} & \text{if } c_{k,3} \le n < b_{k,4} \\ \{11k+3, 11k+4, 11k+5\} & \text{if } b_{k,4} \le n < c_{k,4} \\ \{11k+5\} & \text{if } c_{k,4} \le n < b_{k,5} \\ \{11k+5, \dots, 11(k+1)\} & \text{if } b_{k,5} \le n < c_{k,5} = a_{k+1}. \end{cases}$$

See Figure 1 for a depiction of some of the structures involved in each step here.

With this preparation, we now state our result regarding the concentration of $\alpha_{r+1}(G_{n,1/2})$.

Theorem 2. *Fix $r \ge 1$. Then whp $\alpha_{r+1}(G_{n,1/2}) \in I_n$.*

Proof. Theorem 2 will follow immediately from Theorem 1. Indeed, setting $j = j_{i-1} + 1$ we have $\mu_j = \mu_{j_i}$ and

$$\begin{aligned} & n < b_{k,i} \quad \Longrightarrow \quad \mathbb{E}[Z_{k+1,\mu_{j_i}}] < \varepsilon \quad \Longrightarrow \quad \mathbb{E}[Z_{k+1,\mu_j}] < \varepsilon \\ & \quad \Longrightarrow \quad \mathbb{P}(X > rk + j_{i-1}) = \mathbb{P}(X \ge rk + j) \le \mathbb{P}(\mathrm{Poiss}(\varepsilon) \ge 1) + o_n(1) = 1 - e^{-\varepsilon} + o_n(1) = O(\varepsilon). \end{aligned}$$

And, for k sufficiently large we have

$$\begin{aligned} & n \ge c_{k,i} \quad \Longrightarrow \quad \mathbb{E}[Z_{k+1,\mu_{j_i}}] > 1/\varepsilon \\ & \qquad \Longrightarrow \quad \mathbb{P}(X \ge rk + j_i) \ge \mathbb{P}(\mathrm{Poiss}(1/\varepsilon) \ge r) - o_n(1) = 1 - \frac{(1/\varepsilon)^r}{r!} e^{-1/\varepsilon} - o_n(1) = 1 - o_k(1). \end{aligned}$$

Consequently, the implications above imply that if $c_{k,i-1} \le n < b_{k,i}$, then

$$\mathbb{P}(X = kr + j_{i-1}) = \mathbb{P}(X \ge kr + j_{i-1}) - \mathbb{P}(X > kr + j_{i-1}) \ge 1 - o_k(1) - O(\varepsilon)$$

and if $b_{k,i} \le n < c_{k,i}$, then, since $c_{k,i-1} \le b_{k,i} \le n < c_{k,i} \le b_{k,i+1}$, we have

$$\mathbb{P}(X > kr + j_i) = O(\varepsilon) \qquad \text{and} \qquad \mathbb{P}(X < kr + j_{i-1}) = o_k(1). \qquad \square$$

The proof of Theorem 1 is split into upper and lower bounds. Showing a lower bound on $X \ge kr + j$ will be fairly simple: we show that one can take ξ_j many $(k+1)$-sets with μ_j defects, $j - \xi_j$ many $(k+1)$-sets with $\mu_j + 1$ defects, and $r - j$ many k independent sets and combine them to get a K_{r+1}-free set on $kr + j$ vertices. The upper bound is more delicate. One important tool in the proof of the upper bound is a result of Balogh and Samotij [3] which counts the number of K_{r+1}-free graphs on k vertices that are far from being r-partite. Their argument is an application of an efficient hypergraph container lemma (see Lemma 17) and can be viewed as a generalization and refinement of the classical works of Erdős-Kleitman-Rothschild [10] and Kolaitis-Prömel-Rothschild [14] who showed that almost all K_{r+1}-free graphs are r-partite.

1.4 Color-critical graphs and other properties

Rather than looking only at the largest K_r-free subgraph, we can similarly examine the largest F-free graph, where we use F-free to mean that there is no (not necessarily induced) subgraph isomorphic to F. With this in mind, define $\alpha_F(G)$ to be the size of the largest F-free subgraph of G.

Say that a graph is *r-color-critical* (henceforth *r-critical*) if $\chi(F) = r + 1$ and F contains an edge whose deletion reduces its chromatic number to r. For example, C_5 is 2-critical. While we are not able to deduce as refined a statement for α_F for color-critical graphs, we are able to show concentration on an interval of size at most $r+1$.

Theorem 3. *Fix $r \geq 2$ and an r-critical graph F. There is a sequence m_n so that with high probability $\alpha_F(G_{n,1/2}) \in [m_n - r, m_n]$.*

Remark 4. The proof of Theorem 3 yields a somewhat explicit choice of m_n. Let m_0 be the smallest integer so that the expected number of F-free graphs in $G_{n,1/2}$ whose vertex set has size m_0 is at most 1. Then we take either $m_n = m_0 - 1$ or $m_n = m_0 - 2$ depending on the expected number of F-free subgraphs of size m_0.

As an example, Theorem 3 implies that whp $\alpha_{C_5}(G_{n,1/2}) \in \{m_n - 2, m_n - 1, m_n\}$. The proof of Theorem 3 is much simpler: the upper bound is given by using a theorem of Prömel and Steger enumerating such graphs along with a basic union bound; for the lower bound, the theorem of Prömel and Steger will essentially say that the number of such graphs is the same as the number of r-partite graphs, and so we will find an r-partite graph using the same ideas as in the proof of Theorem 1.

More generally, a *property* $\mathcal{P}$ of graphs is a class of graphs closed under isomorphism. Given a property $\mathcal{P}$ and a graph G, let $\alpha_{\mathcal{P}}(G)$ be the maximum number of vertices in an induced subgraph of G that is isomorphic to some member of $\mathcal{P}$. A graph property $\mathcal{P}$ is *monotone* (respectively *hereditary*) if every (induced) subgraph of a graph with property $\mathcal{P}$ also has property $\mathcal{P}$. For instance, being a bipartite graph or being a triangle-free graph is monotone, while being perfect or chordal is hereditary. To every monotone (hereditary) property $\mathcal{P}$, there exists a collection $\mathcal{F}$ of graphs such that $G \in \mathcal{P}$ if and only if every (induced) subgraph of G is not isomorphic to any member of $\mathcal{F}$. We end with the following problem, which asks if for every monotone or hereditary property there is concentration on a constant-length interval.

Problem 5. *Let $\mathcal{P}$ be a monotone or hereditary property. Does there exist a positive integer $R = R_{\mathcal{P}}$ such that with high probability $\alpha_{\mathcal{P}}(G_{n,1/2})$ lies in an interval of length at most R?*

We believe this is true for all properties $\mathcal{P}$ for which the number of graphs on n vertices in $\mathcal{P}$ is $2^{\Omega(n^2)}$.

2 Preparations for proof of Theorem 1

We now begin discussing various ingredients needed for our proof of Theorem 1. Throughout the rest of the paper we will assume that n is sufficiently large. We will first compute various relevant scales and then import a few useful tools such as Janson's inequality and a consequence of the Stein-Chen method on Poisson approximation.

2.1 Useful asymptotics

Recall that $Z_{k,i}$ is the number of k-sets with exactly i edges and $Y_k = Z_{k,0}$ is the number of independent sets of size k. We start with a basic calculation for understanding the scales at play.

Fact 6. *We have $\mathbb{E}[Y_k] = n^{O(1)}$ if and only if $k = 2\log_2 n - 2\log_2\log_2 n + O(1)$. If $k = 2\log_2 n + O(\log\log n)$ then we also have $\mathbb{E}[Y_{k+1}]/\mathbb{E}[Y_k] = n^{-1+o(1)}$. If we further have $i = O(\log\log n)$ then*

$$\mathbb{E}[Z_{k,i}] = k^{2i(1+o(1))}\mathbb{E}[Y_k]$$

Proof. Write $k = 2\log_2 n - 2\log_2\log_2 n - t$ and compute

$$\mathbb{E}[Y_k] = \binom{n}{k}2^{-\binom{k}{2}} = \exp_2\left((1+o(1))k[\log_2 n - \log_2 k - k/2 + O(1)]\right) = n^{(1+o(1))t+O(1)}\,.$$

The second assertion is similar. For the third assertion, note that

$$\mathbb{E}[Z_{k,i}] = \binom{n}{k}\binom{\binom{k}{2}}{i}2^{-\binom{k}{2}} = k^{2i(1+o(1))}\mathbb{E}[Y_k]$$

completing the proof. $\square$

We will also make frequent use of a second moment-type calculation:

Fact 7. *If $k = 2\log_2 n + O(\log\log n)$ then*

$$\binom{n}{k}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}2^{-2\binom{k}{2}+\binom{j}{2}} \le n^{-1+o(1)}(\mathbb{E}[Y_k]^2 + \mathbb{E}[Y_k])\,.$$

Proof. For $j \in [k-1]$, write $C_j = \binom{k}{j}\binom{n-k}{k-j}2^{\binom{j}{2}}$ so that

$$\binom{n}{k}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}2^{-2\binom{k}{2}+\binom{j}{2}} = \binom{n}{k}2^{-2\binom{k}{2}}\sum_{j=1}^{k-1}C_j.$$

Bound

$$\frac{C_{j+1}}{C_j} = \frac{k-j}{j+1}\frac{\binom{n-k}{k-j-1}}{\binom{n-k}{k-j}}2^j \le \log^3 n \cdot n^{-1}2^j\,.$$

Above we added an extra factor of $\log n$ to absorb all implicit constants. For $2 \le j \le (3/4)k$ we then have

$$\frac{C_j}{C_1} = \prod_{i=1}^{j-1}\frac{C_{i+1}}{C_i} \lesssim \left(\frac{\log^3 n \cdot 2^{j/2}}{n}\right)^{j-1} \le n^{-1+o(1)}\,.$$

Similarly bound

$$\frac{C_{k-(j+1)}}{C_{k-j}} \le \log^3 n \cdot n \cdot 2^{-(k-j)}.$$

For $2 \le k-j \le (3/4)k$ we then have

$$\frac{C_{k-j}}{C_{k-1}} \le n^{-1+o(1)}\,.$$

This implies

$$\begin{aligned}\binom{n}{k}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}2^{-2\binom{k}{2}+\binom{j}{2}} &\le (1+o(1))\binom{n}{k}2^{-2\binom{k}{2}}(C_1 + C_{k-1}) \\ &= (1+o(1))\binom{n}{k}2^{-2\binom{k}{2}}\left(k\binom{n-k}{k-1} + k(n-k)2^{\binom{k-1}{2}}\right) \\ &= n^{-1+o(1)}(\mathbb{E}[Y_k]^2 + \mathbb{E}[Y_k])\,. \end{aligned}$$

