A survey of quantitative bounds for hypergraph Ramsey problems

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Abstract

The classical *hypergraph Ramsey number* $r_k(s,n)$ is the minimum N such that for every red-blue coloring of the k-tuples of $\{1,\ldots,N\}$, there are s integers such that every k-tuple among them is red, or n integers such that every k-tuple among them is blue. We survey a variety of problems and results in hypergraph Ramsey theory that have grown out of understanding the quantitative aspects of $r_k(s,n)$. Our focus is on recent developments. We also include several new results and proofs that have not been published elsewhere.

1 Introduction

A k-uniform hypergraph H (k-graph for short) with vertex set V is a collection of k-element subsets of V. We write $K_n^{(k)}$ for the complete k-uniform hypergraph on an n-element vertex set. The *Ramsey number* $r_k(s,n)$ is the minimum N such that every red-blue coloring of the edges of $K_N^{(k)}$ contains a monochromatic red copy of $K_s^{(k)}$ or a monochromatic blue copy of $K_n^{(k)}$.

In this survey we focus on problems and results related to generalizations and extensions of $r_k(s,n)$ in the *hypergraph* case, i.e., when $k \geq 3$ (we refer the reader to [22] for a survey of Graph Ramsey theory). Our emphasis is on recent results and although we believe we have touched on most important developments in this area, this survey is not an exhaustive compendium of all work in hypergraph Ramsey theory.

2 Diagonal Ramsey numbers

Diagonal Ramsey numbers refer to the special case when $s = n$, i.e. $r_k(n,n)$, and have been studied extensively over the past 80 years. Classic results of Erdős and Szekeres [42] and Erdős [32] imply that $2^{n/2} < r_2(n,n) \leq 2^{2n}$ for every integer $n > 2$. While small improvements have been made in both the upper and lower bounds for $r_2(n,n)$ (see [81, 13]), the constant factors in the exponents have not changed over the last 70 years.

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Unfortunately for 3-uniform hypergraphs, our understanding of $r_3(n,n)$ is much less. A result of Erdős, Hajnal, and Rado [39] gives the best known lower and upper bounds for $r_3(n,n)$,

$$2^{c_1 n^2} < r_3(n,n) < 2^{2^{c_2 n}},$$

where c_1 and c_2 are absolute constants. For $k \geq 4$, there is also a difference of one exponential between the known lower and upper bounds for $r_k(n,n)$, that is,

$$\mathrm{twr}_{k-1}(c_1 n^2) \leq r_k(n,n) \leq \mathrm{twr}_k(c_2 n), \tag{1}$$

where the *tower function* $\mathrm{twr}_k(x)$ is defined by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$ (see [42, 40, 38]). A notoriously difficult conjecture of Erdős, Hajnal, and Rado states that the upper bound in (1) is essentially the truth, that is, there are constructions which demonstrates that $r_k(n,n) > \mathrm{twr}_k(cn)$, where $c = c(k)$. The crucial case is when $k = 3$, since a double exponential lower bound for $r_3(n,n)$ would verify the conjecture for all $k \geq 4$ by using the well-known stepping-up lemma of Erdős and Hajnal (see [47]).

Conjecture 2.1 (Erdős)**.** *For $n \geq 4$ we have $r_3(n,n) > 2^{2^{cn}}$, where c is an absolute constant.*

It is worth mentioning that Erdős offered a \$500 reward for a proof of this conjecture (see [11]).

3 Off-Diagonal Ramsey numbers

Off-diagonal Ramsey numbers, $r_k(s,n)$, refer to the special case when k, s are fixed and n tends to infinity. It is known [2, 51, 5, 7] that $r_2(3,n) = \Theta(n^2/\log n)$, and more generally for fixed $s > 3$, $r_2(s,n) = n^{\Theta(1)}$. For 3-uniform hypergraphs, the following result yields the current best known bounds for this problem.

Theorem 3.1 (Conlon-Fox-Sudakov [19])**.** *For $s \geq 4$, there are constants c_1 and c_2 such that*

$$2^{c_1 n \log n} < r_3(s,n) < 2^{c_2 n^{s-2} \log n}.$$

For k-uniform hypergraphs, where $k \geq 4$, it is known that $r_k(s,n) \leq \mathrm{twr}_{k-1}(n^c)$, where $c = c(s)$ [40]. Erdős and Hajnal proved that

$$r_k(s,n) > \mathrm{twr}_{k-1}(c'n), \tag{2}$$

for $k \geq 4$ and $s \geq 2^{k-1} - k + 3$, where $c' = c'(s)$. They conjectured that a similar bound should hold for smaller s as follows.

Conjecture 3.2 (Erdős-Hajnal [38])**.** *Fix $4 \leq k < s$. There are constants c and c' such that*

$$\mathrm{twr}_{k-1}(cn) < r_k(s,n) < \mathrm{twr}_{k-1}(c'n).$$

Actually, this was part of a more general conjecture that they posed in that paper that will be discussed in Section 4. In [18], Conlon, Fox, and Sudakov verified the Erdős-Hajnal conjecture for all $s \geq \lceil 5k/2 \rceil - 3$. Recently, the current authors [74] and independently Conlon, Fox, and Sudakov [20] verified the conjecture for all $s \geq k+3$ (using different constructions). Since $2^{k-1} - k + 3 = \lceil 5k/2 \rceil - 3 = k + 3 = 7$ when $k = 4$, all three of these approaches succeed in proving a double exponential lower bound for $r_4(7, n)$ but fail for $r_4(6, n)$ and $r_4(5, n)$.

