New lower bounds for hypergraph Ramsey numbers

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Abstract

The *Ramsey number* $r_k(s,n)$ is the minimum N such that for every red-blue coloring of the k-tuples of $\{1,\dots,N\}$, there are s integers such that every k-tuple among them is red, or n integers such that every k-tuple among them is blue. We prove the following new lower bounds for 4-uniform hypergraph Ramsey numbers:

$$r_4(5,n) > 2^{n^{c\log n}} \qquad \text{and} \qquad r_4(6,n) > 2^{2^{cn^{1/5}}},$$

where c is an absolute positive constant. This substantially improves the previous best bounds of $2^{n^{c\log\log n}}$ and $2^{n^{c\log n}}$, respectively. Using previously known upper bounds, our result implies that the growth rate of $r_4(6,n)$ is double exponential in a power of n.

As a consequence, we obtain similar bounds for the k-uniform Ramsey numbers $r_k(k+1,n)$ and $r_k(k+2,n)$ where the exponent is replaced by an appropriate tower function. This almost solves the question of determining the tower growth rate for *all* classical off-diagonal hypergraph Ramsey numbers, a question first posed by Erdős and Hajnal in 1972. The only problem that remains is to prove that $r_4(5,n)$ is double exponential in a power of n.

1 Introduction

A k-uniform hypergraph H with vertex set V is a collection of k-element subsets of V. We write $K_n^{(k)}$ for the complete k-uniform hypergraph on an n-element vertex set. The *Ramsey number* $r_k(s,n)$ is the minimum N such that every red-blue coloring of the edges of $K_N^{(k)}$ contains a monochromatic red copy of $K_s^{(k)}$ or a monochromatic blue copy of $K_n^{(k)}$.

Diagonal Ramsey numbers refer to the special case when $s=n$, i.e. $r_k(n,n)$, and have been studied extensively over the past 80 years. Classic results of Erdős and Szekeres [12] and Erdős [8] imply that $2^{n/2} < r_2(n,n) \leq 2^{2n}$ for every integer $n>2$. While small improvements have been made in both the upper and lower bounds for $r_2(n,n)$ (see [18, 4]), the constant factors in the exponents have not changed over the last 70 years.

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Unfortunately, for 3-uniform hypergraphs, our understanding of $r_3(n,n)$ is much less. Results of Erdős, Hajnal, and Rado [10] gives the best known lower and upper bounds for $r_3(n,n)$,

$$2^{c_1 n^2} < r_3(n,n) < 2^{2^{c_2 n}},$$

where c_1 and c_2 are positive constants. For $k \geq 4$, there is also a difference of one exponential between the known lower and upper bounds for $r_k(n,n)$, that is,

$$\mathrm{twr}_{k-1}(c_1 n^2) \leq r_k(n,n) \leq \mathrm{twr}_k(c_2 n), \tag{1}$$

where the tower function $\mathrm{twr}_k(x)$ is defined by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$ (see [12, 11, 9]). A notoriously difficult conjecture of Erdős, Hajnal, and Rado states that the upper bound in (1) is essentially the truth, that is, there are constructions which demonstrates that $r_k(n,n) > \mathrm{twr}_k(cn)$, where $c = c(k)$. The crucial case is when $k = 3$, since a double exponential lower bound for $r_3(n,n)$ would verify the conjecture for all $k \geq 4$ by using the well-known stepping-up lemma of Erdős and Hajnal (see [13]).

Conjecture 1.1 (Erdős)**.** *For $n \geq 4$, $r_3(n,n) > 2^{2^{cn}}$ where c is an absolute constant.*

Off-diagonal Ramsey numbers, $r_k(s,n)$, refer to the special case when k, s are fixed and n tends to infinity. It is known [1, 14, 2, 3] that $r_2(3,n) = \Theta(n^2/\log n)$, and more generally for fixed $s > 3$, $r_2(s,n) = n^{\Theta(1)}$. For 3-uniform hypergraphs, a result of Conlon, Fox, and Sudakov [6] shows that

$$2^{c_1 n \log n} \leq r_3(s,n) \leq 2^{c_2 n^{s-2} \log n},$$

where c_1 and c_2 depend only on s. For k-uniform hypergraphs, where $k \geq 4$, it is known that $r_k(s,n) \leq \mathrm{twr}_{k-1}(n^c)$, where $c = c(s)$ [11]. By applying the Erdős-Hajnal stepping up lemma in the off-diagonal setting, it follows that

$$r_k(s,n) \geq \mathrm{twr}_{k-1}(c'n), \tag{2}$$

for $k \geq 4$ and $s \geq 2^{k-1} - k + 3$, where $c' = c'(s)$. In 1972, Erdős and Hajnal [9] conjectured that (2) holds for every fixed $k \geq 4$ and $s \geq k+1$. Actually, this was part of a more general conjecture that they posed in that paper (see [16, 17] for details). In [5], Conlon, Fox, and Sudakov verified the Erdős-Hajnal conjecture for all $s \geq \lceil 5k/2 \rceil - 3$. Very recently, the current authors [16] and independently Conlon, Fox, and Sudakov [7] verified the conjecture for all $s \geq k+3$ (using different constructions). Since $2^{k-1} - k + 3 = \lceil 5k/2 \rceil - 3 = k+3 = 7$ when $k = 4$, all three of these approaches succeed in proving a double exponential lower bound for $r_4(7,n)$ but fail for $r_4(6,n)$ and $r_4(5,n)$. Just as for diagonal Ramsey numbers, a double exponential in n^c lower bound for $r_4(5,n)$ and $r_4(6,n)$ would imply $r_k(k+1,n) > \mathrm{twr}_{k-1}(n^{c'})$ and $r_k(k+2,n) > \mathrm{twr}_{k-1}(n^{c'})$ respectively, for all fixed $k \geq 5$. This follows from a variant of the stepping-up lemma that we will describe in Section 2. Therefore, the difficulty in verifying (2) for the two remaining cases, $s = k+1$ and $k+2$, is due to our lack of understanding of $r_4(5,n)$ and $r_4(6,n)$. Consequently, showing double exponential lower bounds for $r_4(5,n)$ and $r_4(6,n)$ are the only two problems that remain to determine the tower growth rate for all off-diagonal hypergraph Ramsey numbers.

Until very recently, the only lower bound for both $r_4(5,n)$ and $r_4(6,n)$ was 2^{cn}, which was implicit in the paper of Erdős and Hajnal [9]. Our results in [15, 16] improved both these bounds to

$$r_4(5,n) > 2^{n^{c \log\log n}} \qquad \text{and} \qquad r_4(6,n) > 2^{n^{c \log n}} \tag{3}$$

and these are the current best known bounds. As mentioned above, the bounds in (3) imply the corresponding improvements to the lower bounds for $r_k(k+1,n)$ and $r_k(k+2,n)$. In this paper we further substantially improve both lower bounds in (3).

Theorem 1.2. *For all $n \geq 6$,*

$$r_4(5,n) > 2^{n^{c\log n}} \qquad \text{and} \qquad r_4(6,n) > 2^{2^{cn^{1/5}}},$$

where $c > 0$ is an absolute constant.

Using the stepping-up lemma (see Section 2) we obtain the following.

Corollary 1.3. *For $n > k \geq 5$, there is a $c = c(k) > 0$ such that*

$$r_k(k+1,n) > \mathrm{twr}_{k-2}(n^{c\log n}) \qquad \text{and} \qquad r_k(k+2,n) > \mathrm{twr}_{k-1}(cn^{1/5}).$$

A standard argument in Ramsey theory together with results in [6] for 3-uniform hypergraph Ramsey numbers yields the upper bound $r_k(k+2,n) < \mathrm{twr}_{k-1}(c'n^3\log n)$, so we now know the tower growth rate of $r_k(k+2,n)$.

In [16], we established a connection between diagonal and off-diagonal Ramsey numbers. In particular, we showed that a solution to Conjecture 1.1 implies a solution to the following conjecture.

Conjecture 1.4. *For $n \geq 5$, there is an absolute constant $c > 0$ such that $r_4(5,n) > 2^{2^{n^c}}$.*

The main idea in our approach is to apply stepping-up starting from a graph to construct a 4-uniform hypergraph, rather than the usual method of going from a 3-uniform hypergraph to a 4-uniform hypergraph. Although this approach was implicitly developed in [16], here we use it explicitly.