□

2.2 Poisson Approximation

Define $\mathrm{Poiss}(\lambda)$ to be the Poisson distribution of mean λ and set $d_{\mathrm{TV}}(X, \mathrm{Poiss}(\lambda))$ to be the total variation distance to the Poisson of mean λ, i.e.

$$d_{\mathrm{TV}}(X, \mathrm{Poiss}(\lambda)) = \sum_{k\ge 0}\left|\mathbb{P}(X = k) - \frac{e^{-\lambda}\lambda^k}{k!}\right|.$$

The Stein-Chen method will allow us to show that the number of copies of certain subgraphs will be approximately Poisson. We use the following simplified version from [1, Thm. 1]

Theorem 8. *For $\alpha \in \mathcal{A}$, let X_α be a Bernoulli random variable and set $p_\alpha = \mathbb{E}[X_\alpha] = \mathbb{P}(X_\alpha = 1)$. Let $W = \sum_{\alpha\in\mathcal{A}} X_\alpha$. For each α, let $B_\alpha = \{\beta : X_\alpha$ and X_β are not independent$\}$. Then*

$$d_{\mathrm{TV}}(W, \mathrm{Poiss}(\mathbb{E}[W])) \le 2\left(\sum_{\alpha\in\mathcal{A}}\sum_{\beta\in B_\alpha}p_\alpha p_\beta + \sum_{\alpha\in\mathcal{A}}\sum_{\beta\in B_\alpha\setminus\alpha}\mathbb{E}[X_\alpha X_\beta]\right)\,.$$

Our main use of Theorem 8 is to show Poissonian statistics for $Z_{k,i}$.

Lemma 9. *If* $k = 2\log_2 n + O(\log\log n)$ *and* $i = O(\log n \log n)$ *then*

$$d_{\mathrm{TV}}(Z_{k,i}, \mathrm{Poiss}(\mathbb{E}[Z_{k,i}])) \le n^{-1+o(1)}(\mathbb{E}[Z_{k,i}]^2 + \mathbb{E}[Z_{k,i}]) .$$

Proof. Let $\lambda = \mathbb{E}[Z_{k,i}] = \mathbb{E}[Y_k]n^{o(1)}$. Set $\mathcal{A} = \binom{[n]}{k}$ to be the collection of k sets of $[n]$ and for each $\alpha \in \mathcal{A}$ set X_α to be the indicator that α has exactly i edges; note that $Z_{k,i} = \sum_{\alpha\in\mathcal{A}} X_\alpha$. Further, note that p_α is constant over all α and we have $p_\alpha = \lambda/\binom{n}{k} = \binom{\binom{k}{2}}{i}2^{-\binom{k}{2}}$. Note that we have $\binom{\binom{k}{2}}{i} = n^{o(1)}$. For each α, we have that B_α is the collection of sets that overlap with α in at least 2 vertices. We can then bound

$$\sum_{\alpha\in\mathcal{A}}\sum_{\beta\in B_\alpha\setminus\alpha} \mathbb{E}[X_\alpha X_\beta] \le n^{o(1)}\binom{n}{k}\sum_{j=2}^{k-1}\binom{k}{j}\binom{n-k}{k-j}2^{-2\binom{k}{2}+\binom{j}{2}} \le n^{-1+o(1)}(\mathbb{E}[Z_{k,i}]^2 + \mathbb{E}[Z_{k,i}])$$

by Fact 7. A similar argument shows

$$\sum_{\alpha\in\mathcal{A}}\sum_{\beta\in B_\alpha} p_\alpha p_\beta \le n^{-1+o(1)}(\mathbb{E}[Z_{k,i}]^2 + \mathbb{E}[Z_{k,i}])$$

and so Theorem 8 completes the proof. □

2.3 Janson's inequality

We make use of not only the classical Janson's inequality but also a generalization due to Riordan and Warnke [17]. The set-up for our application of these inequalities is as follows.

Let S be a finite set, consider the probability space on the $\Omega = \{0,1\}^S$ given by the product measure, and let $\mathcal{I}$ be the collection of all decreasing events in Ω. This implies that for all $A \cap B \in \mathcal{I}$, we have $\mathbb{P}(A\cap B) \ge \mathbb{P}(A)\mathbb{P}(B)$, $A \cup B \in \mathcal{I}$, and $A \cup B \in \mathcal{I}$. Given $A_1, \dots, A_k \in \mathcal{I}$ we let I_i be the indicator random variable for the event A_i and set

$$X = \sum_{i=1}^{k} I_i \quad \text{and} \quad \Delta = \sum_i \sum_{j\sim i} \mathbb{P}(A_i \cap A_j)$$

where we write $i \sim j$ if $i \ne j$ and A_i and A_j are dependent.

Theorem 10 (Janson; Riordan and Warnke)**.** *Under the conditions above, for any* $0 \le t \le \mu$ *we have*

$$\mathbb{P}(X \le \mathbb{E}[X] - t) \le e^{-\frac{t^2}{2(\mu+\Delta)}}.$$

Note that we recover Janson's inequality by specifying a collection of sets $B_1, B_2 \dots, B_k \subset \Omega$ and letting the event A_i be the event no element of B_i appears.

3 The lower bound: covering defects with independent sets

Our goal is to prove the following that gives the lower bound for Theorem 1. Recall that $\mu_j = \lfloor (r-j)/j \rfloor$ and $\xi_j = (\mu_j + 2)j - r$.

Lemma 11. *Let n be sufficiently large and suppose $k = 2\log_2 n - 2\log_2 \log n + O(1)$, $r = o(\log\log n)$. If $0 < j \le r$ then*

$$\mathbb{P}(X \ge kr + j) \ge \mathbb{P}(Z_{k+1,\mu_j} \ge \xi_j) - \frac{1}{\log n}.$$

We begin by noting that we may assume

$$\frac{1}{2\log n} < \mathbb{E}[Z_{k+1,\mu_j}] < \log^{4r} n \tag{1}$$

Indeed, if $\mathbb{E}[Z_{k+1,\mu_j}] \le (2\log n)^{-1}$ then, by Lemma 9 we have

$$\mathbb{P}(Z_{k+1,\mu_j} \ge \xi_j) \le \mathbb{P}(Z_{k+1,\mu_j} \ge 1) \le n^{-1+o(1)} + \left(1 - e^{-\mathbb{E}[Z_{k+1,\mu_j}]}\right) \le \frac{1}{\log n},$$

and the conclusion of Lemma 11 is trivial. To see the upper bound, note that we have

$$\frac{\mathbb{E}[Z_{k+1,i+1}]}{\mathbb{E}[Z_{k+1,i}]} = \frac{\binom{k}{2} - i}{i+1} \le \log^4 n. \tag{2}$$

It follows that if $\mathbb{E}[Z_{k+1,\mu_j}] \ge \log^{4r} n$ then $\mathbb{E}[Z_{k+1,0}] = \mathbb{E}[Y_{k+1}] > \log^4 n$. We may assume $\mathbb{E}[Y_{k+1}] \le n^{1/4}$ by simply restricting our attention to a subset of the vertex set. Then Lemma 9 implies that it is likely that there are at r independent sets of size $k+1$, and a first moment calculation using Fact 7 shows that these independent sets are disjoint whp. So we have a K_{r+1}-free set by taking the union of r pairwise-disjoint independent sets of size $k+1$; to be precise, we have

$$\mathbb{P}(X \ge kr + j) \ge \mathbb{P}(X \ge (k+1)r) \ge 1 - n^{-1/2+o(1)}.$$

As $\mathbb{E}[Z_{k+1,i}] \ge \log^4 n =: \lambda$ implies

$$\mathbb{P}(Z_{k+1,i} \ge r) \ge 1 - \sum_{\ell=0}^{r-1} e^{-\lambda}\frac{\lambda^\ell}{\ell!} \ge 1 - e^{-(\log n)^2 + O((\log\log n)^2)},$$

this bound suffices to establish the conclusion of Lemma 11 in this case.

Note that assumption (1) implies

$$\mathbb{E}[Y_k] = n^{1+o(1)}.$$

Now, for each k and i, define $\mathcal{Z}_{k,i}$ to be the collection of k-sets of $[n]$ that induced exactly i edges so that $Z_{k,i} = |\mathcal{Z}_{k,i}|$. Similarly define $\mathcal{Y}_k = \mathcal{Z}_{k,0}$ to be the collection of independent sets of size k. We will show that elements of $\mathcal{Z}_{k,i} \cup \mathcal{Z}_{k,i+1}$ are pairwise disjoint in the relevant regime. Define the "bad" event $\mathcal{B}_1$ via

$$\mathcal{B}_1 = \{\exists\, S_1, S_2 \in \mathcal{Z}_{k+1,i} \cup \mathcal{Z}_{k+1,i+1}, S_1 \ne S_2, S_1 \cap S_2 \ne \emptyset\}.$$

A simple union bound will handle $\mathbb{P}(\mathcal{B}_1)$:

Fact 12. *For $k = 2\log_2 n + O(\log\log n)$ and $i = o(\log n/\log\log n)$ we have*

$$\mathbb{P}(\mathcal{B}_1) \le n^{-1+o(1)}(\mathbb{E}[Z_{k+1,i}]^2 + \mathbb{E}[Z_{k+1,i}]).$$

Proof. As $i = o(\log n/\log\log n)$, we have $k^i = n^{o(1)}$. We union bound over all possible choices and use Fact 7 to bound

$$\begin{aligned}\mathbb{P}(\mathcal{B}_1) &\le n^{o(1)}\binom{n}{k+1}\sum_{j=1}^{k}\binom{k+1}{j}\binom{n-k-1}{k+1-j}2^{-2\binom{k+1}{2}+\binom{j}{2}} \\ &\le n^{-1+o(1)}(\mathbb{E}[Z_{k+1,i}]^2 + \mathbb{E}[Z_{k+1,i}])\,. \end{aligned}$$