Just as for diagonal Ramsey numbers, a double exponential in n^c lower bound for $r_4(5, n)$ and $r_4(6, n)$ would imply $r_k(k+1, n) > \mathrm{twr}_{k-1}(n^{c'})$ and $r_k(k+2, n) > \mathrm{twr}_{k-1}(n^{c'})$ respectively, for all fixed $k \geq 5$, by a variant of the Erdős-Hajnal stepping up lemma. Therefore, the difficulty in verifying (2) for the two remaining cases, $s = k+1$ and $k+2$, is due to our lack of understanding of $r_4(5, n)$ and $r_4(6, n)$.

For many decades, the only lower bound for both $r_4(5, n)$ and $r_4(6, n)$ was 2^{cn}, which was implicit in the paper of Erdős and Hajnal [38]. The current authors improved both exponents to n^2 in [74]. Then in [73], these were further improved to

$$r_4(5, n) > 2^{n^{c \log \log n}} \qquad \text{and} \qquad r_4(6, n) > 2^{n^{c \log n}}. \tag{3}$$

Very recently, the authors established a double exponential in n^c lower bound for $r_4(6, n)$ and further improved the lower bound for $r_4(5, n)$. These represent the current best bounds.

Theorem 3.3 (Mubayi-Suk [72]). *For all $n \geq 6$,*

$$r_4(5, n) > 2^{n^{c \log n}} \qquad \text{and} \qquad r_4(6, n) > 2^{2^{cn^{1/5}}},$$

where $c > 0$ is an absolute constant. More generally, for $n > k \geq 5$, there is a $c = c(k) > 0$ such that

$$r_k(k+1, n) > \mathrm{twr}_{k-2}(n^{c \log n}) \qquad \text{and} \qquad r_k(k+2, n) > \mathrm{twr}_{k-1}(cn^{1/5}).$$

A standard argument in Ramsey theory together with results in [19] for 3-uniform hypergraph Ramsey numbers yields the upper bound $r_k(k+2, n) < \mathrm{twr}_{k-1}(c' n^3 \log n)$, so we now know the tower growth rate of $r_k(k+2, n)$. The only problem that now remains is to prove that $r_4(5, n)$ is double exponential in a power of n.

Conjecture 3.4. *For $n \geq 5$, there is an absolute constant $c > 0$ such that $r_4(5, n) > 2^{2^{n^c}}$.*

In [74], we established a connection between diagonal and off-diagonal Ramsey numbers. In particular, we showed that a solution to Conjecture 2.1 implies a solution to Conjecture 3.4 (see Section 7 for more details).

4 The Erdős-Hajnal Problem

As mentioned in previous sections, it is a major open problem to determine if $r_3(n, n)$ and $r_4(5, n)$ are double exponential in a power of n. In order to shed more light on these questions, Erdős and Hajnal [38] in 1972 considered the following more general parameter.

Definition 4.1. For integers $2 \leq k < s < n$ and $2 \leq t \leq \binom{s}{k}$, let $r_k(s, t; n)$ be the minimum N such that every red/blue coloring of the edges of $K_N^{(k)}$ results in a monochromatic blue copy of $K_n^{(k)}$ or has a set of s vertices which contains at least t red edges.

The function $r_k(s,t;n)$ encompasses several fundamental problems which have been studied for a while. Clearly $r_k(s,n) = r_k(s,\binom{s}{k};n)$ so $r_k(s,t;n)$ includes classical Ramsey numbers. In addition to off-diagonal and diagonal Ramsey numbers already mentioned, the function $r_k(k+1,k+1;k+1)$ has been studied in the context of the Erdős-Szekeres theorem and Ramsey numbers of ordered tight-paths by several researchers [29, 31, 46, 64, 63], the more general function $r_k(k+1,k+1;n)$ is related to high dimensional tournaments [59], and even the very special case $r_3(4,3;n)$ has tight connections to quasirandom hypergraph constructions [4, 52, 57, 58].

The main conjecture Erdős and Hajnal [38] states for $r_k(s,t;n)$ is that, as t grows from 1 to $\binom{s}{k}$, there is a well-defined value $t_1 = h_1^{(k)}(s)$ at which $r_k(s,t_1-1;n)$ is polynomial in n while $r_k(s,t_1;n)$ is exponential in a power of n, another well-defined value $t_2 = h_2^{(k)}(s)$ at which it changes from exponential to double exponential in a power of n and so on, and finally a well-defined value $t_{k-2} = h_{k-2}^{(k)}(s) < \binom{s}{k}$ at which it changes from twr_{k-2} to twr_{k-1} in a power of n. They were not able to offer a conjecture as to what $h_i^{(k)}(s)$ is in general, except when $i=1$ and when $s=k+1$.