For more related Ramsey-type results for hypergraphs, we refer the interested reader to [16, 15, 17]. All logarithms are in base 2 unless otherwise stated. For the sake of clarity of presentation, we omit floor and ceiling signs whenever they are not crucial.

2 The stepping-up lemma and proof of Lemma 2.1

The proof of our main result, Theorem 1.2, follows by applying a variant of the classic Erdős-Hajnal stepping-up lemma. In this section, we describe the stepping-up procedure and sketch the proof of Lemma 2.1 below which is used to prove Corollary 1.3. The particular case below can be found in [15], though a special case of Lemma 2.1 was communicated to us independently by Conlon, Fox, and Sudakov [7].

Lemma 2.1. *For $k \geq 5$ and $n \geq s \geq k+1$, $r_k(s,2kn) > 2^{r_{k-1}(s-1,n)-1}$.*

Proof. Let $k \geq 5$, $n \geq s \geq k+1$, and set $A = \{0,1,\ldots,N-1\}$ where $N = r_{k-1}(s-1,n)-1$. Let $\phi : \binom{A}{k-1} \to \{\text{red, blue}\}$ be a red/blue coloring of the $(k-1)$-tuples of A such that there is no monochromatic red copy of $K_{s-1}^{(k-1)}$ and no monochromatic blue copy of $K_n^{(k-1)}$. We know ϕ exists by the definition of N. Set $V = \{0, 1, \dots, 2^N - 1\}$. In what follows, we will use ϕ to define a red/blue coloring $\chi : \binom{V}{k} \to \{\text{red, blue}\}$ of the k-tuples of V such that χ does not contain a monochromatic red copy of $K_s^{(k)}$, and does not contain a monochromatic blue copy of $K_{2kn}^{(k)}$.

For any $v \in V$, write $v = \sum_{i=0}^{N-1} v(i)2^i$ with $v(i) \in \{0, 1\}$ for each i. For $u \neq v$, let $\delta(u, v) \in A$ denote the largest i for which $u(i) \neq v(i)$. Notice that we have the following stepping-up properties (see [13])

Property I: For every triple $u < v < w$, $\delta(u, v) \neq \delta(v, w)$.

Property II: For $v_1 < \cdots < v_r$, $\delta(v_1, v_r) = \max_{1 \leq j \leq r-1} \delta(v_j, v_{j+1})$.

We will also use the following two stepping-up properties, which are easy consequences of Properties I and II.

Property III: For every 4-tuple $v_1 < \cdots < v_4$, if $\delta(v_1, v_2) > \delta(v_2, v_3)$, then $\delta(v_1, v_2) \neq \delta(v_3, v_4)$. Note that if $\delta(v_1, v_2) < \delta(v_2, v_3)$, it is possible that $\delta(v_1, v_2) = \delta(v_3, v_4)$.

Property IV: For $v_1 < \cdots < v_r$, set $\delta_j = \delta(v_j, v_{j+1})$ and suppose that $\delta_1, \dots, \delta_{r-1}$ forms a monotone sequence. Then for every subset of k-vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$, where $v_{i_1} < \cdots < v_{i_k}$, $\delta(v_{i_1}, v_{i_2}), \delta(v_{i_2}, v_{i_3}), \dots, \delta(v_{i_{k-1}}, v_{i_k})$ forms a monotone sequence. Moreover, for every subset of $k-1$ vertices $\delta_{j_1}, \delta_{j_2}, \dots, \delta_{j_{k-1}}$, there are k vertices $v_{i_1}, \dots, v_{i_k}$ such that $\delta(v_{i_t}, v_{i_{t+1}}) = \delta_{j_t}$.

Given any k-tuple $v_1 < v_2 < \ldots < v_k$ of V, consider the integers $\delta_i = \delta(v_i, v_{i+1}), 1 \leq i \leq k-1$. We say that δ_i is a *local minimum* if $\delta_{i-1} > \delta_i < \delta_{i+1}$, a *local maximum* if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and a *local extremum* if it is either a local minimum or a local maximum. Since $\delta_{i-1} \neq \delta_i$ for every i, every nonmonotone sequence $\delta_1, \dots, \delta_{k-1}$ has a local extremum.

Using $\phi : \binom{A}{k-1} \to \{\text{red, blue}\}$, we define $\chi : \binom{V}{k} \to \{\text{red, blue}\}$ as follows. For $v_1 < \cdots < v_k$ and $\delta_i = \delta(v_i, v_{i+1})$, we define $\chi(v_1, \dots, v_k) = \text{red}$ if

(a) $\delta_1, \dots, \delta_{k-1}$ forms a monotone sequence and $\phi(\delta_1, \dots, \delta_{k-1}) = \text{red}$, or if

(b) $\delta_1, \dots, \delta_{k-1}$ forms a *zig-zag* sequence such that δ_2 is a local maximum. In other words, $\delta_1 < \delta_2 > \delta_3 < \delta_4 > \cdots$.

Otherwise $\chi(v_1, \dots, v_k) = \text{blue}$.

For the sake of contradiction, suppose χ produces a monochromatic red copy of $K_s^{(k)}$ on vertices $v_1 < \cdots < v_s$, and let $\delta_i = \delta(v_i, v_{i+1})$. If $\delta_1, \delta_2, \dots, \delta_{s-1}$ forms a monotone sequence, then by Property IV, ϕ colors every $(k-1)$-tuple in the set $\{\delta_1, \dots, \delta_{s-1}\}$ red, which is a contradiction. Let δ_i denote the first local extremum in the sequence $\delta_1, \dots, \delta_{s-1}$. It is easy to see that δ_i is a local maximum since otherwise we would get a contradiction. Suppose $i + k - 1 \leq s$. If δ_{i+1} is not a local extremum, then $\chi(v_{i-1}, v_i, v_{i+1}, \dots, v_{i+k-2}) = \text{blue}$ which is a contradiction. If δ_{i+1} is a local extremum, then it must be a local minimum which implies that $\chi(v_i, v_{i+1}, \ldots, v_{i+k-1}) =$ blue, contradiction. Therefore we can assume that $i + k - 1 > s$, which implies $i \geq 3$ since $s \geq k + 1$. However, this implies that either $\chi(v_{i-2}, v_{i-1}, \ldots, v_{i+k-3}) =$ blue or $\chi(v_{s-k+1}, v_{s-k+2}, \ldots, v_s) =$ blue, contradiction. Hence, χ does not produce a monochromatic red copy of $K_s^{(k)}$ in V.

Let $m = 2kn$. For the sake of contradiction, suppose χ produces a monochromatic blue copy of $K_m^{(k)}$ on vertices $v_1, \ldots, v_m$ and let $\delta_i = \delta(v_i, v_{i+1})$. By Property IV, there is no x such that $\delta_x, \delta_{x+1}, \ldots, \delta_{x+n-1}$ forms a monotone sequence. Indeed, otherwise ϕ would produce a monochromatic blue copy of $K_n^{(k-1)}$ on vertices $\delta_x, \delta_{x+1}, \ldots, \delta_{x+n-1}$. Therefore, we can set $\delta_{i_1}, \ldots, \delta_{i_k}$ to be the first k local minimums in the sequence $\delta_1, \ldots, \delta_{m-1}$. However, by Property II, χ colors the first k vertices in the set $\{v_{i_1}, v_{i_1+1}, v_{i_2}, v_{i_2+1}, \ldots, v_{i_k}, v_{i_k+1}\}$ red which is a contradiction. This completes the proof of Lemma 2.1. □

3 A double exponential lower bound for $r_4(6, n)$

The lower bound for $r_4(6, n)$ follows by applying a variant the Erdős-Hajnal stepping up lemma. We start with the following simple lemma which is a straightforward application of the probabilistic method.