□

For $S \in \mathcal{Z}_{k,i}$ let $\mathcal{D}(S)$ denote the set of defects in S. Recall that an independent set $T \in \mathcal{Y}_k$ *covers* an edge e if there is no triangle consisting of a vertex in T together with e. We prove that each defect is covered by many independent sets of size k. Set $\mathcal{C}(e) = \{T \in \mathcal{Y}_k : T \text{ covers } e\}$. Define the bad events $\mathcal{B}_2$ and $\mathcal{B}_3$ via

$$\begin{aligned}\mathcal{B}_2 &:= \left\{\exists\ S \in \mathcal{Z}_{k+1,i} \cup \mathcal{Z}_{k+1,i+1}, e \in \mathcal{D}(S) : |\mathcal{C}(e)| \le (3/4)^k\mathbb{E}[Y_k]/2\right\} \\ \mathcal{B}_3 &:= \{\exists\ S \in \mathcal{Z}_{k+1,i} \cup \mathcal{Z}_{k+1,i+1}, e \in \mathcal{D}(S), T_1, T_2 \in \mathcal{C}(e) : T_1 \cap T_2 \ne \emptyset, T_1 \ne T_2\}\ .\end{aligned}$$

Lemma 13. *Suppose that $\mathbb{E}[Y_k] \in [n^{5/6}, n^{6/5}]$. Then for $i = O(\log\log n)$ we have*

$$\mathbb{P}(\mathcal{B}_2) \le n^{-1/100} \quad \text{and} \quad \mathbb{P}(\mathcal{B}_3) \le n^{-1/100}\,.$$

Proof. Let $S \in \mathcal{Z}_{k+1,i} \cup \mathcal{Z}_{k+1,i+1}$ and $e \in \mathcal{D}(S)$. We begin with a union bound:

$$\mathbb{P}(\mathcal{B}_2) \le (i\mathbb{E}[Z_{k+1,i}] + (i+1)\mathbb{E}[Z_{k+1,i+1}])\mathbb{P}_{n-(k+1)}\left(|\mathcal{C}(e)| \le (3/4)^k\mathbb{E}[Y_k']/2\right) \tag{3}$$

where we write $\mathbb{P}_{n-(k+1)}$ for the distribution of the random graph on $\binom{V}{2} \setminus \binom{S}{2}, e \in S$ and Y_k' is the number of independent sets of size k that do not intersect S.

To handle this probability, we will show by the second moment method that the number of independent sets that cover the disjoint edge e is near its mean. Define $N = n - (k+1)$. Set $X_S = |\{T \in \mathcal{C}(e) : S \cap T = \emptyset\}|$ and note

$$\mathbb{E}[X_S] = (3/4)^k\binom{N}{k}2^{-\binom{k}{2}} = (1+o(1))(3/4)^k\mathbb{E}[Y_k']\,. \tag{4}$$

We also can bound

$$\begin{aligned}\mathrm{Var}[X_S] - \mathbb{E}[X_S] &\le \binom{N}{k}\sum_{j=1}^{k-1}\binom{k}{j}\binom{N-k}{k-j}2^{-2\binom{k}{2}+\binom{j}{2}}(3/4)^{2k-j} \\ &= N^{-1+o(1)}\left(\mathbb{E}[X_S]^2 + \mathbb{E}[X_S]\right)\end{aligned}$$

where the second line is by adapting Fact 7 and we recall $\mathbb{E}[X_S] = (1+o(1))(3/4)^k \mathbb{E}[Y'_k]$ by (4). By Chebyshev's inequality, we then see

$$\mathbb{P}(X_S \le (3/4)^k \mathbb{E}[Y'_k]/2) \le (4+o(1)) \frac{\mathrm{Var}[X_S]}{\mathbb{E}[X_S]^2} \le 4\left(N^{-1+o(1)} + \frac{1+o(1)}{\mathbb{E}[X_S]}\right) \le n^{-1/99}$$

where in the last line we used $5/6 + 2\log_2(3/4) \ge 1/99$. Consequently,

$$\mathbb{P}_{n-(k+1)}\left(|\mathcal{C}(e)| \le (3/4)^k \mathbb{E}[Y'_k]/2\right) = \mathbb{P}(X_S \le (3/4)^k \mathbb{E}[Y'_k]/2) \le n^{-1/99}.$$

Finally, since $\mathbb{E}[Z_{k+1,i}], \mathbb{E}[Z_{k+1,i+1}]$, and i are each $n^{o(1)}$, we obtain $\mathbb{P}(\mathcal{B}_2) \le n^{-1/100}$ from (3).

For the bound on $\mathcal{B}_3$ we may again union bound to see

$$\begin{aligned}\mathbb{P}(\mathcal{B}_3) &\le (i\mathbb{E}[Z_{k+1,i}] + (i+1)\mathbb{E}[Z_{k+1,i+1}]) \binom{N}{k} \sum_{j=1}^{k-1} \binom{k}{j}\binom{N-k}{k-j} 2^{-2\binom{k}{2}+\binom{j}{2}} (3/4)^{2k-j} \\ &= n^{-1+o(1)}(i\mathbb{E}[Z_{k+1,i}] + (i+1)\mathbb{E}[Z_{k+1,i+1}]) \left(\mathbb{E}[X_S]^2 + \mathbb{E}[X_S]\right).\end{aligned}$$

As $\mathbb{E}[Y'_k] = (1+o(1))\mathbb{E}[Y_k]$, recalling (4) we have

$$\mathbb{E}[X_S] = n^{2\log_2(3/4)+o(1)}\mathbb{E}[Y_k] \le n^{2\log_2(3/4)+6/5+o(1)} \le n^{0.38+o(1)}.$$

As $\mathbb{E}[Z_{k+1,i}], \mathbb{E}[Z_{k+1,i+1}]$ are each $n^{-1+o(1)}\mathbb{E}[Y_k]$ this completes the proof. □

As our structure may include some $(k+1)$-sets with $\mu_j + 1$ defects, we include a fourth bad event $\mathcal{B}_4$, which we define to be the event $Z_{k+1,\mu_j+1} < j - \xi_j$. Note that assumption (1) and observation (2) imply $E[Z_{k+1,\mu_j+1}] \ge (\log n)/(2r)$. We then apply Lemma 9 to conclude

$$\mathbb{P}(\mathcal{B}_4) < n^{-1+o(1)} + \mathbb{P}(\mathrm{Poiss}[(\log n)/(2r)] \le r) < n^{-1+o(1)} + e^{-(\log n)/(2r)} r(\log n)^r < e^{-(\log n)^{1/2}}. \quad (5)$$

Proof of Lemma 11. Set $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_4$ and note that by Fact 12, Lemma 13 and (5) we have that $\mathbb{P}(\mathcal{B}) \le e^{-(\log n)^{1/2}+o(1)}$. Define the "good" event $\mathcal{G}$ to be the event $Z_{k+1,\mu_j} \ge \xi_j$. Then Lemma 9 and Fact 6 show

$$\mathbb{P}(\mathcal{G}) \ge \mathbb{P}(\mathrm{Poiss}(\mathbb{E}[Z_{k+1,\mu_{j_i}}]) \ge \xi_j) - n^{-1/3}. \quad (6)$$

We now note that on the event $\mathcal{G} \cap \mathcal{B}^c$, we have at least ξ_j elements of $\mathcal{Z}_{k+1,\mu_j}$ and at least $j - \xi_j$ elements of $\mathcal{Z}_{k+1,\mu_j+1}$ and these sets are pairwise disjoint by event $\mathcal{B}_1^c$. Let $A_1, \ldots, A_j$ be this collection of $(k+1)$-sets. Each defect in these sets is covered by at least $(3/4)^k\mathbb{E}[Y_k]/2 = n^{1+2\log_2(3/4)+o(1)} \ge n^{3/20}$ elements of $\mathcal{Y}_k$ by $\mathcal{B}_2^c$; by event $\mathcal{B}_3^c$, each pair of independent sets that both cover a defect among $\mathcal{Z}_{k+1,\mu_j} \cup \mathcal{Z}_{k+1,\mu_j+1}$ are disjoint. We may thus construct a K_{r+1}-free graph on $kr + j$ vertices by combining the sets $A_1, \ldots, A_j$ with

$$\xi_j \mu_j + (j - \xi_j)(\mu_j + 1) = j(\mu_j + 1) - \xi_j = r - j$$

covering independent k-sets.

To see that there is no K_{r+1} among these vertices, suppose K is a set of $r+1$ vertices in this structure and assume for the sake of contradiction that K spans a copy of K_{r+1}. For $i = 1, \ldots, j$ let $|K \cap A_j| = k_j$. As the graph induced by K is complete this implies that $K \cap A_j$ contains $\binom{k_j}{2}$ defects. The k-sets in our structure that cover these defects are not in K. As each k-set intersects K in at most one vertex, it follows that we have

$$r + 1 = |K| \leq \sum_{i=1}^{j} k_i + r - j - \sum_{i=1}^{j} \binom{k_i}{2} = r + \sum_{i=1}^{j} k_i - 1 - \binom{k_i}{2} \leq r.$$

We conclude that

$$\mathbb{P}(X \geq kr + j) \geq \mathbb{P}(\mathcal{G} \cap \mathcal{B}^c) \geq \mathbb{P}(\mathcal{G}) - \mathbb{P}(\mathcal{B}).$$

Applying (6) completes the proof. $\square$

4 The upper bound: not covering too many defects

The main goal of this section is to prove the following probabilistic upper bound.

Lemma 14. *Let n be sufficiently large, $k = 2\log_2 n + O(\log\log n)$ and $r = o(\log\log n / \log\log\log n)$. Then*

$$\mathbb{P}(X \geq kr + j) \leq \mathbb{P}(Z_{k+1,\mu_j} \geq \xi_j) + (\log n)^{-1/2}.$$

Remark 15. We note that the error here is only polylogarithmic rather than polynomial. While we make no effort to optimize the error, the error in the upper bound of Lemma 14 in some instances must be only polylogarithmic. This is because the proof of Lemma 11 can also be used to show

$$\mathbb{P}(X \geq kr + j) \geq \mathbb{P}(Z_{k+1,\mu_j} \geq \xi_j) - \mathbb{P}(Z_{k+1,\mu_j+1} < j - \xi_j) - n^{-\Omega(1)}.$$

Depending on the value of n, j_i and j_{i-1}, it is possible for both of the probabilities on the right-hand-side to be polylogarithmic. The error in Lemma 14 is driven by this event and comes into play with Lemma 16.