• When $i=1$, they conjectured that $t_1 = h_1^{(k)}(s)$ is one more than the number of edges in the k-graph obtained by taking a complete k-partite k-graph on s vertices with almost equal part sizes, and repeating this construction recursively within each part. Erdős offered $500 for a proof of this (see [11]).

• When $s=k+1$, they conjectured that $h_i^{(k)}(k+1) = i+2$, that is, $r_k(k+1,2;n)$ is polynomial in n, $r_k(k+1,3;n)$ is exponential in a power of n, $r_k(k+1,4;n)$ is double exponential in a power of n, and etc. such that at the end, both $r_k(k+1,k;n)$ and $r_k(k+1,k+1;n)$ are twr_{k-1} in a power of n. They proved this for $i=1$ via the following:

Theorem 4.2 (Erdos-Hajnal [38])**.** *For $k \geq 3$, there are positive $c = c(k)$ and $c' = c'(k)$ such that*

$$r_k(k+1,2;n) < cn^{k-1} \qquad \text{and} \qquad r_k(k+1,3;n) > 2^{c'n}.$$

Results of Rödl-Šinajová [79] on partial Steiner systems, and more recently Kostochka-Mubayi-Verstraëte [56] on independent sets in hypergraphs determine the order of magnitude of the function $r_k(k+1,2;n)$.

Theorem 4.3 (Rödl-Šinajová [79] lower bound, Kostochka-Mubayi-Verstraëte [56] upper bound)**.** *For each $k \geq 3$ there exist positive $c = c_k$ and $c' = c'(k)$ such that*

$$c'n^{k-1}/\log n < r_k(k+1,2;n) < c\,n^{k-1}/\log n.$$

For the $t=3$ case, the authors in [75] showed the following.

Theorem 4.4 (Mubayi-Suk [75])**.** *For $k \geq 3$, there are positive $c = c(k)$ and $c' = c'(k)$ such that*

$$2^{cn^{k-2}} \leq r_k(k+1,3;n) \leq 2^{c'n^{k-2}\log n}.$$

For general t, the methods of Erdős and Rado [40] show that there exists $c = c(k,t) > 0$ such that

$$r_k(k+1,t;n) \leq \mathrm{twr}_{t-1}(n^c), \tag{4}$$

for $3 \leq t \leq k$. Erdős and Hajnal conjectured that this upper bound is the correct tower growth rate for $r_k(k+1,t;n)$.

Conjecture 4.5 (Erdos-Hajnal [38]). *For $k \geq 3$ and $2 \leq t \leq k$, there exists $c = c(k,t) > 0$ such that*

$$r_k(k+1,t;n) \geq \mathrm{twr}_{t-1}(c\,n).$$

Note that when $t = k+1$, the results from the previous section states that $r_k(k+1,k+1;n) = r_k(k+1,n) \leq \mathrm{twr}_{k-1}(n^{c'})$ where $c' = c'(k,t)$.

Hence for 3-uniform hypergraphs, $r_3(4,t;n)$ is fairly well understood. We know that $r_3(4,2;n)$ is polynomial in n, and both $r_3(4,3;n)$ and $r_3(4,4;n)$ are exponential in $n^{1+o(1)}$. Unfortunately for 4-uniform hypergraphs, we do not have a good understanding of $r_4(5,t;n)$ when $4 \leq t \leq 5$. The best known upper and lower bounds for $r_4(5,4;n)$ are obtained by Theorem 4.4 and (4) which give $2^{cn^2} < r_4(5,4;n) < 2^{2^{n^c}}$. Notice that Conjecture 4.5 states that $r_4(5,4;n)$ is double exponential in a power of n, but we don't even know if $r_4(5,5;n) = r_4(5,n)$ is double exponential in a power of n. Likewise, for 5-uniform hypergraphs, not much is known about $r_5(6,4;n)$ and $r_5(6,5;n)$, and Theorem 4.4 and (4) imply that

$$2^{c'n^3} < r_5(6,4;n) < 2^{2^{n^c}} \qquad \text{and} \qquad 2^{c'n^3} < r_5(6,5;n) < 2^{2^{2^{n^c}}}. \tag{5}$$

Problem 4.6. *Determine the tower growth rate of $r_4(5,4;n)$, $r_5(6,4;n)$, and $r_5(6,5;n)$.*

However for k-uniform hypergraphs, when $k \geq 6$, the authors in [75] settled Conjecture 4.5 in almost all cases in a strong form, by determining the correct tower growth rate, and in half of the cases also determining the correct power of n within the tower.