Lemma 3.1. *There is an absolute constant $c > 0$ such that the following holds. For every $n \geq 6$, there is a red/blue coloring ϕ of the pairs of $\{0, 1, \ldots, \lfloor 2^{cn} \rfloor - 1\}$ such that*

1. *there are no two disjoint n-sets $A, B \subset \{0, 1, \ldots, \lfloor 2^{cn} \rfloor - 1\}$, such that $\phi(a, b) =$ red for every $a \in A$ and $b \in B$, or $\phi(a, b) =$ blue for every $a \in A$ and $b \in B$ (i.e., no monochromatic $K_{n,n}$),*
2. *there is no n-set $A \subset \{0, 1, \ldots, \lfloor 2^{cn} \rfloor - 1\}$ such that every triple $a_i, a_j, a_k \in A$, where $a_i < a_j < a_k$, avoids the pattern $\phi(a_i, a_j) = \phi(a_j, a_k) =$ blue and $\phi(a_i, a_k) =$ red.*

Proof. Set $N = \lfloor 2^{cn} \rfloor$, where c is a sufficiently small constant that will be determined later. Consider the red/blue coloring ϕ of the pairs (edges) of $\{0, 1, \ldots, N - 1\}$, where each edge has probability 1/2 of being a particular color independent of all other edges. Then the expected number of monochromatic copies of the complete bipartite graph $K_{n,n}$ is at most

$$\binom{N}{n}^2 2^{-n^2+1} < 1/3,$$

for c sufficiently small and $n \geq 6$.

We call a triple $a_i, a_j, a_k \in \{0, 1, \ldots, N-1\}$ *bad* if $a_i < a_j < a_k$ and $\phi(a_i, a_j) = \phi(a_j, a_k) =$ blue and $\phi(a_i, a_k) =$ red. Otherwise, we call the triple (a_i, a_j, a_k) *good*. Now, let us estimate the expected number of sets $A \subset \{0, 1, \ldots, N - 1\}$ of size n such that every triple in A is good. For a given triple $a_i, a_j, a_k \in \{0, 1, \ldots, N - 1\}$, where $a_i < a_j < a_k$, the probability that (a_i, a_j, a_k) is good is 7/8. Let $A = \{a_1, \ldots, a_n\}$ be a set of n vertices in $\{0, 1, \ldots, N - 1\}$, where $a_1 < \cdots < a_n$. Let S be a partial Steiner $(n, 3, 2)$-system with vertex set A, that is, S is a 3-uniform hypergraph such that each 2-element set of vertices is contained in at most one edge in S. Moreover, S satisfies

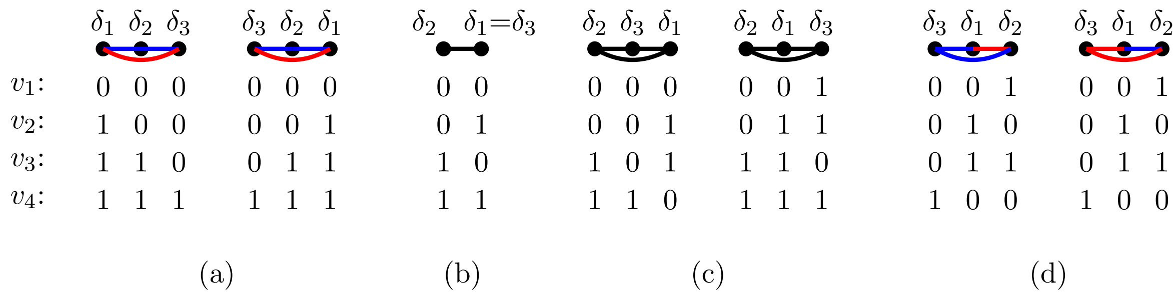

Figure 1: Examples of $v_1 < v_2 < v_3 < v_4$ and $\delta_1 = \delta(v_1, v_2), \delta_2 = \delta(v_2, v_3), \delta_3 = \delta(v_3, v_4)$, such that $\chi(v_1, v_2, v_3, v_4) = \text{red}$. For each case, v_i is represented in binary form with the left-most entry being the most significant bit.

$|S| = c'n^2$. It is known that such a system exists. Then the probability that every triple in A is good is at most the probability that every edge in S is good. Since the edges in S are independent, that is no two edges have more than one vertex in common, the probability that every triple in A is good is at most $\left(\frac{7}{8}\right)^{|S|} \leq \left(\frac{7}{8}\right)^{c'n^2}$. Therefore, the expected number of sets of size n with every triple being good is at most

$$\binom{N}{n}\left(\frac{7}{8}\right)^{c'n^2} < 1/3,$$

for an appropriate choice for c. By Markov's inequality and the union bound, we can conclude that there is a coloring ϕ with the desired properties. $\square$

Let $c > 0$ be the constant from the lemma above, and let $A = \{0, 1, \ldots, \lfloor 2^{cn} \rfloor - 1\}$ and $\phi : \binom{A}{2} \to \{\text{red, blue}\}$ be a 2-coloring of the pairs of A with the properties described above. Let $V = \{0, 1, \ldots, N-1\}$, where $N = 2^{\lfloor 2^{cn} \rfloor}$. In what follows, we will use ϕ to define a red/blue coloring $\chi : \binom{V}{4} \to \{\text{red, blue}\}$ of the 4-tuples of V such that χ does not produce a monochromatic red copy of $K_6^{(4)}$ and does not produce a monochromatic blue copy of $K_{32n^5}^{(4)}$. This would imply the desired lower bound for $r_4(6, n)$. For $v_1 < v_2 < v_3 < v_4$ and $\delta_i = \delta(v_i, v_{i+1})$, we set $\chi(v_1, v_2, v_3, v_4) = \text{red}$ if

(a) $\delta_1, \delta_2, \delta_3$ forms a monotone sequence and the triple $(\delta_1, \delta_2, \delta_3)$ is *bad*, that is, $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$, or

(b) $\delta_1 < \delta_2 > \delta_3$ and $\delta_1 = \delta_3$, or

(c) $\delta_1 < \delta_2 > \delta_3$, $\delta_1 \neq \delta_3$, and the set $\{\delta_1, \delta_2, \delta_3\}$ is monochromatic with respect to ϕ, or

(d) $\delta_1 > \delta_2 < \delta_3$, $\delta_1 < \delta_3$, and $\phi(\delta_3, \delta_1) = \phi(\delta_3, \delta_2)$ and $\phi(\delta_1, \delta_2) \neq \phi(\delta_3, \delta_1)$.

See Figure 1 for small examples. Otherwise, $\chi(v_1, v_2, v_3, v_4) = \text{blue}$.

For the sake of contradiction, suppose that the coloring χ produces a red $K_6^{(4)}$ on vertices $v_1 < \cdots < v_6$, and let $\delta_i = \delta(v_i, v_{i+1})$, $1 \leq i \leq 5$. Let us first consider the following cases for $\delta_1, \ldots, \delta_4$, which corresponds to the vertices, $v_1, \ldots, v_5$.

Case 1. Suppose that $\delta_1, \ldots, \delta_4$ forms a monotone sequence. If $\delta_1 > \cdots > \delta_4$, then we have $\phi(\delta_1, \delta_3) = \text{red}$ since $\chi(v_1, v_2, v_3, v_4) = \text{red}$. However, this implies that $\chi(v_1, v_3, v_4, v_5) = \text{blue}$ since $\delta(v_1, v_3) = \delta_1$ by Property II, contradiction. A similar argument follows if $\delta_1 < \cdots < \delta_4$.

Case 2. Suppose $\delta_1 > \delta_2 > \delta_3 < \delta_4$. By Property III, $\delta_4 \neq \delta_2, \delta_1$. Since $\delta_1 > \delta_2 > \delta_3$, this implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$. Since $\delta(v_2, v_4) = \delta_2$ and $\chi(v_1, v_2, v_4, v_5) = \text{red}$, we have $\delta_4 > \delta_1$. Hence $\phi(\delta_4, \delta_3) = \phi(\delta_4, \delta_2) = \text{red}$. However, since $\delta(v_1, v_3) = \delta_1$ by Property II, we have $\chi(v_1, v_3, v_4, v_5)$ is blue, contradiction.