Note that the desired conclusion of Lemma 14 is equivalent, by taking complements, to showing

$$\mathbb{P}(Z_{k+1,\mu_{j_i}} < \xi_{j_i}) \leq \mathbb{P}(X < kr + j_i) + (\log n)^{-1/2}.$$

If $\mathbb{E}[Z_{k+1,\mu_j}] = \lambda \geq \log n$ note first that we may assume that $\lambda \leq n^{o(1)}$ by restricting our attention to a subset of the vertex set so that this occurs. Then by Lemma 9 we have

$$\mathbb{P}(Z_{k+1,\mu_j} < \xi_j) \leq \sum_{t=0}^{\xi_j - 1} \frac{e^{-\lambda}\lambda^t}{t!} + d_{\mathrm{TV}}(Z_{k+1,\mu_j}, \mathrm{Poiss}(\lambda)) \leq n^{-1+o(1)},$$

where the last inequality holds as $\lambda < n^{o(1)}$. This shows the conclusion of Lemma 14. Consequently, to prove Lemma 14 we may assume that $\mathbb{E}[Z_{k+1,\mu_j}] \leq \log n$.

To begin with, we show that each $(k+1)$-set has at least μ_j defects. Set

$$\mathcal{B}_1' = \bigvee_{\mu<\mu_j} \{Z_{k+1,\mu} > 0\}\,. \tag{7}$$

Lemma 16. *Let $k = 2\log_2 n + O(\log\log n)$, $r = o(\log\log n/\log\log\log n)$ and μ_j satisfy $\mathbb{E}[Z_{k+1,\mu_j}] \le \log n$. Then*

$$\mathbb{P}(\mathcal{B}_1') \le (\log n)^{-1+o(1)}\,.$$

Proof. By Fact 6 we may bound

$$\mathbb{P}(\mathcal{B}_1') \le \sum_{\mu<\mu_j} \mathbb{E}[Z_{k+1,\mu}] \le \sum_{\mu<\mu_j} \mathbb{E}[Z_{k+1,\mu_j}] \cdot k^{-2(\mu_j-\mu)+o(1)} \le (\log n)^{1-2+o(1)} < (\log n)^{-1+o(1)}\,. \quad \square$$

We prove Lemma 14 by a stability argument. The first step in the proof of Lemma 14 is an application of an argument of Balogh and Samotij [3] that bounds the number of K_{r+1}-free graphs on a given set of vertices. As we need to adapt their argument for our purpose, we begin with one of their definitions. We say that a graph G is *t-close to r-partite* if G can be made r-partite by removing t edges. We say that G is *t-far from r-partite* if there is no such set of t edges.

Define $\mathcal{B}_2'$ to be the event that for some $kr \le u \le (k+1)r$ there is a K_{r+1}-free subgraph on u vertices that is $k^{3/4}$-far from being r-partite.

Lemma 17. *If $k = 2\log_2 n + O(\log\log n)$ and $r = o(\log\log n/\log\log\log n)$ then we have $\mathbb{P}(\mathcal{B}_2') \le e^{-\omega(\log n)}$.*

Lemma 17 is essentially implicit in the work [3]. More specifically, [3] implicitly proves the following counting statement. Write $\mathrm{ex}(m, K_{r+1})$ for the maximum number of edges in a K_{r+1}-free graph on m vertices.

Proposition 18. *Let $\mathcal{G}$ be the collection of K_{r+1}-free graphs on m vertices that are $m^{3/5}$-far from being r-partite with $r = o(\log m/\log\log m)$. Then*

$$|\mathcal{G}| \le 2^{\mathrm{ex}(m,K_{r+1})} e^{-m(\log m)^{3/2}}. \tag{8}$$

We deduce Proposition 18 from [3] in Appendix A, where we make no attempt to optimize the logarithms. From here, a simple union bound handles Lemma 17.

Proof of Lemma 17. We will handle a fixed $u \in [kr, (k+1)r]$ first and let $\mathcal{B}_2'(u)$ denote the event that there is a K_{r+1}-free subgraph on u vertices that is $k^{3/4}$-far from being r-partite. Write $u = kr + a$. Let us argue that

$$\frac{\binom{n}{u}}{\binom{n}{k}^{r-a}\binom{n}{k+1}^{a}} \mathbb{E}[Y_k]^{r-a}\mathbb{E}[Y_{k+1}]^a < \mathbb{E}[Y_k]^r. \tag{9}$$

First, assume $a = 0$. In this case we are to show that $\binom{n}{kr} < \binom{n}{k}^r$. As $1 \ll k = O(\log n)$, we certainly have $\binom{n}{k} > (2n/k)^k$, and since $r \geq 2$, this yields $\binom{n}{kr} \leq (ne/kr)^{kr} < (2n/k)^{kr} < \binom{n}{k}^r$. Next, assume $a > 0$. In this case, apply Fact 6 to obtain

$$\frac{\binom{n}{u}}{\binom{n}{k}^{r-a}\binom{n}{k+1}^a}\mathbb{E}[Y_k]^{r-a}\mathbb{E}[Y_{k+1}]^a < \frac{\binom{n}{u}}{\binom{n}{k}^{r-a}\binom{n}{k+1}^a}\mathbb{E}[Y_k]^r n^{-a+o(a)}.$$

The power of n in the expression on the right above (excluding the $\mathbb{E}[Y_k]$ term) is

$$u - a + o(a) - k(r-a) - (k+1)a = kr - k(r-a) - (k+1)a + o(a) = -a + o(a) < 0,$$

and hence (9) holds.

Next, observe that $\binom{u}{2} = \mathrm{ex}(u, K_{r+1}) + (r-a)\binom{k}{2} + a\binom{k+1}{2}$. Let $\mathcal{F}$ denote the set of K_{r+1}-free graphs on u vertices that are $k^{3/4}$-far from being r-partite. Then (9) yields

$$\begin{aligned}
\mathbb{P}(\mathcal{B}_2'(u)) &\leq \binom{n}{u}|\mathcal{F}|\,2^{-\binom{u}{2}} \\
&= |\mathcal{F}|\binom{n}{u}2^{-\mathrm{ex}(u,K_{r+1})-(r-a)\binom{k}{2}-a\binom{k+1}{2}} \\
&= |\mathcal{F}|2^{-\mathrm{ex}(u,K_{r+1})}\frac{\binom{n}{u}}{\binom{n}{k}^{r-a}\binom{n}{k+1}^a}\mathbb{E}[Y_k]^{r-a}\mathbb{E}[Y_{k+1}]^a \\
&\leq \frac{|\mathcal{F}|}{2^{\mathrm{ex}(u,K_{r+1})}}\mathbb{E}[Y_k]^r.
\end{aligned}$$

Every graph in $\mathcal{F}$ is certainly $u^{3/5}$-far from being r-partite as $u^{3/5} < ((k+1)r)^{3/5} < k^{3/4}$. Hence by Proposition 18, $|\mathcal{F}| \leq 2^{\mathrm{ex}(u,K_{r+1})}e^{-u(\log u)^3}$. As $k \sim 2\log_2 n$ and $r = o(\log\log n)$, we finally obtain

$$\mathbb{P}(\mathcal{B}_2'(u)) \leq e^{-u(\log u)^3}\mathbb{E}[Y_k]^r \leq \exp(-kr(\log kr)^3 + 2r\log n) \leq \exp(-\omega(\log n))\,. \qquad \square$$

In light of Lemma 17 we restrict our attention to potential K_{r+1}-free graphs with vertex sets $A_1\dot\cup A_2\dot\cup\cdots\dot\cup A_r$ where each set A_i contains at most $k^{3/4}$ edges. We say a set is *light* if it has at most $k^{3/4}$ edges.

A simple first moment computation shows that with high probability there are no light sets on m vertices where $m \geq k+2$. Let $\mathcal{B}_3'$ be the event that there is a light $(k+2)$-set.

Fact 19. *If* $\mathbb{E}[Y_k] \leq n^{1+o(1)}$ *then* $\mathbb{P}(\mathcal{B}_3') \leq n^{-1+o(1)}$.

Proof. By Fact 6,

$$\mathbb{P}(\mathcal{B}_3') = \binom{n}{k+2}\binom{\binom{k+2}{2}}{k^{3/4}}\left(\frac{1}{2}\right)^{\binom{k+2}{2}} = \mathbb{E}[Y_{k+2}]e^{O((\log k)k^{3/4})} = \mathbb{E}[Y_{k+2}]n^{o(1)} \leq n^{-1+o(1)}\,. \qquad \square$$

Since we assume that $|A_1 \cup \cdots \cup A_r| \geq kr$ and $\max|A_i| \leq k+1$ we can restrict our attention to light sets A_i with $k-(r-1) \leq |A_i| \leq k+1$. Further, recall that we refer to each edge in such a light set as a *defect*.

We now make observations that put additional conditions on the defects in the sets $A_1, \dots, A_r$. We start by noting that a collection of $r-1$ pairwise-disjoint light sets spans many copies of K_{r-1}. Let $\mathcal{B}_4'$ be the event that there are pairwise disjoint light sets $A_1, A_2, \dots, A_{r-1}$ and sets $B_i \subset A_i$ such that $k-(r-2) \le |A_i| \le k+1$ and $B_i \ge k^{2/3}$ such that there is no copy of K_{r-1} with exactly one vertex in each B_i

Lemma 20. *If $k = 2\log_2 n - 2\log_2 \log_2 n + O(1)$ and $r = o(\log\log n / \log\log\log n)$ then we have $\mathbb{P}(\mathcal{B}_4') = o(1/n)$.*

Proof. Note that the claim holds trivially for $r=2$. For larger r the proof is a straightforward application of Janson's inequality. We will union bound over choices of $A_1, \dots, A_{r-1}$ and $B_1, \dots, B_{r-1}$.

Let $r \ge 3$ and let $B_1, \dots, B_{r-1}$ be fixed, disjoint sets of $k^{2/3}$ vertices. Let the random variable W count the number of K_{r-1}'s among these sets with one vertex in each B_i. We have $\mathbb{E}[W] = k^{2(r-1)/3} 2^{-\binom{r-1}{2}}$. Furthermore, letting Δ denote the corresponding quantity for an application of Janson's inequality, we may bound

$$\Delta \le \mathbb{E}[W] \sum_{j=2}^{r-2} \binom{r-1}{j} k^{\frac{2(r-j-1)}{3}} \left(\frac{1}{2}\right)^{\binom{r-1}{2} - \binom{j}{2}} = \mathbb{E}[W]^2 \sum_{j=2}^{r-2} \binom{r-1}{j} 2^{\binom{j}{2}} k^{-\frac{2j}{3}} = r^{O(1)} \mathbb{E}[W]^2 k^{-4/3}.$$

Janson's inequality (Theorem 10) then implies that the probability that the sets $B_1, \dots, B_{r-1}$ span no copy of K_{r-1} is at most $e^{-\Omega(k^{5/4})}$.