Theorem 4.7 (Mubayi-Suk [75]). *For $k \geq 6$ and $4 \leq t \leq k-2$, there are positive $c = c(k,t)$ and $c' = c'(k,t)$ such that*

$$\mathrm{twr}_{t-1}(c' n^{k-t+1}\log n) \;\geq\; r_k(k+1,t;\,n) \;\geq\; \begin{cases} \mathrm{twr}_{t-1}(c\,n^{k-t+1}) & \text{if } k-t \text{ is even} \\ \mathrm{twr}_{t-1}(c\,n^{(k-t+1)/2}) & \text{if } k-t \text{ is odd.} \end{cases}$$

When $k \geq 6$ and $t \in \{k-1,k\}$, Conjecture 4.5 remains open. We note that the upper bound in Theorem 4.7 also holds when $k-1 \leq t \leq k$. The best known upper and lower bounds for $r_k(k+1,k-1;n)$ and $r_k(k+1,k;n)$, also due to the authors [75], are

$$\mathrm{twr}_{k-3}(c\,n^3) \leq r_k(k+1,k-1;\,n) \leq \mathrm{twr}_{k-2}(c'\,n^2),$$

and

$$\mathrm{twr}_{k-3}(c\,n^3) \leq r_k(k+1,k;\,n) \leq \mathrm{twr}_{k-1}(c'\,n).$$

In fact, by using the stepping-up lemma established in [75], any improvement in the lower bound for $r_5(6,4;n)$ and $r_5(6,5;n)$ in (5) would imply a better lower bound for $r_k(k+1,k-1;\,n)$ and $r_k(k+1,k;\,n)$ respectively.

5 The Erdős-Rogers Problem

An s-independent set in a k-graph H is a vertex subset that contains no copy of $K_s^{(k)}$. So if $s = k$, then it is just an independent set. Let $\alpha_s(H)$ denote the size of the largest s-independent set in H.

Definition 5.1. For $k \le s < t < N$, the Erdős-Rogers function $f_{s,t}^k(N)$ is the minimum of $\alpha_s(H)$ taken over all $K_t^{(k)}$-free k-graphs H of order N.

To prove the lower bound $f_{s,t}^k(N) \ge n$, one must show that every $K_t^{(k)}$-free k-graph on N vertices contains an s-independent set with n vertices. On the other hand, to prove the upper bound $f_{s,t}^{(k)}(N) < n$, one must construct a $K_t^{(k)}$-free k-graph H of order N with $\alpha_s(H) < n$.

The problem of determining $f_{s,t}^k(n)$ extends that of finding Ramsey numbers. Formally,

$$r_k(s,n) = \min\{N : f_{k,s}^k(N) \ge n\}.$$

For $k = 2$, the above function was first considered by Erdős and Rogers [41] only for $t = s+1$, which is perhaps the most interesting case. So in this case we wish to construct a K_{s+1}-free graph on N vertices such that the s-independence number is as small as possible. Since then the function has been studied by several researchers culminating in the work of Wolfowitz [84] and Dudek, Retter and Rödl [28] who proved the upper bound that follows (the lower bound is due to Dudek and the first author [27]): for every $s \ge 3$ there are positive constants c_1 and $c_2 = c_2(s)$ such that

$$c_1 \left(\frac{N \log N}{\log \log N} \right)^{1/2} < f_{s,s+1}^2(N) < c_2 (\log N)^{4s^2} N^{1/2}. \tag{6}$$

The problem of estimating the Erdős-Rogers function for $k > 2$ appears to be much harder. Let us denote

$$g(k,N) = f_{k+1,k+2}^k(N).$$

In other words, $g(k,N)$ is the minimum n such that every $K_{k+2}^{(k)}$-free k-uniform hypergraph on N vertices has the property that every n-set of vertices has a copy of $K_{k+1}^{(k)}$. With this notation, the bounds in (6) for $s = 3$ imply that $g(2,N) = N^{1/2+o(1)}$.

Dudek and the first author [27] proved that $(\log N)^{1/4+o(1)} < g(3,N) < O(\log N)$, and more generally, that there are positive $c_1 = c_1(k)$ and $c_2 = c_2(k)$ with

$$c_1 (\log_{(k-2)} N)^{1/4} < g(k,N) < c_2 (\log N)^{1/(k-2)}, \tag{7}$$

where $\log_{(i)}$ is the log function iterated i times. The exponent $1/4$ in (7) was improved to $1/3$ by Conlon-Fox-Sudakov [21]. Both sets of authors asked whether the upper bound could be improved (presumably to an iterated log function). This was achieved in the result below, where the number of iterations is $k - O(1)$.

Theorem 5.2 (Mubayi-Suk [73])**.** *Fix $k \ge 14$. Then $g(k,N) = O(\log_{(k-13)} N)$.*

It remains an open problem to determine the correct number of iterations (which may well be $k-2$). We pose this as a conjecture.

Conjecture 5.3. *For all $k \ge 3$, there are $c_1, c_2 > 0$ such that*

$$c_1 \log_{(k-2)} N < g(k,N) < c_2 \log_{(k-2)} N.$$

6 The Erdős-Gyárfás Problem

A (p,q)-coloring of $K_N^{(k)}$ is an edge-coloring of $K_N^{(k)}$ that gives every copy of $K_p^{(k)}$ at least q colors. Let $f_k(N,p,q)$ be the minimum number of colors in a (p,q)-coloring of K_N^k.