Case 3. Suppose $\delta_1 < \delta_2 < \delta_3 > \delta_4$. This implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \text{blue}$ and $\phi(\delta_1, \delta_3) = \text{red}$. Suppose $\delta_4 = \delta_2$. Since $\delta(v_1, v_3) = \delta_2$ and $\delta_2 < \delta_3 > \delta_4$, this implies the triple $(\delta_1, \delta_2, \delta_4)$ forms a monochromatic blue set with respect to ϕ, which is a contradiction. A similar argument follows in the case that $\delta_4 = \delta_1$. So we can assume $\delta_4 \neq \delta_1, \delta_2$. Since $\chi(v_2, v_3, v_4, v_5) = \text{red}$, the triple $\{\delta_2, \delta_3, \delta_4\}$ forms a monochromatic blue set with respect to ϕ. By Property II we have $\delta(v_2, v_4) = \delta_3$ and $\delta_1 < \delta(v_2, v_4) > \delta_4$. This implies that $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction

Case 4. Suppose $\delta_1 < \delta_2 > \delta_3 > \delta_4$. This implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) = \text{blue}$ and $\phi(\delta_2, \delta_4) = \text{red}$. Suppose $\delta_1 = \delta_3$. By Property II, we have $\delta(v_2, v_4) = \delta_2$. However, $\chi(v_1, v_2, v_4, v_5) = \text{red}$ and $\delta_1 < \delta_2 > \delta_4$ implies that the triple $(\delta_1, \delta_2, \delta_4)$ must form a monochromatic set with respect to ϕ, contradiction. A similar argument follows if $\delta_1 = \delta_4$. Therefore, we can assume that $\delta_1 \neq \delta_3, \delta_4$. Since $\chi(v_1, v_2, v_3, v_4) = \text{red}$, the triple $(\delta_1, \delta_2, \delta_3)$ forms a monochromatic blue set with respect to ϕ. By Property II we have $\delta(v_2, v_4) = \delta_2$ and $\delta_1 < \delta(v_2, v_4) > \delta_4$. This implies $\chi(v_1, v_2, v_4, v_5) = \text{blue}$, contradiction.

Case 5. Suppose $\delta_1 > \delta_2 < \delta_3 < \delta_4$. Note that by Property III, $\delta_1 \neq \delta_3, \delta_4$. Since $\delta_1, \delta_2, \delta_3$ forms a monotone sequence, this implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) = \text{blue}$ and $\phi(\delta_2, \delta_4) = \text{red}$. Moreover, we must have $\delta_1 < \delta_3$ since $\chi(v_1, v_2, v_3, v_4) = \text{red}$. Hence $\phi(\delta_3, \delta_1) = \text{blue}$ and $\phi(\delta_1, \delta_2) = \text{red}$. However, since $\delta(v_3, v_5) = \delta_4$, we have $\delta_1 > \delta_2 < \delta(v_3, v_5)$ and $\chi(v_1, v_2, v_3, v_5) = \text{blue}$, contradiction.

Case 6. Suppose $\delta_1 < \delta_2 > \delta_3 < \delta_4$. Then we must also have $\delta_4 > \delta_2$ since $\chi(v_2, v_3, v_4, v_5) = \text{red}$. By Property II, $\delta(v_3, v_5) = \delta_4$ and we have $\delta_1 < \delta_2 < \delta(v_3, v_5)$. Since $\chi(v_1, v_2, v_3, v_5) = \text{red}$, we have $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_4) = \text{blue}$ and $\phi(\delta_1, \delta_4) = \text{red}$. Since $\chi(v_1, v_2, v_3, v_4) = \text{red}$, the triple $(\delta_1, \delta_2, \delta_3)$ forms a monochromatic blue set. However, this implies that $\chi(v_2, v_3, v_4, v_5) = \text{blue}$, contradiction.

Now if $v_1, \ldots, v_5$ and $\delta_1, \ldots, \delta_4$ does not fall into one of the 6 cases above, then we must have $\delta_1 > \delta_2 < \delta_3 > \delta_4$. However, this implies that $v_2, \ldots, v_6$ and $\delta_2, \ldots, \delta_5$ does fall into one of the 6 cases above, which implies our contradiction. Therefore, χ does not produce a monochromatic red copy of $K_6^{(4)}$ in our 4-uniform hypergraph.

Next we show that there is no blue $K_m^{(4)}$ in coloring χ, where $m = 32n^5$. For the sake of contradiction, suppose we have vertices $v_1, \ldots, v_m \in V$ such that $v_1 < \cdots < v_m$, and χ colors every 4-tuple in the set $\{v_1, \ldots, v_m\}$ blue. Let $\delta_i = \delta(v_i, v_{i+1})$ for $1 \leq i \leq m - 1$.

Set $\delta_1^* = \max\{\delta_1, \ldots, \delta_m\}$, where $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$. Set

$$V_1 = \{v_1, v_2, \ldots, v_{i_1}\} \quad \text{and} \quad V_2 = \{v_{i_1+1}, v_{i_1+2}, \ldots, v_m\}.$$

Now we establish the following lemma.

Lemma 3.2. *For any $W \subset \{v_1, \ldots, v_m\}$, where $W = W_1 \cup W_2$ is a partition of W described as above, either $|W_1| < m/2n$ or $|W_2| < m/2n$. In particular, either $|V_1| < m/2n$ or $|V_2| < m/2n$.*

Before we prove Lemma 3.2, let us finish the argument that χ does not color every 4-tuple in the set $\{v_1, \ldots, v_m\}$ blue via the following lemma which will also be used later in the paper.

Lemma 3.3. *If Lemma 3.2 holds, then χ colors a 4-tuple in the set $\{v_1, \ldots, v_m\}$ red.*

Proof. We greedily construct a set $D_t = \{\delta_1^*, \delta_2^*, \ldots, \delta_t^*\} \subset \{\delta_1, \delta_2, \ldots, \delta_m\}$ and a set $S_t \subset \{v_1, \ldots, v_m\}$ such that the following holds.

1. We have $\delta_1^* > \cdots > \delta_t^*$, where $\delta_j^* = \delta(v_{i_j}, v_{i_j+1})$.
2. The indices of the vertices in S_t are consecutive, that is, $S_t = \{v_r, v_{r+1}, \ldots, v_{s-1}, v_s\}$ for $1 \leq r < s \leq n$. Moreover, $\delta_t^* > \max\{\delta_r, \delta_{r+1}, \ldots, \delta_{s-1}\}$.
3. $|S_t| > m - tm/2n$.
4. For each $\delta_j^* = \delta(v_{i_j}, v_{i_j+1}) \in D_t$, consider the set of vertices

   $$S = \{v_{i_{j+1}}, v_{i_{j+1}+1}, v_{i_{j+2}}, v_{i_{j+2}+1} \ldots, v_{i_t}, v_{i_t+1}\} \cup S_t.$$

   Then either every element in S is greater than v_{i_j} or every element in S is less than v_{i_j+1}. In the former case we will label δ_j^* *white*, in the latter case we label it *black*.

We start with the $D_0 = \emptyset$ and $S_0 = \{v_1, \ldots, v_m\}$. Having obtained $D_{t-1} = \{\delta_1^*, \ldots, \delta_{t-1}^*\}$ and $S_{t-1} = \{v_r, \ldots, v_s\}$, where $1 \leq r < s \leq n$, we construct D_t and S_t as follows. Let $\delta_t^* = \delta(v_{i_t}, v_{i_t+1})$ be the unique largest element in $\{\delta_r, \delta_{r+1}, \ldots, \delta_{s-1}\}$, and set $D_t = D_{t-1} \cup \delta_t^*$. The uniqueness of δ_t^* follows from Properties I and II. We partition $S_{t-1} = T_1 \cup T_2$, where $T_1 = \{v_r, v_{r+1}, \ldots, v_{i_t}\}$ and $T_2 = \{v_{i_t+1}, v_{i_t+2}, \ldots, v_s\}$. By Lemma 3.2, either $|T_1| < m/2n$ or $|T_2| < m/2n$. If $|T_1| < m/2n$, we set $S_t = T_2$ and label δ_t^* white. Likewise, if $|T_2| < m/2n$, we set $S_t = T_1$ and label δ_t^* black. By induction, we have

$$|S_t| > |S_{t-1}| - m/2n \geq (m - (t-1)m/2n) - m/2n = m - tm/2n.$$

Since $|S_0| = m$ and $|S_t| \geq 1$ for $t = 2n$, we can construct $D_{2n} = \{\delta_1^*, \ldots, \delta_{2n}^*\}$ with the desired properties. By the pigeonhole principle, there are at least n elements in D_{2n} with the same label, say *white*. The other case will follow by a symmetric argument. We remove all black labeled elements in D_{2n}, and let $\{\delta_{j_1}^*, \ldots, \delta_{j_n}^*\}$ be the resulting set.