Next we note that the expected number of collections of pairwise disjoint light sets $A_1, \dots, A_{r-1}$ that contain sets $B_1, \dots, B_{r-1}$, respectively, such that $|B_i| = k^{2/3}$ is at most

$$\left[\sum_{\rho=-1}^{r-2} \mathbb{E}[Y_{k-\rho}] \binom{\binom{k-\rho}{2}}{k^{3/4}} \binom{k-\rho}{k^{2/3}} \right]^{r-1} < \left[r\mathbb{E}[Y_{k-r+1}] \binom{\binom{k+1}{2}}{k^{3/4}} \binom{k+1}{k^{2/3}} \right]^{r-1} = e^{O(r^2 k)}.$$

The desired bound on $\mathbb{P}(\mathcal{B}_4')$ then follows from the first moment. ◻

Observe that if we have a defect in one of the sets $A_1, \dots, A_r$ then we expect that edge to form copies of K_3 with many (about $k/4$) of the vertices in each of the other A_i. Lemma 20 then implies that in such a situation we find a copy of K_{r+1} in $A_1 \dot\cup A_2 \dot\cup \cdots \dot\cup A_r$. So, if we find defects among the sets $A_1, \dots, A_r$ then we would be in the situation (which we might naively anticipate is unlikely) that for each of the defects there is one of the sets not containing the defect with the property that the defect forms few copies of K_3 with the vertices in that set. Now, it is important to note that the probability that a defect makes *no copies* of K_3 with one of the sets is not negligible. Indeed, we take advantage of this fact in the proof of Lemma 11.

Given a collection of pairwise disjoint light sets $A_1, \dots, A_r$ we say that a set A_j *weakly covers* a defect in a set A_i if the defect forms a K_3 with at most $k^{2/3}$ of the vertices in A_j. In light of Lemma 20, we may assume that if $A_1 \dot\cup \cdots \dot\cup A_r$ defines a K_{r+1}-free subgraph of $G_{n,1/2}$ then every defect in $A_1, \dots, A_r$ is weakly covered by one of the sets in the collection.

We first observe that only sets of size smaller than $k+1$ can weakly cover a defect in a $(k+1)$-set. Let $\mathcal{B}_5'$ be the event that there exist light $(k+1)$-sets A and B such that A weakly covers a defect in B.

Fact 21. *If $\mathbb{E}[Y_k] \leq n^{1+o(1)}$ then $\mathbb{P}(\mathcal{B}_5') = e^{-\Omega(k)}$*

Proof. The probability that a given light $(k+1)$-set A weakly covers a defect e in a light $(k+1)$-set B is at most

$$\binom{k+1}{k^{2/3}}\left(\frac{3}{4}\right)^{k+1-k^{2/3}} = \left(\frac{3}{4}\right)^{k(1+o(1))}.$$

We now union bound over all potential configurations using $\mathbb{E}[Y_{k+1}] = n^{-1+o(1)}\mathbb{E}[Y_k] = n^{o(1)}$:

$$\mathbb{P}(\mathcal{B}_5') \leq \mathbb{E}[Y_{k+1}]^2 \binom{\binom{k+1}{2}}{k^{3/4}}^2 k^{3/4}\left(\frac{3}{4}\right)^{k(1+o(1))} = \left(\frac{3}{4}\right)^{k(1+o(1))}. \qquad \square$$

As our goal here is to bound X relative to the simple lower bound kr, our focus is on defects in sets A_i such that $|A_i| = k+1$. Our canonical picture is that each of these defects is covered by an independent set A_ℓ such that $|A_\ell| = k$, where each such k-set covers at most one defect. But we emphasize that other structures are possible that occur with roughly the same probability.

For example, consider $j \in \mathcal{J}$ such that $\xi_j \leq j-2$ (e.g. the case $r = 11, j = 3, \mu_3 = 2, \xi_3 = 1$ depicted in Figure 1 and shown again in Figure 2).

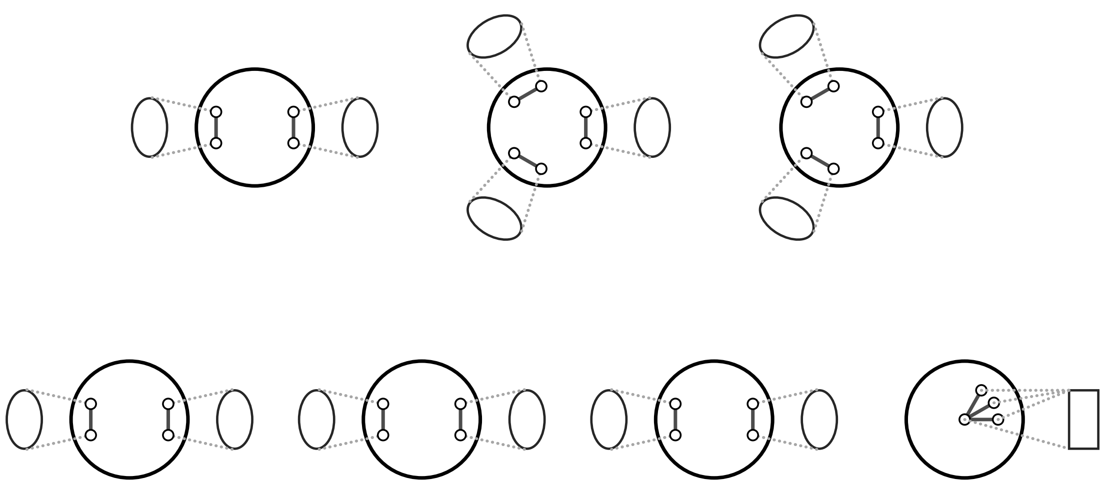

Figure 2: Two structures that could achieve $X \geq rk+3$ in the case $r = 11$ (where we seek a K_{12}-free set) are shown. Recall that $\mu_3 = 2, \xi_3 = 1$, and whp $Z_{k+1,2} \geq 1$ implies that the standard structure, which is the structure in the first row, appears whp. However, if $Z_{k+1,2} \geq 3$ we could also achieve $X \geq kr+3$ with the second structure depicted here. In this second structure the gray rectangle is an independent set of size $k-1$ that covers the three defects in the fourth $(k+1)$-set.

Our canonical picture, shown on the top line of Figure 2, consists of $\xi_j = 1$ many $(k+1)$-sets with $\mu_j = 2$ defects each together with $j - \xi_j = 2$ many $(k+1)$-sets with $\mu_j + 1 = 3$ defects. Each of the defects is covered by its own k-set. However, there are other ways to obtain a K_{r+1}-free set of size $kr + j$. One of them is to combine three $(k + 1)$-sets with $\mu_j = 2$ defects, each of which is covered by a k-set, together with one $(k + 1)$-set with 3 defects all of which are covered by a single $(k - 1)$-set. A typical k-set is not likely to cover more than one defect, however there are sufficiently many $(k - 1)$-sets that it is possible for one of these to cover more than one defect. Such an adjustment does not influence X but leads to flexibility in the maximum structure. This trade-off—of introducing some $(k + 1)$-sets with more than μ_j defects at the expense of using sets smaller than k to cover them—is captured in Lemma 22, which is the technical core of the proof of Lemma 14.

For the remainder of the proof we fix a set of at most $\mu < r$ defects in each of the $(k + 1)$-sets among the A_i. We prove that each $(k-\rho)$-set A_ℓ weakly covers at most $1 + (\mu + 1)\rho$ of the specified defects in the $(k + 1)$-sets. To be precise, let $\mathcal{E} = \mathcal{E}_{\mu,\rho,\beta}$ be the event that there is a light set A such that $|A| = k - \rho$ where $0 \le \rho \le r - 2$ and edges $e_1, e_2, \ldots, e_\beta$ not contained in A such that

1. e_i is an edge in a light set A_i such that $|A_i| = k + 1$ and $A \cap A_i = \emptyset$ for $i = 1, \ldots, \beta$
2. if $i \ne \ell$ then A_i and A_ℓ are either equal or disjoint,
3. each set A_i contains at most μ of the edges $e_1, \ldots, e_\beta$,
4. at most $k^{2/3}$ of the vertices in A form a triangle with e_i for $i = 1, \ldots, \beta$, and
5. $\beta > 1 + (\mu + 1)\rho$.

Lemma 22. *If* $\mathbb{E}[Y_k] \le n^{1+o(1)}$ *then* $\mathbb{P}[\mathcal{E}] \le n^{-1/8+o(1)}$.

Proof. We bound the expected number of such configurations. Consider a fixed ρ and the connected components in the graph formed by the β defects. Let β_1 be the number of isolated edges in this graph and let β_2 be the number of connected components with at least 2 edges. We claim that Item 3 and Item 5 imply that we have

$$\frac{\beta_1}{2} + \beta_2 \ge \rho + 1. \tag{10}$$

Indeed, if $\mu = 1$ then $\beta_2 = 0$ and (10) follows immediately from Item 5. If $\mu \ge 2$ then observe that we have

$$\beta_1 + \mu\beta_2 \ge \beta \ge 2 + (\mu + 1)\rho \quad \Rightarrow \quad \frac{\beta_1}{2} + \beta_2 \ge \rho + \frac{\rho + 2 - \beta_1}{\mu} + \frac{\beta_1}{2}$$

Noting that the left hand side of (10) is in $\mathbb{Z}^+ \cup \{0, 1/2\}$, we see that (10) follows by considering the cases $\beta_1 = 0$; $\beta_1 = 1$; $2 \le \mu \le 4$; and finally $\mu \ge 5$ and $\beta_1 \ge 2$.