The problem of determining $f_k(N,p,q)$ for fixed k,p,q has a long history, beginning with its introduction by Erdős and Shelah [33, 35], and subsequent investigation (for graphs) by Erdős and Gyárfás [37]. We will use the notation

$$r_k(n;t) = r_k(\underbrace{n,\ldots,n}_{t \text{ times}}).$$

Since

$$f_k(N,p,2) = t \qquad \Longleftrightarrow \qquad r_k(p;t) \geq N+1 \quad \text{and} \quad r_k(p;t-1) \leq N,$$

most of the effort on determining $f_k(N,p,q)$ has been for $q > 2$. As mentioned above, Erdős and Gyárfás [37] initiated a systematic study of this parameter for graphs and posed many open problems. One main question was to determine the minimum q such that $f_2(N,p,q) = N^{o(1)}$ and $f_2(N,p,q+1) > N^{c_p}$ for some $c_p > 0$. For $p = 3$ this value is clearly $q = 2$ as $f_2(N,3,2) = O(\log N)$ due to the easy bound $r_2(3;t) > 2^t$, while $f_2(N,3,3) = \chi'(K_N) \geq N-1$. Erdős and Gyárfás proved that $f_2(N,p,p) > N^{c_p}$ and asked whether $f_2(N,p,p-1) = N^{o(1)}$. The first open case was $f_2(N,4,3)$, which was shown to be $N^{o(1)}$ by the first author [66] and later $\Omega(\log N)$ (see [45, 54]). The same upper bound was shown for $f(N,5,4)$ in [30]. Conlon-Fox-Lee-Sudakov [16] recently extended this construction considerably by proving that $f_2(N,p,p-1) = N^{o(1)}$ for all fixed $p \geq 4$. Their result is sharp in the sense that $f_2(N,p,p) = \Omega(N^{1/(p-2)})$. The exponent $1/(p-2)$ was shown to be sharp for $p = 4$ by the first author [67] and recently also for $p = 5$ by Cameron and Heath [10] via explicit constructions.

The first nontrivial hypergraph case is $f_3(N,4,3)$ and this function has tight connections to Shelah's breakthrough proof [80] of primitive recursive bounds for the Hales-Jewett numbers. Answering a question of Graham-Rothschild-Spencer [47], Conlon et al. [15] proved that

$$f_3(N,4,3) = N^{o(1)}.$$

They also posed a variety of basic questions about $f_k(N,p,q)$ and related parameters including the following generalization of the Erdős-Gyárfás problem for hypergraphs. Using a variant of the pigeonhole argument for hypergraph Ramsey numbers due to Erdős and Rado, [15] proved that

$$f_k\left(N, p, \binom{p-i}{k-i} + 1\right) = \Omega(\log_{(i-1)} N^{c_{p,k,i}})$$

where $\log_{(0)}(x) = x$ and, as usual, $\log_{(i+1)} x = \log\log_{(i)} x$ for $i \geq 0$.

Problem 6.1 (Conlon-Fox-Lee-Sudakov [15]). *For $p > k \geq 3$ and $0 < i < k$ prove that $f_k(N,p,\binom{p-i}{k-i})$ is substantially smaller than $f_k(N,p,\binom{p-i}{k-i}+1)$, in particular, prove that $f_k(N,p,\binom{p-i}{k-i})$ is much smaller than $\log_{(i-1)} N$.*

One natural way to interpret this problem is that it asks whether

$$f_k\left(N, p, \binom{p-i}{k-i}\right) = (\log_{(i-1)} N)^{o(1)}?$$

The case $k = 2$ is precisely the Erdős-Gyárfás problem and the case $k = 3, p = 4, i = 1$ is to prove that $f_3(N, 4, 3) = N^{o(1)}$ which was established in [15]. The next open case is $k = 3, p = 5, i = 2$, which asks whether $f_3(N, p, p-2) = (\log N)^{o(1)}$. This was solved with a better bound in the following theorem of the first author [68].

Theorem 6.2 (Mubayi [68]). $f_3(N, 5, 3) = e^{O(\sqrt{\log\log N})} = (\log N)^{O(1/\sqrt{\log\log N})}$.

No other nontrivial cases of Problem 6.1 have been solved. We refer the reader to [15] for related problems and results.

7 Ordered Hypergraph Ramsey Problems

7.1 Tight-paths and cliques in hypergraphs

Consider an *ordered* N-vertex k-uniform hypergraph H, that is, a hypergraph whose vertex set is $[N] = \{1, 2, \ldots, N\}$. A *tight-path of size* s in H, denoted by $P_s^{(k)}$, comprises a set of s vertices $v_1, \ldots, v_s \in [N]$, $v_1 < \cdots < v_s$, such that $(v_j, v_{j+1}, \ldots, v_{j+k-1}) \in E(H)$ for $j = 1, 2, \ldots, s-k+1$. The *length* of a tight-path $P_s^{(k)}$ is the number of edges it contains, that is, $s - k + 1$. In order to avoid the excessive use of superscripts, we write $P_s = P_s^{(k)}$ when the uniformity is already implied.