Now consider the vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_n}, v_{j_n+1} \in V$. By construction and by Property II, we have $v_{j_1} < v_{j_2} < \cdots < v_{j_n} < v_{j_n+1}$ and $\delta(v_{j_1}, v_{j_2}) = \delta_{i_{j_1}}^*, \delta(v_{j_2}, v_{j_3}) = \delta_{i_{j_2}}^*, \ldots, \delta(v_{j_n}, v_{j_{n+1}}) = \delta_{i_{j_n}}^*$. Therefore, we have a monotone sequence

$$\delta(v_{j_1}, v_{j_2}) > \delta(v_{j_2}, v_{j_3}) > \cdots > \delta(v_{j_n}, v_{j_{n+1}}).$$

By Lemma 3.1, there is a bad triple in the set $\{\delta^*_{j_1}, \ldots, \delta^*_{j_n}\}$ with respect to ϕ. By Property IV, χ does not color every 4-tuple in $V = \{v_1, \ldots, v_m\}$ blue, which completes the proof of Lemma 3.3. $\square$

Now let us go back and prove Lemma 3.2. First, we make the following observation.

Observation 3.4. *Let $v_1 < \cdots < v_m \in V$ such that χ colors every 4-tuple in the set $\{v_1, \ldots, v_m\}$ blue. Then for $\delta_i = \delta(v_i, v_{i+1})$, $\delta_i \neq \delta_j$ for $1 \leq i < j < m$.*

Proof. For the sake of contradiction, suppose $\delta_i = \delta_j$ for $i \neq j$. By Property I, $j \neq i+1$. Without loss of generality, we can assume that for all r such that $i < r < j$, $\delta_r \neq \delta_i$. Set $\delta_r = \max\{\delta_{i+1}, \delta_{i+2}, \ldots, \delta_{j-1}\}$, and notice that $\delta(v_{i+1}, v_j) = \delta_r$ by Property II. Now if $\delta_r > \delta_i = \delta_j$, then $\chi(v_i, v_{i+1}, v_j, v_{j+1}) =$ red and we have a contradiction. If $\delta_r < \delta_i$, then this would contradict Property III. Hence, the statement follows. $\square$

Proof of Lemma 3.2. It suffices to show that the statement holds when $W_1 = V_1$ and $W_2 = V_2$. For the sake of contradiction, suppose $|V_1|, |V_2| \geq m/2n = 16n^4$. Recall that $\delta^*_1 = \delta(v_{i_1}, v_{i_1+1})$, $V_1 = \{v_1, v_2, \ldots, v_{i_1}\}$, $V_2 = \{v_{i_1+1}, v_{i_1+1}, \ldots, v_m\}$, and set $A_1 = \{\delta_1, \ldots, \delta_{i_1-1}\}$ and $A_2 = \{\delta_{i_1+1}, \ldots, \delta_{m-1}\}$. For $i \in \{1, 2\}$, let us partition $A_i = A^r_i \cup A^b_i$ where

$$A^r_i = \{\delta_j \in A_i : \phi(\delta^*_1, \delta_j) = \text{ red}\} \qquad \text{and} \qquad A^b_i = \{\delta_j \in A_i : \phi(\delta^*_1, \delta_j) = \text{ blue}\}.$$

By the pigeonhole principle, either $|A^b_2| \geq 8n^4$ or $|A^r_2| \geq 8n^4$. Without loss of generality, we can assume that $|A^b_2| \geq 8n^4$ since a symmetric argument would follow otherwise.

Fix $\delta_{j_1} \in A^b_1$ and $\delta_{j_2} \in A^b_2$, and recall that $\delta_{j_1} = \delta(v_{j_1}, v_{j_1+1})$ and $\delta_{j_2} = \delta(v_{j_2}, v_{j_2+1})$. By Observation 3.4, $\delta_{j_1} \neq \delta_{j_2}$, and by Property II, we have $\delta(v_{j_1+1}, v_{j_2}) = \delta^*_1$. Since $\chi(v_{j_1}, v_{j_1+1}, v_{j_2}, v_{j_2+1}) =$ blue, this implies that $\phi(\delta_{j_1}, \delta_{j_2}) =$ red. By Lemma 3.1 and Observation 3.4, we have $|A^b_1| < n$. Indeed, otherwise we would have a monochromatic red copy of $K_{n,n}$ in A with respect to ϕ. Therefore we have $|A^r_1| \geq 16n^4 - n - 1$. Again by the pigeonhole principle, there is a subset $B \subset A^r_1$ of size at least $(16n^4 - n - 1)/n \geq 8n^3 - 1$, such that $B = \{\delta_j, \delta_{j+1}, \ldots, \delta_{j+8n^3-2}\}$, and whose corresponding vertices are $U = \{v_j, v_{j+1}, \ldots, v_{j+8n^3-1}\}$. For simplicity and without loss of generality, let us rename $U = \{u_1, \ldots, u_{8n^3}\}$ and $\delta_i = \delta(u_i, u_{i+1})$ for $1 \leq i \leq 8n^3 - 1$.

Just as before, we greedily construct a set $D_t = \{\delta^*_1, \ldots, \delta^*_t\} \subset \delta^*_1 \cup \{\delta_1, \ldots, \delta_{8n^3-1}\}$ and a set $S_t \subset \{u_1, \ldots, u_{8n^3}\}$ such that the following holds.

1. We have $\delta^*_1 > \cdots > \delta^*_t$, where $\delta^*_j = \delta(u_{i_j}, u_{i_j+1})$ for $i \geq 2$.
2. For each $\delta^*_j = \delta(u_{i_j}, u_{i_j+1}) \in D_t$, consider the set of vertices

   $$S = \{u_{i_{j+1}}, u_{i_{j+1}+1}, \ldots, u_{i_h}, u_{i_h+1}\} \cup S_t.$$

   Then either every element in S is greater than u_{i_j} or every element in S is less than u_{i_j+1}. In the former case we will label $\delta^*_{i_j}$ *white*, in the latter case we label it *black*.
3. The indices of the vertices in S_t are consecutive, that is, $S_t = \{u_r, u_{r+1}, \ldots, u_{s-1}, u_s\}$ for $1 \leq r < s \leq n$. Set $B_t = \{\delta_r, \delta_{r+1}, \ldots, \delta_{u_{s-1}}\}$.

4. for each $\delta_j^* \in D_t$, either $\phi(\delta_j^*, \delta) = $ red for every $\delta \in \{\delta_{j+1}^*, \delta_{j+2}^*, \ldots, \delta_t^*\} \cup B_t$, or $\phi(\delta_j^*, \delta) =$ blue for every $\delta \in \{\delta_{j+1}^*, \delta_{j+2}^*, \ldots, \delta_t^*\} \cup B_t$.

5. We have $|S_t| \geq 8n^3 - (t-1)2n^2$.

We start with $S_1 = U = \{u_1, \ldots, u_{8n^3}\}$ and $D_1 = \{\delta_1^*\}$, where we recall that $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$. Having obtained $D_{t-1} = \{\delta_1^*, \ldots, \delta_{t-1}^*\}$ and $S_{t-1} = \{u_r, \ldots, u_s\}$, $1 \leq r < s \leq n$, we construct D_t and S_t as follows. Let $\delta_t^* = \delta(u_{i_t}, u_{i_t+1})$ be the unique largest element in $\{\delta_r, \delta_{r+1}, \ldots, \delta_{s-1}\}$, and set $D_t = D_{t-1} \cup \delta_t^*$. The uniqueness of δ_t^* follows from Properties I and II. Let us partition $S_t = T_1 \cup T_2$, where $T_1 = \{u_r, u_{r+1}, \ldots, u_{i_t}\}$ and $T_2 = \{u_{i_t+1}, u_{i_{h+1}+2}, \ldots, u_s\}$. Now we make the following observation.