Recall that the probability that a fixed edge makes no copy of K_3 with a set of t vertices is $(3/4)^t$. Similarly, the probability that no edge in some fixed connected graph with at least two edges makes no copy of K_3 with the vertices in a set of t vertices is at most $(5/8)^t$. Thus, for a given light set A on $k - \rho$ vertices and a given collection of $(k + 1)$-sets with defect graphs having β_1 isolated edges and β_2 connected components with at least 2 edges, the probability that at most $k^{2/3}$ of the vertices in A form a triangle with the defect graph is bounded above by

$$\begin{aligned}\binom{k-\rho}{k^{2/3}}^{\beta}\left(\frac{3}{4}\right)^{\beta_1(k-k^{2/3})}\left(\frac{5}{8}\right)^{\beta_2(k-k^{2/3})} &\leq \left(\frac{9}{16}\right)^{(\beta_1+o(1))\log_2 n}\left(\frac{25}{64}\right)^{(\beta_2+o(1))\log_2 n}\\ &\leq \left(\frac{2}{5}\right)^{(\frac{\beta_1}{2}+\beta_2+o(1))\log_2 n}.\end{aligned}$$

Write $Z_{k,\leq i}$ for the number of k-sets with at most i edges. Then by Fact 6

$$\mathbb{E}[Z_{k-\rho,\leq k^{3/4}}] \leq n^{o(1)}\mathbb{E}[Y_{k-\rho}] = n^{1+\rho+o(1)} \qquad \text{and} \qquad \mathbb{E}[Z_{k+1,\leq k^{3/4}}] = n^{o(1)}\,.$$

The number of choices for the set A together with the tuple of at most $\beta_1+\beta_2$ light sets of size $k+1$ together with choices for the configuration of defect graph is bounded above by

$$\mathbb{E}[Z_{k-\rho,\leq k^{3/4}}]\sum_{j=1}^{\beta_1+\beta_2}\left[\mathbb{E}[Z_{k+1,\leq k^{3/4}}]\binom{k^{3/4}}{\mu}\right]^j = n^{1+\rho+o(1)}\cdot\sum_{j=1}^{\beta_1+\beta_2}\left[2^{O((\log k)k^{3/4})}\right]^j = n^{1+\rho+o(1)}$$

where we used that $\mu,\rho<r$. Combining the two displayed equations provides the bound

$$\mathbb{P}[\mathcal{E}] \leq \left(\frac{2}{5}\right)^{(\frac{\beta_1}{2}+\beta_2+o(1))\log_2 n}\cdot n^{1+\rho+o(1)} \leq \left(n^{\log_2(2/5)+1}\right)^{(\frac{\beta_1}{2}+\beta_2+o(1))} \leq n^{-1/8+o(1)}$$

where in the first inequality we used (10) and the second we used that $\frac{1}{2}(\log_2(2/5)+1) < -1/8$. $\square$

Remark 23. We note that some simple counting illustrates that the bound on β in the definition of $\mathcal{E}$ should be sufficient for our purposes. We would like to conclude that we cannot gain an advantage in the random variable X by replacing a k-set in the canonical structure with a smaller set. If we replace a k-set with a $(k-\rho)$-set then we should also replace at least ρ other k-sets with $(k+1)$-sets in order to achieve a balance relative to X. Thus, the new set of size $k-\rho$ would need to weakly cover the defects in ρ new $(k+1)$-sets (of which there are at least $\rho\mu$). Furthermore, the new $(k-\rho)$-set would need to cover the defects that were potentially weakly covered by the $\rho+1$ sets of size k that were previously in the canonical structure (one that was replaced with a smaller set and ρ that were replaced with larger sets). Thus, in order to be useful, this $(k-\rho)$-set should weakly cover at least $\mu\rho+\rho+2$ defects. If the event $\mathcal{E}$ does not hold, then this type of exchange is not possible.

Remark 24. The bound in Item 5 is not tight in general, it is just what is needed for the proof of Lemma 14. For example, if $\mu\geq 2$ then we could replace Item 5 with $\beta>1+\mu\rho$ and we see that using smaller sets generally loses ground. However, we cannot replace Item 5 with $\beta>1+\mu\rho$ when $\mu=1$ as this stops working for some moderate ρ.

As a last quasirandomness event, we define

$$\mathcal{B}_6' = \bigcup_{\mu\geq\mu_j,\rho\in[0,r-2],\beta\in[0,r^2]}\mathcal{E}_{\mu,\rho,\beta}$$

where we are union bounding over all choices of μ,ρ and β that (naively) are possible. An immediate corollary to Lemma 22 by using the union bound is the following:

Corollary 25. *If* $\mathbb{E}[Y_k] \le n^{1+o(1)}$ *then we have* $\mathbb{P}(\mathcal{B}_6') \le n^{-1/8+o(1)}$.

Proof. There are at most $O(r^3)$ choices for β, ρ. By Item 5, for each choice of $\beta \in [0, r^2]$ and $\rho \neq 0$ we have at most $O(r^2)$ choices of μ. Taking a union bound over these choices and applying Lemma 22 completes the proof. $\square$

We now have all the ingredients needed to quickly complete the proof of Lemma 14.

Proof of Lemma 14. We assume that none of the events $\mathcal{B}_1', \ldots, \mathcal{B}_6'$ hold by Lemma 16, Lemma 17, Fact 19, Lemma 20, Fact 21, and Corollary 25. We also assume that $Z_{k+1,\mu_j} \le \xi_j - 1$ and want to show that $X \le kr + (j-1)$. By event $\mathcal{B}_2'^c$, if we have $X \ge rk$ then we must have that it can be partitioned into r light sets. By $\mathcal{B}_3'^c$, each set in the partition has size at most $k+1$. Suppose we have a partition of pairwise disjoint light sets $A_1, \ldots, A_r$ so that $k - (r-1) \le |A_i| \le k+1$ for all i and so that $A_1 \cup A_2 \cup \ldots \cup A_r$ is K_{r+1} free. Let b be the number of A_j with size at most k and reorder the sets so that $A_1, \ldots, A_b$ are these sets; for $i = 1, \ldots, b$ define $\rho_i = k - |A_i|$ so that $|A_i| = k - \rho_i$. Let a be the number of A_i of size $k+1$. The size of the set $A_1 \cup A_2 \cup \ldots \cup A_r$ is

$$rk + a - \sum_{i=1}^{b} \rho_i =: rk + \delta\,.$$

Our goal is to show that $\delta \le j - 1$. As such, we may assume that $a \ge j$ since $\rho_i \ge 0$ for all i. Write $\mu = \mu_j$ for simplicity and note that no $(k+1)$-set has fewer than μ defects by $\mathcal{B}_1'^c$. Since $Z_{k+1,\mu} \le \xi_j - 1$, and a sets among the A_i have size $k+1$, at least $a - (\xi_j - 1)$ sets among the A_i have at least $\mu + 1$ defects. Consequently, the total number of defects is at least

$$(\xi_j - 1)\mu + (a - (\xi_j - 1))(\mu + 1)\,.$$

Applying event $\mathcal{B}_6'^c$, each A_i covers at most $1 + (\mu+1)\rho_i$ defects. Since all defects are covered by $\{A_i\}_{i=1}^{b}$ we must have

$$(\xi_j - 1)\mu + (a - (\xi_j - 1))(\mu + 1) \le \sum_{i=1}^{b} (1 + (\mu+1)\rho_i)\,.$$

Rearranging and recalling $a - \sum_i \rho_i = \delta$ gives

$$\delta \le \frac{b + \xi_j - 1}{\mu + 1} = \frac{b + j(\mu+2) - r - 1}{\mu + 1} = j + \frac{b + j - r - 1}{\mu + 1}\,.$$

To complete the proof recall that we assume $a \ge j$ which implies $b + j \le r$. Therefore, $\delta < j$, as desired. $\square$

5 Proof of Theorem 3

In this section, we give the proof of Theorem 3. When we say that G is F-free, we mean that G contains no subgraph isomorphic to F. We do not impose any requirement on the subgraph being induced.

We prove Theorem 3 via a first moment calculation. We need the following result of Prömel and Steger [16] on the number of F-free graphs.

Theorem 26 (Prömel-Steger [16]). *Fix $r \geq 2$ and an r-critical graph F. Then for m sufficiently large, the number of F-free graphs on vertex set $[m]$ is at most*

$$2^{\left(1-\frac{1}{r}\right)\frac{m^2}{2}+m\log_2 r+\Theta(\log_2 m)}.$$

With this in mind, define

$$N_m := 2^{\left(1-\frac{1}{r}\right)\frac{m^2}{2}+m\log_2 r}, \qquad \overline{N}_{n,m} = \binom{n}{m} N_m 2^{-\binom{m}{2}}.$$

We will compare $\overline{N}_{n,m}$—and thus the expected number of F-free sets—to the expected number of k independent sets. Recall that Y_k is the number of independent sets of size k in $G_{n,1/2}$.

Lemma 27. *Suppose that $m = rk$ is so that $\overline{N}_{n,m} = n^{\Theta(1)}$. Then $\overline{N}_{n,m} = n^{o(1)} \cdot (\mathbb{E}[Y_k])^r$.*

Proof. First write

$$\overline{N}_{n,m} \cdot (\mathbb{E}[Y_k])^{-r} = \binom{n}{m} \exp_2\left(\left(1-\frac{1}{r}\right)\frac{m^2}{2} + m\log_2 r - \binom{m}{2}\right) \cdot \binom{n}{k}^{-r} \exp_2\left(r\binom{k}{2}\right).$$

Since $\overline{N}_{n,m} = n^{\Theta(1)}$ we have $m = O(\log n)$ and so we have

$$\binom{n}{m} \cdot \binom{n}{k}^{-r} = n^{o(1)} \left(\frac{en}{m}\right)^m \left(\frac{en}{k}\right)^{-kr} = n^{o(1)} r^{-m}$$

where in the last equality we recalled that $m = kr$. Noting that

$$\left(1-\frac{1}{r}\right)\frac{m^2}{2} - \binom{m}{2} + r\binom{k}{2} = 0$$

completes the proof. $\square$

Using Theorem 26 we will be able to find an expression for the value of m_0 in Theorem 3 up to lower order terms.

Lemma 28. *Let X_m be the number of F-free sets of size m. Define $m_0 = \min\{m : \mathbb{E}[X_m] \leq 1\}$. Then $m_0 = 2r\log_2 n + O(\log\log n)$.*

Proof. Set $m = \alpha \log_2 n$ for $\alpha = O(1)$ to be chosen later and note that

$$\begin{aligned}
\overline{N}_{n,m} &= \binom{n}{m} \exp_2\left(\left(1 - \frac{1}{r}\right)\frac{m^2}{2} + m\log_2 r - \binom{m}{2}\right) \\
&= \exp_2\left(m\left(\log_2 e + \log_2 n - \log_2 m + \left(1 - \frac{1}{r}\right)\frac{m}{2} + \log_2 r - \frac{m}{2} + \frac{1}{2} + o(1)\right)\right) \\
&= \exp_2\left(m\left(\log_2 n - \log_2 m - \frac{m}{2r} + O_r(1)\right)\right) \\
&= \exp_2\left(m\left(\left(1 - \frac{\alpha}{2r}\right)\log_2 n - \log_2 m + O_r(1)\right)\right).
\end{aligned}$$

Applying Theorem 26 completes the proof. $\square$

Our last calculation before proving Theorem 3 is to show that $\mathbb{E}[X_m]$ changes by a factor of $n^{1+o(1)}$ as m changes, provided m is near m_0.