The Ramsey number $r_k(P_s, P_n)$ is the minimum integer N such that for every red/blue coloring of the k-tuples of $[N]$ contains either a red copy of P_s or a blue copy of P_n. Two famous theorems of Erdős and Szekeres in [42], known as the monotone subsequence theorem and the cups-caps theorem, imply that $r_2(P_s, P_n) = (n-1)(s-1) + 1$ and $r_3(P_s, P_n) = \binom{n+s-4}{s-2} + 1$. Fox, Pach, Sudakov, and Suk [46] later extended their results to k-uniform hypergraphs, and gave a geometric application related to the Happy Ending Theorem.[1] Moshkovitz and Shapira [64] determined an exact formula for $r_3(P_s, \ldots, P_s)$ with q colors in terms of the number of $(q-1)$-dimensional integer partitions of $[s] \times \cdots \times [s]$ with entries from $\{0, 1, \ldots, s\}$ and also determined the correct tower height of $r_k(P_s, \ldots, P_s)$ when $k > 3$ (see also the much earlier paper [29]). Soon after, Milans-Stolee-West [63] obtained an exact formula for $r_k(P_{s_1}, \ldots, P_{s_q})$ for all $k, q \geq 2$, and $s_i \geq k$ (see [25] for some related results).

Theorem 7.1 (Milans-Stolee-West [63]). *Let $k, q \geq 2$, and $s_i > k$ for all $i \in [q]$. Let J_1 be the poset comprising disjoint chains $C_1, \ldots, C_q$, with $|C_i| = s_i - k$ for $i \in [q]$ and for $i \geq 1$, let J_{i+1} be the poset whose elements are the ideals (down sets) of J_i with order defined by containment. Then*

$$r_k(P_{s_1}, \ldots, P_{s_q}) = |J_k| + 1.$$

Let $r_k(P_s, n)$ be the minimum integer N such that every for every red/blue coloring of the k-tuples of $[N]$ contains either a red copy of P_s or a blue copy of K_n. Interestingly, the proof of the Erdős-Szekeres monotone subsequence theorem [42] (see also Dilworth's Theorem [26]) actually implies

[1] The main result in [42], known as the Happy Ending Theorem, states that for any positive integer n, any sufficiently large set of points in the plane in general position has a subset of n members that form the vertices of a convex polygon.

that $r_2(P_s, n) = (n-1)(s-1)+1$. For $k \geq 3$, estimating $r_k(P_s, n)$ appears to be more difficult. Clearly we have

$$r_k(P_s, n) \leq r_k(s, n) \leq \mathrm{twr}_{k-1}(O(n^{s-2}\log n)). \tag{8}$$

In [74], the authors established the following connection between $r_k(P_s, n)$ and the multi-color Ramsey number $r_k(n; q) = r_k(\underbrace{n, \ldots, n}_{q \text{ times}})$.

Theorem 7.2 (Mubayi-Suk [74]). *Let $k \geq 2$ and $s \geq k+1$. Then for $q = s - k + 1$, we have*

$$r_{k-1}(\lfloor n/q \rfloor; q) \leq r_k(P_s, n) \leq r_{k-1}(n; q).$$

The upper bound in Theorem 7.2 follows from the following argument. Let $q = s - k + 1$, $N = r_{k-1}(n; q)$, and suppose χ is a red/blue coloring on the k-tuples of $[N]$. We can assume χ does not produce a red tight-path of length q, since otherwise we would have a red P_s and be done. We define the coloring $\phi : \binom{[N]}{k-1} \to \{0, 1, \ldots, q-1\}$ on the $(k-1)$-tuples of $[N]$, where $\phi(i_1, \ldots, i_{k-1}) = j$ if the longest red tight-path ending in vertices $(i_1, \ldots, i_{k-1})$ has length j. Since $N = r_{k-1}(n; q)$, by Ramsey's theorem, we have a monochromatic clique of size n in color j for some $j \in \{0, 1, \ldots, q-1\}$. However, this clique would correspond to a blue clique with respect to χ. For the lower bound, set $N = r_{k-1}(\lfloor n/q \rfloor; q) - 1$, and let χ be a q coloring on the $(k-1)$-tuples of $[N]$ with colors $\{1, 2, \ldots, q\}$, such that χ does not produce a monochromatic clique of size $\lfloor n/q \rfloor$. Then let $\phi : \binom{[N]}{k} \to \{\text{red,blue}\}$ such that for $i_1 < \cdots < i_k$, $\phi(i_1, \ldots, i_k)$ is red if and only if $\chi(i_1, \ldots, i_{k-1}) < \chi(i_2, \ldots, i_k)$. It is easy to see that ϕ does not produce a red tight-path P_s. With a slightly more complicated argument, one can show by contradiction that ϕ also does not produce a monochromatic blue clique of size n.

The arguments above can be easily extended to obtain the following result for multiple colors.

Theorem 7.3 ([74]). *Let $k \geq 2$ and $s_1, \ldots, s_t \geq k+1$. Then for $q = (s_1 - k + 1) \cdots (s_t - k + 1)$, we have*

$$r_{k-1}(\lfloor n/q \rfloor; q) \leq r_k(P_{s_1}, \ldots, P_{s_t}, n) \leq r_{k-1}(n; q).$$

Theorem 7.2 has several consequences. First, we can considerably improve the upper bound for $r_k(P_s, n)$ in (8).