Observation 3.5. *We have $|T_1| < 2n^2$ or $|T_2| < 2n^2$.*

Proof. For the sake of contradiction, suppose $|T_1|, |T_2| \geq 2n^2$ and let $B_1 = \{\delta_r, \delta_{r+1}, \ldots, \delta_{i_t-1}\}$ and $B_2 = \{\delta_{i_1+1}, \delta_{i_t+2}, \ldots, \delta_{s-1}\}$. Notice that for every $\delta \in B_2$ we have $\phi(\delta_t^*, \delta) =$ red. Indeed, suppose for $\delta = \delta(u_\ell, u_{\ell+1}) \in B_2$ we have $\phi(\delta_t^*, \delta) =$ blue. Recall $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$, $\delta_t^* = \delta(u_{i_t}, u_{i_t+1})$, where

$$u_{i_t} < u_{i_t+1} < u_\ell < u_{\ell+1} < v_{i_1} < v_{i_1+1}.$$

Consider the vertices $v_{i_1+1}, u_{i_t}, u_\ell, u_{\ell+1}$. By definition of χ, we have $\chi(u_{i_t}, u_\ell, u_{\ell+1}, v_{i_1+1}) =$ red, contradiction. Therefore, by the same argument as above, there are less than n elements $\delta \in B_1$ such that $\phi(\delta_t^*, \delta) =$ red. Since $|T_1| > 2n^2$, by the pigeonhole principle, there is a set of $n+1$ consecutive vertices $\{u_\ell, u_{\ell+1}, \ldots, u_{\ell+n}\} \subset T_1$ and the subset $\{\delta_\ell, \delta_{\ell+1}, \ldots, \delta_{\ell+n-1}\} \subset B_1$ such that $\phi(\delta_t^*, \delta) =$ blue for every $\delta \in \{\delta_\ell, \delta_{\ell+1}, \ldots, \delta_{\ell+n-1}\}$. Notice that

$$\delta_\ell < \delta_{\ell+1} < \cdots < \delta_{\ell+n-1}.$$

Indeed, suppose $\delta_r > \delta_{r+1}$ for some $r \in \{\ell, \ell+1, \ldots, \ell+n-2\}$. Then $\phi(\delta_r, \delta_{r+1}) =$ red implies that $\chi(u_{i_t+1}, u_r, u_{r+1}, u_{r+2}) =$ red, contradiction. Likewise if $\phi(\delta_r, \delta_{r+1}) =$ blue, then $\chi(v_{i_1+1}, u_r, u_{r+1}, u_{r+2}) =$ red, contradiction. However, by Lemma 3.1, there is a bad triple in $\{\delta_\ell, \delta_{\ell+1}, \ldots, \delta_{\ell+n-1}\}$ with respect to ϕ. Since $\delta_\ell, \delta_{\ell+1}, \ldots, \delta_{\ell+n-1}$ forms a monotone sequence, by Property IV, χ colors some 4-tuple in the set $\{u_\ell, u_{\ell+1}, \ldots, u_{\ell+n}\}$ red, contradiction. Hence the statement follows. □

If $|T_1| < 2n^2$, we set $S_t = T_2$. Otherwise by Observation 3.5 we have $|T_2| < 2n^2$ and we set $S_t = T_1$. Hence $|S_t| > |S_{t-1}| - 2n^2$.

Since $|S_1| = |U| = 8n^3$, we have $|S_t| > 0$ for $t = 2n$. Therefore, we can construct $D_{2n} = \{\delta_1^*, \ldots, \delta_{2n}^*\}$ with the desired properties. By the pigeonhole principle, at least n elements in D_{2n} have the same label, say *white*. The other case will follow by a symmetric argument. We remove all black labeled elements in D_{2n}, and let $\{\delta_{j_1}^*, \ldots, \delta_{j_n}^*\}$ be the resulting set, and for simplicity, let $\delta_{j_r}^* = \delta(v_{j_r}, v_{j_r+1})$.

Now consider the vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_n}, v_{j_n+1} \in V$. By construction and by Property II, we have $v_{j_1} < v_{j_2} < \cdots < v_{j_n} < v_{j_n+1}$ and $\delta(v_{j_1}, v_{j_2}) = \delta_{i_{j_1}}^*, \delta(v_{j_2}, v_{j_3}) = \delta_{i_{j_2}}^*, \ldots, \delta(v_{j_n}, v_{j_{n+1}}) = \delta_{i_{j_n}}^*$. Therefore, we have a monotone sequence

$$\delta(v_{j_1}, v_{j_2}) > \delta(v_{j_2}, v_{j_3}) > \cdots > \delta(v_{j_n}, v_{j_{n+1}}).$$

By Lemma 3.1, there is a bad triple in the set $\{\delta^*_{j_1}, \dots, \delta^*_{j_n}\}$ with respect to ϕ. By Property IV, χ does not color every 4-tuple in $V = \{v_1, \dots, v_m\}$ blue which is a contradiction. □

4 A new lower bound for $r_4(5, n)$

Again we apply a variant to the Erdős-Hajnal stepping up lemma in order to establish a new lower bound for $r_4(5, n)$. We will use the following lemma.

Lemma 4.1. *For $n \geq 5$, there is an absolute constant $c > 0$ such that the following holds. For $N = \lfloor n^{c \log n} \rfloor$, there is a red/blue coloring ϕ on the pairs of $\{0, 1, \dots, N-1\}$ such that*

1. *there is no monochromatic red copy of $K_{\lfloor \log n \rfloor}$,*
2. *there are no two disjoint n-sets $A, B \subset \{0, 1, \dots, N-1\}$, such that $\phi(a, b) =$ blue for every $a \in A$ and $b \in B$ (i.e. no blue $K_{n,n}$).*
3. *there is no n-set $A \subset \{0, 1, \dots, N-1\}$ such that every triple $a_i, a_j, a_k \in A$, where $a_i < a_j < a_k$, avoids the pattern $\phi(a_i, a_j) = \phi(a_j, a_k) =$ blue and $\phi(a_i, a_k) =$ red.*

We omit the proof of Lemma 4.1, which follows by the same probabilistic argument used for Lemma 3.1. For the reader's convenience, let us restate the result that we are about to prove.

Theorem 4.2. *For $n \geq 5$, there is an absolute constant $c > 0$ such that $r_4(5, n) > 2^{n^{c \log n}}$.*

Proof. Let $c > 0$ be the constant from Lemma 4.1, and set $A = \{0, 1, \dots, \lfloor n^{c \log n} \rfloor - 1\}$. Let ϕ be the red/blue coloring on the pairs of A with the properties described in Lemma 4.1. Set $N = 2^{\lfloor n^{c \log n} \rfloor}$ and let $V = \{0, 1, \dots, N-1\}$. In what follows, we will use ϕ to define a red/blue coloring $\chi : \binom{V}{4} \to \{\text{red, blue}\}$ of the 4-tuples of V such that χ does not produce a monochromatic red $K_5^{(4)}$, and does not produce a monochromatic blue copy of $K_{2n^4}^{(4)}$. This would imply the desired lower bound for $r_4(5, n)$.

Just as in the previous section, for any $v \in V$, we write $v = \sum_{i=0}^{\lfloor n^{c \log n} \rfloor - 1} v(i) 2^i$ with $v(i) \in \{0, 1\}$ for each i. For $u \neq v$, set $\delta(u, v) \in A$ denote the largest i for which $u(i) \neq v(i)$. Let $v_1, v_2, v_3, v_4 \in V$ such that $v_1 < v_2 < v_3 < v_4$ and set $\delta_i = \delta(v_i, v_{i+1})$. We define $\chi(v_1, v_2, v_3, v_4) =$ red if

(a) $\delta_1, \delta_2, \delta_3$ is monotone and $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) =$ blue and $\phi(\delta_1, \delta_3) =$ red, or

(b) $\delta_1 < \delta_2 > \delta_3$ and $\delta_1 = \delta_3$, or

(c) $\delta_1 < \delta_2 > \delta_3$, $\delta_1 \neq \delta_3$, and $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) =$ blue and $\phi(\delta_1, \delta_3) =$ red, or

(d) $\delta_1 > \delta_2 < \delta_3$, $\delta_1 < \delta_3$, and $\phi(\delta_1, \delta_3) = \phi(\delta_2, \delta_3) =$ red and $\phi(\delta_1, \delta_2) =$ blue, or

(e) $\delta_1 > \delta_2 < \delta_3$, $\delta_1 > \delta_3$, and $\phi(\delta_1, \delta_3) = \phi(\delta_1, \delta_2) =$ red and $\phi(\delta_2, \delta_3) =$ blue.