Lemma 29. *Let X_m be the number of F-free sets of size m. Define $m_0 = \min\{m : \mathbb{E}[X_m] \le 1\}$. For $|\ell| = O(1)$ we have*

$$\frac{\mathbb{E}[X_{m_0}]}{\mathbb{E}[X_{m_0+\ell}]} = n^{\ell+o(1)}.$$

Proof. By Lemma 28 we have that $m_0 = 2r\log_2 n + o(\log n)$. For any $m = (2r + o(1))\log_2 n$, use Theorem 26 to compute

$$\begin{aligned}
\frac{\mathbb{E}[X_m]}{\mathbb{E}[X_{m+1}]} &= n^{o(1)}\frac{\overline{N}_{n,m}}{\overline{N}_{n,m+1}} \\
&= n^{o(1)}\binom{n}{m}\cdot\binom{n}{m+1}^{-1} 2^{-m} \\
&\quad \times \exp_2\left(\left(1 - \frac{1}{r}\right)\frac{m^2}{2} + m\log_2 r - \left(1 - \frac{1}{r}\right)\frac{(m+1)^2}{2} - (m+1)\log_2 r\right) \\
&= n^{o(1)}\cdot\frac{1}{n}\cdot\exp_2\left(\frac{m}{r}\right) \\
&= n^{1+o(1)}.
\end{aligned}$$

Iterating this bound ℓ times completes the proof. $\square$

Proof of Theorem 3. Let X_m be the number of F-free sets of size m and define

$$m_0 := \min\{m : \mathbb{E}[X_m] \le 1\}.$$

We proceed in two cases depending on if $\mathbb{E}[X_{m_0}] \le n^{-1/2r}$ or $\mathbb{E}[X_{m_0}] \ge n^{-1/2r}$. If $\mathbb{E}[X_{m_0}] \le n^{-1/(2r)}$ then set $M = m_0$, otherwise set $M = m_0 + 1$. In either case, Lemma 29 implies that $\mathbb{E}[X_M] \le n^{-1/(2r)}$ and so $\mathbb{P}(\alpha_F(G_{n,1/2}) \ge M) = o(1)$. Now, let k be the largest integer so that $kr < M - 1$. Note that $kr \in [M - r - 1, M - 2]$ so we can write $kr = M - \ell$ where $\ell \in \{2, 3, \dots, r + 1\}$. By Lemma 29 we have

$$\mathbb{E}[X_{kr}] = \mathbb{E}[X_{M-\ell}] = \mathbb{E}[X_M]\, n^{\ell+o(1)} < n^{2+r+o(1)}. \tag{11}$$

We claim that

$$\mathbb{E}[X_{kr}] \geq n^{1/2}\,. \tag{12}$$

To see this, assume first that $M = m_0$ and note that we must have $\mathbb{E}[X_{m_0}] \geq n^{-1+o(1)}$ by Lemma 29. Noting that $\mathbb{E}[X_{kr}] = \mathbb{E}[X_{m_0}]\, n^{\ell+o(1)}$ by (11) shows (12) in this case. If $M = m_0 + 1$, then we have $\mathbb{E}[X_{m_0}] \geq n^{-1/2r}$ implying that $\mathbb{E}[X_{kr}] \geq \mathbb{E}[X_{m_0}]n^{\ell-1+o(1)} > \mathbb{E}[X_{m_0}]n^{1+o(1)}$ thus showing (12).

Note that by Theorem 26 and Lemma 27 we have

$$\mathbb{E}[X_{rk}] = n^{o(1)}\overline{N}_{n,m} = n^{o(1)}(\mathbb{E}[Y_k])^r$$

which implies that $\mathbb{E}[Y_k] \geq n^{1/(2r)+o(1)}$. The proof of Lemma 11 for $j = r$ applied to $(k-1)$ shows that with high probability there are r many disjoint independent sets of size $(k-1)+1 = k$. Taking the induced subgraph with this vertex set yields a graph with kr vertices that has chromatic number at most r, thus implying it is F-free. This shows that $\alpha_F(G_{n,1/2}) \geq kr = M - \ell \geq (M-1) - r$ with high probability. Together with the upper bound, this shows

$$\alpha_F(G_{n,1/2}) \in [(M-1) - r, M-1]$$

with high probability, completing the proof. $\square$

6 Concluding Remarks

Extending beyond $p = 1/2$. It is a natural to study $\alpha_{r+1}(G_{n,p})$ for $p \neq 1/2$. It appears that there are two relevant transitions for p. Define

$$(1-p_-) + p_-(1-p_-)^2 = (1-p_-)^{1/2} \quad \text{and} \quad p_+ = \frac{\sqrt{5}-1}{2}\,.$$

The value $p_- \approx 0.323$ is when it becomes possible for an independent k-set to cover more than one defect; p_+ is when it becomes unlikely for an independent k-set to cover any defects. We suspect the relevant regimes are broken apart in terms of p_- and p_+

Problem 30. *Is it the case that for fixed $p \in (p_-, p_+)$ a version of Theorem 1 and Theorem 2 hold? For fixed $p > p_+$ is it the case that $\alpha_{r+1}(G_{n,p})$ is witnessed by a Turán graph? What happens for $p < p_-$?*

It is possible that new interesting phenomenon may become visible for smaller p that is a function of n, and we leave this as an open question as well.

Sharpness of Theorem 2. We gather here two closing remarks regarding the distribution of $X = \alpha_{r+1}(G_{n,1/2})$ for r fixed.

Theorem 2 is best possible in the sense that the result does not hold with intervals I_n that are smaller. To see this, consider the values of n defined by

$$n_{k,i} = \min\left\{n : \mathbb{E}_n[Z_{k+1,\mu_{j_i}}] > 1\right\}$$

for each $j_i \in \mathcal{J}$. Note that we have $b_{k,i} < n_{k,i} < c_{k,i}$. Set $\lambda = \mathbb{E}_{n_k}[Z_{k+1,\mu_{j_i}}]$ and note that $\lambda = 1 + o(1)$. Then by Theorem 1 we have

$$\begin{aligned}\mathbb{P}_{n_{k,i}}(X = kr + j_{i-1}) = \mathbb{P}_{n_{k,i}}(Z_{k+1,\mu_{j_i}} < \xi_{j_{i-1}+1}) + O(1/\sqrt{\log n}) = \sum_{j=0}^{\xi_{j_{i-1}+1}-1} e^{-\lambda}\frac{\lambda^i}{i!} + O(1/\sqrt{\log n})\\ > \frac{1}{e} + o(1).\end{aligned}$$

Similarly, for $\ell \in \{j_{i-1}+1, \dots, j_i - 1\}$ we have

$$\begin{aligned}\mathbb{P}_{n_{k,i}}(X = kr + \ell) = \mathbb{P}_{n_{k,i}}(X \geq kr + \ell) - \mathbb{P}_{n_{k,i}}(X \geq kr + \ell + 1) = \sum_{t=\xi_\ell}^{\xi_{\ell+1}-1} e^{-\lambda}\frac{\lambda^t}{t!} + O(1/\sqrt{\log n})\\ = \sum_{t=\xi_\ell}^{\xi_{\ell+1}-1} \frac{e^{-1}}{t!} + o(1).\end{aligned}$$

And finally we observe that

$$\begin{aligned}\mathbb{P}_{n_{k,i}}(X = kr + j_i) = \mathbb{P}_{n_{k,i}}(Z_{k+1,\mu_{j_i}} \geq \xi_{j_i}) + O(1/\sqrt{\log n}) > e^{-\lambda}\frac{\lambda^{\xi_{j_i}}}{\xi_{j_i}!} + O(1/\sqrt{\log n})\\ = \frac{e^{-1}}{\xi_{j_i}!} + o(1).\end{aligned}$$

Thus, $\alpha_{r+1}(G_{n_{k,i},1/2})$ takes all values in $\{j_{i-1}, \dots, j_i\}$ with probabilities bounded away from zero, and $\alpha(G_{n,1/2})$ is not concentrated on any smaller interval.

As a second note, if $\ell \notin \mathcal{J}$ then $\mathbb{P}(X \equiv \ell \mod r)$ is bounded away from 1 for n sufficiently large. Indeed, if $j_{i-1} < \ell < j_i$ then we have

$$\mathbb{P}(X = kr + \ell) = \mathbb{P}(X \geq kr + \ell) - \mathbb{P}(X \geq kr + \ell + 1) = \sum_{t=\xi_\ell}^{\xi_{\ell+1}-1} e^{-\lambda}\frac{\lambda^t}{t!} + O(1/\sqrt{\log n})$$

where $\lambda = \mathbb{E}[Z_{k+1,\mu_\ell}]$. As $\xi_\ell > 0$ and $\xi_{\ell+1} \leq r$ this expression is bounded away from 1 for all λ.

A hitting time result. The proof of Theorem 1 can be used to establish a hitting time version of this result. Let r be fixed and let $1 \leq j \leq r$. Define T_1 to the first step n in which we have $X \geq kr+j$ where $X = \alpha_{r+1}(G_{n,1/2})$, and define T_2 to be the first step n at which we have $Z_{k+1,\mu_j} \geq \xi_j$.

Corollary 31. *With high probability we have $T_1 = T_2$.*

Proof. Recall that we define $b_{k,j} = \min\{n : \mathbb{E}[Z_{k+1,\mu_j}] \geq \varepsilon\}$ and $c_{k,j} = \min\{n : \mathbb{E}[Z_{k+1,\mu_j}] \geq 1/\varepsilon\}$ where $\varepsilon = o_k(1)$. First observe that $\mathbb{P}(T_2 \notin [b_{k+1,j}, c_{k+1,j}]) = o(1)$, and we can thus restrict our attention to n in this interval.