Corollary 7.4 ([74]). *For fixed $k \geq 3$ and $s \geq k+1$, we have $r_k(P_s, n) \leq \mathrm{twr}_{k-1}(O(sn\log s))$.*

Combining the lower bounds in Theorem 7.2 with the known lower bounds for $r_{k-1}(n, n, n, n)$ in [38], $r_{k-1}(n, n, n)$ in [19], and $r_{k-1}(n, n)$ in [38], we establish the following inequalities. There is an absolute constant $c > 0$ such that for all $k \geq 4$ and $n > 3k$

$$r_k(P_{k+3}, n) \geq r_{k-1}\left(\frac{n}{4}, \frac{n}{4}, \frac{n}{4}, \frac{n}{4}\right) \geq \mathrm{twr}_{k-1}(cn),$$

$$r_k(P_{k+2}, n) \geq r_{k-1}\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right) \geq \mathrm{twr}_{k-1}(c\log^2 n),$$

$$r_k(P_{k+1}, n) \geq r_{k-1}\left(\frac{n}{2}, \frac{n}{2}\right) \geq \mathrm{twr}_{k-2}(cn^2).$$

Summarizing, below are the best known lower bounds for $r_k(P_s, n)$ for fixed s. Note that these lower bounds also provide a lower bound for off-diagonal Ramsey numbers as $r_k(s, n) \geq r_k(P_s, n)$.

Theorem 7.5 ([74]). *There is a positive constant $c > 0$ such that $r_3(P_4, n) > 2^{cn}$, and for $k \geq 4$ and $n > 3k$,*

1. $r_k(P_{k+3}, n) \geq \mathrm{twr}_{k-1}(cn)$,
2. $r_k(P_{k+2}, n) \geq \mathrm{twr}_{k-1}(c \log^2 n)$,
3. $r_k(P_{k+1}, n) \geq \mathrm{twr}_{k-2}(cn^2)$.

We conjecture the following strengthening of the Erdős-Hajnal conjecture.

Conjecture 7.6. *For $k \geq 4$ fixed, $r_k(P_{k+1}, n) \geq \mathrm{twr}_{k-1}(\Omega(n))$.*

For $s = k+1$ in Theorem 7.2, we have $r_{k-1}(n/2, n/2) \leq r_k(P_{k+1}, n) \leq r_{k-1}(n, n)$. Hence, we obtain the following corollary which relates $r_4(P_5, n)$ to the diagonal Ramsey number $r_3(n, n)$.

Corollary 7.7 ([74]). *Conjecture 2.1 holds if and only if there is a constant $c > 0$ such that*

$$r_4(P_5, n) \geq 2^{2^{cn}}.$$

For the case when the size of P_s tends to infinity and the size of K_n is fixed, Mubayi showed the following.

Theorem 7.8 (Mubayi [69]). *For s large, $r_3(P_s, 4) < s^{21}$ and more generally for each $k \geq 3$, there exists $c > 0$ such that for s large,*

$$\mathrm{twr}_{k-2}(s^c) < r_k(P_s, k+1) < \mathrm{twr}_{k-2}(s^{62}).$$

Unfortunately much less is known about $r_k(P_s, k+2)$. The main open problem here is to prove that $r_3(P_s, 5)$ has polynomial growth rate, and more generally, that $r_3(P_s, n)$ has polynomial growth rate for all fixed $n > 4$. The corresponding results for higher uniformity follow easily from the case $k = 3$.

We next consider a version of the Erdős-Hajnal hypergraph Ramsey problem with respect to tight-paths.

Definition 7.9. For integers $2 \leq k < s < n$ and $2 \leq t \leq \binom{s}{k}$, let $r_k(s, t; P_n)$ be the minimum N such that every red/blue coloring of the k-sets of $[N]$ results in a monochromatic blue copy of P_n or has a set of s vertices which induces at least t red edges.

Of course, $r_k(s, \binom{s}{k}; P_n) = r_k(s, P_n)$. We will focus our attention on the smallest case $s = k + 1$. The following conjecture which parallels the Erdős-Hajnal conjecture for cliques was posed in [69].

Conjecture 7.10 (Mubayi [69]). *For $3 \le t \le k$, there are positive $c = c(k,t)$ and $c' = c'(k,t)$ such that*

$$\mathrm{twr}_{t-2}(n^c) < r_k(k+1, t; P_n) < \mathrm{twr}_{t-2}(n^{c'}).$$

This conjecture seems more difficult than the original problem of Erdős and Hajnal. The current best lower bound is only an exponential function; unfortunately the constructions used for Theorem 4.7 fail. Standard arguments yield an upper bound of the form $\mathrm{twr}_{t-1}(n^c)$ for Conjecture 7.10. This upper bound was improved in [69] to $\mathrm{twr}_{t-2}(n^c)$. Some further minor progress towards Conjecture 7.10 was made in [69] for the cases $t = 3$ and $(k,t) = (4,4)$.

7.2 A Ramsey-type result for nonincreasing sets

Inspired by the proof of Theorem 7.2, the authors introduced the following variant of the classical Ramsey number $r_k(n,n)$.