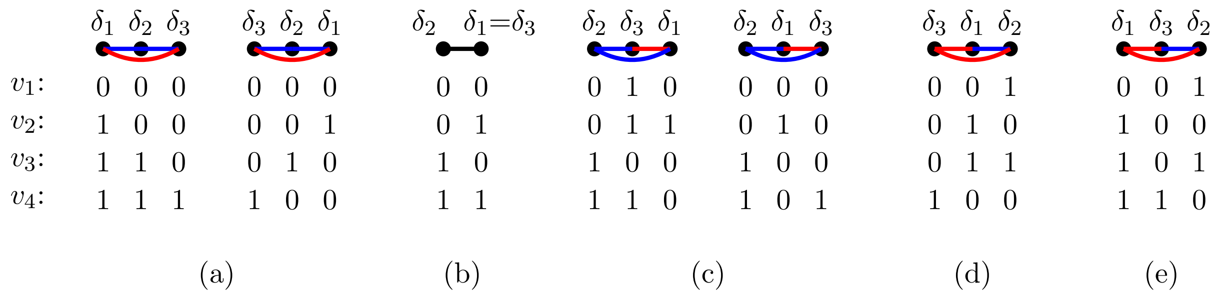

Figure 2: Examples of $v_1 < v_2 < v_3 < v_4$ and $\delta_1 = \delta(v_1, v_2), \delta_2 = \delta(v_2, v_3), \delta_3 = \delta(v_3, v_4)$, such that $\chi(v_1, v_2, v_3, v_4) =$ red. For each case, v_i is represented in binary form with the left-most entry being the most significant bit.

See Figure 2 for small examples. Otherwise, $\chi(v_1, v_2, v_3, v_4) =$ blue.

For the sake of contradiction, suppose that the coloring χ produces a red $K_5^{(4)}$ on vertices $v_1 < \cdots < v_5$, and let $\delta_i = \delta(v_i, v_{i+1})$, $1 \leq i \leq 4$. The proof now falls into the following cases, similar to the previous section.

Case 1. Suppose that $\delta_1, \ldots, \delta_4$ forms a monotone sequence. If $\delta_1 > \cdots > \delta_4$, then we have $\phi(\delta_1, \delta_3) =$ red since $\chi(v_1, v_2, v_3, v_4) =$ red. However, this implies that $\chi(v_1, v_3, v_4, v_5) =$ blue since $\delta(v_1, v_3) = \delta_1$ by Property II, contradiction. A similar argument follows if $\delta_1 < \cdots < \delta_4$.

Case 2. Suppose $\delta_1 > \delta_2 > \delta_3 < \delta_4$. By Property III, $\delta_4 \neq \delta_2, \delta_1$. Since $\delta_1 > \delta_2 > \delta_3$, this implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) =$ blue and $\phi(\delta_1, \delta_3) =$ red. Now consider the following subcases for δ_4.

Case 2.a. Suppose $\delta_4 > \delta_1$. By Property II, $\delta(v_2, v_4) = \delta_2$. Since $\chi(v_1, v_2, v_4, v_5) =$ red, this implies that $\phi(\delta_4, \delta_1) = \phi(\delta_4, \delta_2) =$ red. However, since $\delta_1 = \delta(v_1, v_3)$, this implies $\chi(v_1, v_3, v_4, v_5) =$ blue, contradiction.

Case 2.b. Suppose $\delta_2 < \delta_4 < \delta_1$. Since $\chi(v_2, v_3, v_4, v_5) =$ red, we have $\phi(\delta_4, \delta_2) = \phi(\delta_4, \delta_3) =$ red. However, this implies that $\chi(v_1, v_2, v_4, v_5) =$ blue since $\delta(v_2, v_4) = \delta_2$, contradiction.

Case 2.c. Suppose $\delta_3 < \delta_4 < \delta_2$. Then this would imply $\chi(v_2, v_3, v_4, v_5) =$ blue, contradiction.

Case 3. Suppose $\delta_1 < \delta_2 < \delta_3 > \delta_4$. This implies that $\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) =$ blue and $\phi(\delta_1, \delta_3) =$ red. Hence we have $\delta_4 \neq \delta_1, \delta_2$. Since $\delta(v_2, v_4) = \delta_3$, we have $\chi(v_1, v_2, v_4, v_5) =$ blue, contradiction.

Case 4. Suppose $\delta_1 < \delta_2 > \delta_3 > \delta_4$. This implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) =$ blue and $\phi(\delta_2, \delta_4) =$ red. Hence we have $\delta_1 \neq \delta_3, \delta_4$. Since $\delta(v_2, v_4) = \delta_2$, we have $\chi(v_1, v_2, v_4, v_5) =$ blue, contradiction.

Case 5. Suppose $\delta_1 > \delta_2 < \delta_3 < \delta_4$. Note that by Property III, $\delta_1 \neq \delta_3, \delta_4$. Since $\delta_2, \delta_3, \delta_4$ forms a monotone sequence, this implies that $\phi(\delta_2, \delta_3) = \phi(\delta_3, \delta_4) =$ blue and $\phi(\delta_2, \delta_4) =$ red. Now we consider the following subcases for δ_1.

Case 5.a. Suppose $\delta_2 < \delta_1 < \delta_3$. Then we have $\chi(v_1, v_2, v_3, v_4) =$ blue which is a contradiction.

Case 5.b. Suppose $\delta_3 < \delta_1 < \delta_4$. Then we have $\phi(\delta_1, \delta_3) = \phi(\delta_1, \delta_2) = $ red. Notice that $\delta(v_2, v_4) = \delta_3$ by Property II. Therefore $\chi(v_1, v_2, v_4, v_5) =$ blue, contradiction.

Case 5.c. Suppose $\delta_1 > \delta_4$. Then we have $\phi(\delta_1, \delta_3) = \phi(\delta_1, \delta_2) =$ red. By Property II, $\delta(v_3, v_5) = \delta_4$ which implies $\chi(v_1, v_2, v_3, v_5) =$ blue, contradiction.

Case 6. Suppose $\delta_1 < \delta_2 > \delta_3 < \delta_4$. Then $\chi(v_1, v_2, v_3, v_4) =$ red implies that $\phi(\delta_2, \delta_1) = \phi(\delta_2, \delta_3) =$ blue and $\phi(\delta_1, \delta_3) =$ red. Now if $\delta_2 < \delta_4$, $\chi(v_2, v_3, v_4, v_5) =$ red implies that $\phi(\delta_4, \delta_2) = \phi(\delta_4, \delta_3) =$ red. By Property II, we have $\delta(v_2, v_4) = \delta_2$, and therefore $\delta_1 < \delta_2 < \delta_4$. However, this implies $\chi(v_1, v_2, v_4, v_5) =$ blue, contradiction. Now if $\delta_4 < \delta_2$, then $\chi(v_2, v_3, v_4, v_5) =$ blue, which is again a contradiction.

Case 7. Suppose $\delta_1 > \delta_2 < \delta_3 > \delta_4$. Then $\chi(v_2, v_3, v_4, v_5) =$ red implies that $\phi(\delta_3, \delta_2) = \phi(\delta_3, \delta_4) =$ blue and $\phi(\delta_2, \delta_4) =$ red. Now if $\delta_1 < \delta_3$, then $\chi(v_1, v_2, v_3, v_4) =$ blue which is a contradiction. Therefore we can assume that $\delta_1 > \delta_3$. Since $\chi(v_1, v_2, v_3, v_4) =$ red we have $\phi(\delta_1, \delta_3) =$ red. By Property II, $\delta(v_1, v_3) = \delta_1$ and $\delta_1 > \delta_3 > \delta_4$. This implies that $\chi(v_1, v_3, v_4, v_5) =$ blue which is a contradiction.

Next we show that there is no blue $K_m^{(4)}$ in coloring χ, where $m = 2n^4$. We will prove this statement via the following claims.

Claim 4.3. *There do not exist vertices $w_1 < \cdots < w_n$ in V such that $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) =$ red for every $i < j < k$.*

Proof. Suppose for contradiction that these vertices $w_1 < \cdots < w_n$ exist. Let $\delta_i = \delta(w_i, w_{i+1})$ and set $\delta_{i_1} = \max_i \delta_i$. Let $W = \{w_i : i \leq i_1\}$ and $W' = \{w_i : i > i_1\}$. By the pigeonhole principle, either $|W| \geq n/2$ or $|W'| \geq n/2$. Assume without loss of generality that $|W| \geq n/2$ and set $W_1 = W$. Observe that by hypothesis and definition of δ_{i_1}, for every $w_i, w_j \in W_1$, with $i < j$, we have

$$\phi(\delta(w_i, w_j), \delta_{i_1}) = \phi(\delta(w_i, w_j), \delta(w_j, w_{i_1+1})) = \text{ red}.$$

Note that we obtain the same conclusion if $|W'| \geq n/2$ and $W_1 = W'$ since

$$\phi(\delta_{i_1}, \delta(w_i, w_j)) = \phi(\delta(w_{i_1}, w_i), \delta(w_i, w_j)) = \text{red}.$$