Next observe that the events $\mathcal{B}'_1, \dots, \mathcal{B}'_6$ are vertex-increasing in the sense that if G is in one of these events and we add a vertex to G to form a graph G' then G' is in the event. We establish above that with high probability none of these events holds when $n = c_{k+1,j}$. Setting $\mathcal{B}' = \vee_{i=1}^{6} \mathcal{B}'_i$, it follows that with high probability $\mathcal{B}'^c$ holds for all $n \in [b_{k+1,r}, c_{k+1,r}]$. Following the proof of Lemma 11, we conclude that on the event $\{n < T_2\} \wedge \mathcal{B}'^c$ we have $X < kr + j$, which implies $n < T_1$. So, with high probability $T_1 \geq T_2$.

We make a similar argument to establish $T_2 \geq T_1$ whp. The events $\mathcal{B}_1$ and $\mathcal{B}_3$ are vertex-increasing while the event $\mathcal{B}_4$ is vertex-decreasing. The event $\mathcal{B}_3$, which is the event that some defect is not covered by enough k-sets is neither vertex-increasing nor vertex-decreasing as defined above. Here we make one small adjustment to the variable: We consider covering k-sets that are restricted to the first $b_{k+1,j}$ vertices of the graph. This does not materially change any estimates for $n \in [b_{k+1,j}, c_{k+1,j}]$ and yields a vertex-increasing event. Letting $\tilde{\mathcal{B}}_3$ be this alternate version of the event $\mathcal{B}_3$ we conclude that with high probability $\mathcal{B} = \mathcal{B}_1 \vee \mathcal{B}_2 \vee \tilde{\mathcal{B}}_3 \vee \mathcal{B}_4$ does not hold for any $n \in [b_{k+1,r}, c_{k+1,r}]$. Then, following the proof of Lemma 14 we conclude that on the event $\{n \geq T_1\} \wedge \mathcal{B}^c$ we have $n \geq T_2$. Thus $T_2 \leq T_1$ with high probability. □

A Proof of Proposition 18

Proof of Proposition 18. Let $\mathcal{F}$ be the set of graphs on $[m]$ that are $m^{3/5}$-far from being r-partite, and recall that we are to show $|\mathcal{F}| \leq 2^{\mathrm{ex}(m,K_{r+1})} e^{-m(\log m)^3/2}$. We begin by gathering some definitions and facts from [3]. For each $G \in \mathcal{F}$ we define $t(G)$ to be the minimum number of edges that need to be deleted from G to make the graph r-partite. We sort $\mathcal{F}$ into a number of subsets. Define

$$\mathcal{F}_{\mathrm{far}} = \left\{G \in \mathcal{F} : t(G) \geq m^2/(8\log m)^{15}\right\} \qquad \text{and} \qquad \mathcal{F}_{\mathrm{close}} = \mathcal{F} \setminus \mathcal{F}_{\mathrm{far}}.$$

For each $G \in \mathcal{F}_{\mathrm{close}}$ we define $\Pi(G)$ to be an arbitrary r-partition of U whose parts contain $t(G)$ edges of G. We set that a r-partition Π is *balanced* if each part contains at least $m/2r$ vertices. Define

$$\mathcal{F}^u_{\mathrm{close}} = \{G \in \mathcal{F}_{\mathrm{close}} : \Pi(G) \text{ is unbalanced}\}$$

The following bounds are established[3] in [3]: for $r \leq \log m/(121 \log\log m)$ we have

$$|\mathcal{F}_{\mathrm{far}}| \leq 2^{\mathrm{ex}(m,K_{r+1})} e^{-\Omega\left(m^{2-1/(8r)}\right)}, \qquad |\mathcal{F}^u_{\mathrm{close}}| \leq 2^{\mathrm{ex}(m,K_{r+1})} e^{-\Omega\left(m^2/r^2\right)}. \tag{13}$$

Now let $\mathcal{P}_b$ be the set of balanced r-partitions of U. For each $\Pi \in \mathcal{P}_b$ and $m^{3/5} < t < m^2/(8\log m)^{15}$ let

$$\mathcal{F}_{t,\Pi} = \{G \in \mathcal{F}_{\mathrm{close}} : t(G) = t \text{ and } \Pi(G) = \Pi\}.$$

Applying (13) gives

$$|\mathcal{F}| \leq |\mathcal{F}_{\mathrm{far}}| + |\mathcal{F}^u_{\mathrm{close}}| + \sum_{\Pi \in \mathcal{P}_b} \sum_{t=m^{3/5}}^{m^2/(8\log m)^{15}} |\mathcal{F}_{t,\Pi}| \leq 2^{\mathrm{ex}(m,K_{r+1}) - m^{3/2}} + r^m \max_{\Pi} \sum_{t=m^{3/5}}^{m^2/(8\log m)^{15}} |\mathcal{F}_{t,\Pi}|. \tag{14}$$

[3] These bounds are corollaries of the proofs of Theorem 6.2 and Lemma 6.5, respectively, from [3]. The bounds we give are stronger than the bounds in the statements of Theorem 6.2 and Lemma 6.5, but the improvements can be directly read off of the final lines off the proofs in [3].

In order to bound $\mathcal{F}_{t,\Pi}$ we note, following Balogh and Samotij, that a graph with t edges has either a vertex of degree D or a matching that consists of t/D edges. We use the following two Lemmas to bound the number of graphs in $\mathcal{F}_{t,\Pi}$ in these two categories. Recall that K_Π is the complete multipartite graph with parts given by Π.

Lemma 32 (Lemma 6.8 of [3])**.** *Let D be an integer satisfying $D \geq 2^r r$. Suppose that Π is an r-partition of U and that S is a copy of $K_{1,D}$ with $V(S)$ contained in some part P of Π. If $v \in P$ is the center vertex of S then*

$$|\{G \subseteq K_\Pi : G \cup S \not\supseteq K_{r+1} \text{ and } \deg_G(v, Q) \geq D \text{ for all } Q \in \Pi \setminus \{P\}\}| \leq 2^{e(K_\Pi) - \frac{D^2}{8r^2}}$$

Lemma 33 (Lemma 6.9 of [3])**.** *Suppose that Π is a balanced r-partition of U and that M is a matching with s edges such that $V(M) \subseteq P$ for some $P \in \Pi$. If $r^2 2^{r+3} \leq m$ then*

$$|\{G \subseteq K_\Pi : G \cup M \not\supseteq K_{r+1}\}| \leq 2^{e(K_\Pi) - \frac{sm}{2^{10} r^4}}$$

We apply these Lemmas with $D = (mt)^{1/3}$. A graph in $\mathcal{F}_{t,\Pi}$ has either a vertex of degree D in its own part, or else a matching of size at least t/D whose edges all lie within the parts; some part has at least $s = t/Dr$ of these edges. Note that the additional condition on the degree of v required for Lemma 32 follows from the choice of Π as a minimizer of the parameter $t(G)$. Indeed, if v had fewer than D neighbors in some other part Q, then we could move v to Q to make the graph closer to being r-partite. Note further, that Lemma 32 and 33 do not specify all edges in the graphs in question and so we have to take this into account in our estimate. Observe that for t in the given range we have $(mt)^{2/3} > t(\log m)^2, m(\log m)^4$. Consequently,

$$|\mathcal{F}_{t,\Pi}| \leq 2^{e(K_\Pi)} \left[(m^2)^t 2^{-\frac{D^2}{8r^2}} + (m^2)^t 2^{-\frac{tm}{2^{10} r^5 D}} \right] \leq 2^{\mathrm{ex}(m, K_{r+1}) - m(\log m)^3}.$$

Combining with (14) completes the proof. ∎

References

[1] R. Arratia, L. Goldstein, and L. Gordon. Poisson approximation and the Chen-Stein method. *Statistical Science*, pages 403–424, 1990.

[2] L. Babai, M. Simonovits, and J. Spencer. Extremal subgraphs of random graphs. *Journal of Graph Theory*, 14(5):599–622, 1990.

[3] J. Balogh and W. Samotij. An efficient container lemma. *Discrete Anal.*, pages Paper No. 17, 56, 2020.

[4] T. Bohman and J. Hofstad. A note on two-point concentration of the independence number of $g\_\{n, m\}$. *arXiv preprint arXiv:2410.05420*, 2024.

[5] T. Bohman and J. Hofstad. Two-point concentration of the independence number of the random graph. *Forum Math. Sigma*, 12:Paper No. e24, 24, 2024.

[6] B. Bollobás and P. Erdős. Cliques in random graphs. *Math. Proc. Cambridge Philos. Soc.*, 80(3):419–427, 1976.

[7] D. Conlon and W. T. Gowers. Combinatorial theorems in sparse random sets. *Ann. of Math. (2)*, 184(2):367–454, 2016.

[8] V. Dani and C. Moore. Independent sets in random graphs from the weighted second moment method. In *International Workshop on Approximation Algorithms for Combinatorial Optimization*, pages 472–482. Springer, 2011.

[9] B. DeMarco and J. Kahn. Mantel's theorem for random graphs. *Random Structures & Algorithms*, 47(1):59–72, 2015.

[10] P. Erdős, D. J. Kleitman, and B. L. Rothschild. Asymptotic enumeration of K_n-free graphs. In *Colloquio Internazionale sulle Teorie Combinatorie (Roma, 1973), Tomo II*, pages 19–27. Accad. Naz. Lincei, Rome, 1976.

[11] A. M. Frieze. On the independence number of random graphs. *Discrete Mathematics*, 81(2):171–175, 1990.

[12] A. Heckel. Non-concentration of the chromatic number of a random graph. *Journal of the American Mathematical Society*, 34(1):245–260, 2021.

[13] A. Heckel and O. Riordan. How does the chromatic number of a random graph vary? *Journal of the London Mathematical Society*, 108(5):1769–1815, 2023.

[14] P. G. Kolaitis, H. J. Prömel, and B. L. Rothschild. K_{l+1}-free graphs: asymptotic structure and a 0-1 law. *Trans. Amer. Math. Soc.*, 303(2):637–671, 1987.

[15] D. W. Matula. On the complete subgraphs of a random graph. In *Proc. Second Chapel Hill Conf. on Combinatorial Mathematics and its Applications (Univ. North Carolina, Chapel Hill, N.C., 1970)*, pages 356–369. University of North Carolina, Chapel Hill, NC, 1970.

[16] H. J. Prömel and A. Steger. The asymptotic number of graphs not containing a fixed color-critical subgraph. *Combinatorica*, 12(4):463–473, 1992.

[17] O. Riordan and L. Warnke. The Janson inequalities for general up-sets. *Random Structures & Algorithms*, 46(2):391–395, 2015.

[18] M. Schacht. Extremal results for random discrete structures. *Ann. of Math. (2)*, 184(2):333–365, 2016.