Now define $\delta_{i_2} = \max_{i < i_1} \delta_i$ and repeat the argument above to obtain W_2 with $|W_2| \geq n/4$ such that $\phi(\delta(w_i, w_j), \delta_{i_2}) =$ red for every $w_i, w_j \in W_2$, with $i < j$. Continuing in this way, we obtain $\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_m}$ for $m = \lfloor \log n \rfloor$, such that ϕ colors every pair in the set $\{\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_m}\}$ red. This contradicts Lemma 4.1, and the statement follows. □

Claim 4.4. *There do not exist vertices $w_1 < \cdots < w_{n^2}$ in V such that every 4-tuple among them is blue under χ and for every $i < j < k$ with $\delta(w_i, w_j) > \delta(w_j, w_k)$ we have $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) =$ red.*

Proof. Suppose for contradiction that these vertices $w_1 < \cdots < w_{n^2}$ exist. Let $\delta_i = \delta(w_i, w_{i+1})$ and set $\delta_{i_1} = \max_i \delta_i$. Let $W = \{w_i : i \leq i_1\}$ and $W' = \{w_i : i > i_1\}$. Let us first suppose that $|W'| \geq n$. Pick $w_i, w_j, w_k \in W'$ with $i < j < k$. If $\delta(w_i, w_j) > \delta(w_j, w_k)$, then $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) =$ red by assumption. If $\delta(w_i, w_j) < \delta(w_j, w_k)$, then consider the 4-tuple w_{i_1}, w_i, w_j, w_k. Since this 4-tuple is blue under χ, and both $\phi(\delta(w_{i_1}, w_i), \delta(w_i, w_j))$ and $\phi(\delta(w_{i_1}, w_i), \delta(w_j, w_k))$ are red, $\phi(\delta(w_i, w_j), \delta(w_j, w_k))$ must also be red. Now we may apply Claim 4.3 to W' to obtain a contradiction.

We may therefore assume that $|W'| < n$ and hence $|W| \geq n^2 - n \geq (n-1)^2$. We repeat the previous argument to W to obtain δ_{i_2} and then $\delta_{i_3}, \ldots, \delta_{i_n}$, such that

$$\delta_{i_1} > \delta_{i_2} > \cdots > \delta_{i_n} \quad \text{ and } \quad i_1 > i_2 > \cdots > i_n.$$

Now consider the set $S = \{w_{i_1+1}, w_{i_2+1}, \ldots, w_{i_n+1}, w_{i_n}\}$, whose corresponding delta set is $A = \{\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_n}\}$. Then A is an n-set that has the properties of Lemma 4.1 part 3. This implies that there are $j < k < l$ such that $\phi(\delta_{i_j}, \delta_{i_k}) = \phi(\delta_{i_k}, \delta_{i_l}) = \text{blue}$ and $\phi(\delta_{i_j}, \delta_{i_l}) = \text{red}$. Consequently, $\chi(w_{i_j}, w_{i_k}, w_{i_l}, w_{i_l+1}) = \text{red}$, a contradiction. □

By copying the proof above almost verbatim, we have the following.

Claim 4.5. *There do not exist vertices $w_1 < \cdots < w_{n^2}$ in V such that every 4-tuple among them is blue under χ and for every $i < j < k$ with $\delta(w_i, w_j) < \delta(w_j, w_k)$ we have $\phi(\delta(w_i, w_j), \delta(w_j, w_k)) =$ red.*

Now we are ready to show that there is no blue $K_m^{(4)}$ in coloring χ, where $m = 2n^4$. For the sake of contradiction, suppose we have vertices $v_1, \ldots, v_m \in V$ such that $v_1 < \cdots < v_m$, and χ colors every 4-tuple in the set $\{v_1, \ldots, v_m\}$ blue. Let $\delta_i = \delta(v_i, v_{i+1})$ for $1 \leq i \leq m-1$. Notice that by Observation 3.4 we have $\delta_i \neq \delta_j$ for $1 \leq i < j < m$.

Let $\delta_1^* = \max\{\delta_1, \ldots, \delta_m\}$, where $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$. Set

$$V_1 = \{v_1, v_2, \ldots, v_{i_1}\} \quad \text{ and } \quad V_2 = \{v_{i_1+1}, v_{i_1+1}, \ldots, v_m\}.$$

Now we establish the following lemma.

Lemma 4.6. *We have either $|V_1| < n^3 = m/2n$ or $|V_2| < n^3 = m/2n$.*

Proof of Lemma 4.6. For the sake of contradiction, suppose $|V_1|, |V_2| \geq n^3$. Recall that $\delta_1^* = \delta(v_{i_1}, v_{i_1+1})$, $V_1 = \{v_1, v_2, \ldots, v_{i_1}\}$, $V_2 = \{v_{i_1+1}, v_{i_1+1}, \ldots, v_m\}$, and set $A_1 = \{\delta_1, \ldots, \delta_{i_1-1}\}$ and $A_2 = \{\delta_{i_1+1}, \ldots, \delta_{m-1}\}$. For $i \in \{1, 2\}$, let us partition $A_i = A_i^r \cup A_i^b$ where

$$A_i^r = \{\delta_j \in A_i : \phi(\delta_1^*, \delta_j) = \text{ red}\} \qquad \text{and} \qquad A_i^b = \{\delta_j \in A_i : \phi(\delta_1^*, \delta_j) = \text{ blue}\}.$$

Let us first suppose that $|A_i^b| \geq n$ for $i = 1, 2$. Fix $\delta_{j_1} \in A_1^b$ and $\delta_{j_2} \in A_2^b$, and recall that $\delta_{j_1} = \delta(v_{j_1}, v_{j_1+1})$ and $\delta_{j_2} = \delta(v_{j_2}, v_{j_2+1})$. By Observation 3.4, $\delta_{j_1} \neq \delta_{j_2}$, and by Property II, we have $\delta(v_{j_1+1}, v_{j_2}) = \delta_1^*$. Since $\chi(v_{j_1}, v_{j_1+1}, v_{j_2}, v_{j_2+1}) = \text{blue}$, this implies that $\phi(\delta_{j_1}, \delta_{j_2}) = \text{blue}$. Consequently, we have a monochromatic blue copy of $K_{n,n}$ in A with respect to ϕ, which contradicts Lemma 4.1 part 2.

Therefore we have $|A_1^b| \leq n$ or $|A_2^b| \leq n$. Let us first suppose that $|A_1^b| \leq n$. Since $|A_1| \geq n^3$, by the pigeonhole principle, there is a subset $R \subset A_1^r$ such that $R = \{\delta_j, \delta_{j+1}, \ldots, \delta_{j+n^2-2}\}$, whose corresponding vertices are $U = \{v_j, v_{j+1}, \ldots, v_{j+n^2-1}\}$. For simplicity and without loss of generality, let us rename $U = \{u_1, \ldots, u_{n^2}\}$ and $\delta_i = \delta(u_i, u_{i+1})$ for $1 \leq i \leq n^2$. Now notice that $\phi(\delta(u_i, u_j), \delta(u_j, u_k)) =$ red for every $i < j < k$ with $\delta(u_i, u_j) > \delta(u_j, u_k)$. Indeed, since $\delta(u_i, u_j), \delta(u_j, u_k) \in R$ we have $\phi(\delta_1^*, \delta(u_i, u_j))) = \phi(\delta_1^*, \delta(u_j, u_k))) =$ red. Since $\chi(u_i, u_j, u_k, v_{i_1+1}) =$ blue, this implies that we must have $\phi(\delta(u_i, u_j), \delta(u_j, u_k)) =$ red by definition of χ. However, by Claim 4.4 we obtain a contradiction.

In the case that $|A_2^b| \leq n$, a symmetric argument follows, where we apply Claim 4.5 instead of Claim 4.4 to obtain the contradiction. □

Now we can finish the argument that χ does not color every 4-tuple in the set $\{v_1, \ldots, v_m\}$ blue by copying the proof of Lemma 3.3. In particular, we will obtain vertices $v_{j_1} < \cdots < v_{j_{n+1}} \in \{v_1, \ldots, v_m\}$ such that $\delta(v_{j_1}, v_{j_2}), \delta(v_{j_2}, v_{j_3}), \ldots, \delta(v_{j_n}, v_{j_{n+1}})$ forms a monotone sequence. By Property IV and Lemma 4.1, χ will color a 4-tuple in the set $\{v_{j_1}, \ldots, v_{j_{n+1}}\}$ red.

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