Definition 7.11. Let $\chi : \binom{[N]}{k} \to \{1,2\}$ be a 2-coloring on the k-tuples of $[N]$. For $T \subset [N]$, we say that T is *nonincreasing*, if for any set of $k+1$ vertices $v_1, \ldots, v_{k+1} \in T$, where $v_1 < \cdots < v_{k+1}$, we have $\chi(v_1, \ldots, v_k) \ge \chi(v_2, \ldots, v_{k+1})$.

One example of a nonincreasing set is a monochromatic clique.

Definition 7.12. Let $g_k(n)$ be the minimum integer N such that for any 2-coloring on the k-tuples of $[N]$, with colors $\{1,2\}$, contains a nonincreasing set T of size n.

Clearly we have $g_k(n) \le r_k(n,n)$, and the proof of Theorem 7.2 shows that

$$r_{k+1}(P_{k+2}, n) \le g_k(n). \tag{9}$$

The classic result of Erdős and Szekeres [42] mentioned in the introduction implies that $g_2(n) \le r_2(n,n) < 4^{n+o(n)}$. Below, we give an exponential improvement for this trivial upper bound.

Theorem 7.13. *We have $g_2(n) \le \left\lceil \left(2+\sqrt{2}\right)^n \right\rceil \approx (3.414)^n$.*

Proof. We proceed by induction on n. The base case $n = 1$ is trivial. Suppose now the statement holds for $n' < n$. Set $N = \left\lceil \left(2+\sqrt{2}\right)^n \right\rceil$, and let $\chi : \binom{[N]}{2} \to \{1,2\}$. We will show that there is a nonincreasing subset of size n. Suppose there is a vertex $v \in [N]$ and a subset $S_v \subset \{1, \ldots, v-1\}$ such that $|S_v| \ge \left\lceil \left(2+\sqrt{2}\right)^{n-1} \right\rceil$, and for every $u \in S_v$ we have $\chi(u,v) = 1$. By the induction hypothesis, S_v contains a nonincreasing set of size $n-1$, and together with v we have a nonincreasing set of size n and we are done. Therefore we can assume no such vertex $v \in [N]$ exist.

Set $L = \left\lceil \left(2+\sqrt{2}\right)^{n-1} \right\rceil$, and let $E_2 \subset \binom{[N]}{2}$ denote the set of pairs in $[N]$ with color 2, and whose left endpoint lies in $\{1, \ldots, N-L\}$. For each $v \in [N]$, let $d_2(v)$ denote the number of edges in E_2 whose right endpoint is v. By the assumption above, the back degree of each $v \in [N]$ in color 1 is at most $L-1$, which implies $d_2(v) \ge v - 1 - (L-1) = v - L$. Thus we have

$$\begin{aligned}
|E_2| &\geq \sum_{v=L+1}^{N} d_2(v) \\
&\geq 1 + 2 + \cdots + (N - 2L + 2) + (L-1)(N - 2L + 2) \\
&\geq \tfrac{N(N-2L+2)}{2}.
\end{aligned}$$

By the pigeonhole principle, there is a vertex $u \in \{1, \ldots, N - L\}$ and a set $T \subset \{u + 1, \ldots, N\}$, such that

$$\begin{aligned}
|T| &\geq \tfrac{N(N-2L+2)}{2(N-L)} \\
&= \tfrac{N(N-2L)}{2(N-L)} + \tfrac{N}{N-L} \\
&\geq \left(\tfrac{\sqrt{2}}{1+\sqrt{2}}\tfrac{N}{2}\right) + \left(\tfrac{2+\sqrt{2}}{1+\sqrt{2}}\right) \\
&\geq (2+\sqrt{2})^{n-1} + \left(\tfrac{2+\sqrt{2}}{1+\sqrt{2}}\right),
\end{aligned}$$

and for all $v \in T$ we have $\chi(u, v) = 2$. Hence, $|T| \geq \left\lceil \left(2 + \sqrt{2}\right)^{n-1} \right\rceil$. By the induction hypothesis, we can find a nonincreasing set in T of size $n - 1$, and together with u we have a nonincreasing set of size n. □

Hence, (9) implies the following.

Corollary 7.14. *We have* $r_3(P_4, n) \leq \left\lceil \left(\frac{2}{2-\sqrt{2}}\right)^n \right\rceil \approx (3.414)^n$.

For the lower bound, the probabilistic construction given in [74] (see Theorem 1.6) shows that $g_2(n) \geq r_3(P_4, n) > 2^{cn}$, where $c \approx 0.0691$.

The main problem here is to understand $g_3(n)$. While we know that $r_4(P_5, n) \leq g_3(n) \leq r_3(n, n)$, we do not know if these functions grow single or double exponentially in a power of n.

Problem 7.15. *Determine the tower growth rate for* $g_3(n)$.

7.3 Ordered ℓ-power paths in graphs

As mentioned above, The proof of Dilworth's theorem shows that $r_2(P_s, n) = r(P_s, P_n) = (s - 1)(n - 1) + 1$. On the other hand, we know that the classical Ramsey number $r_2(n, n)$ grows exponentially in $\Theta(n)$. One can consider the case of ordered graphs that are denser than paths but sparser than cliques.

Definition 7.16. Given $\ell \geq 1$, the ℓth power P_s^ℓ of a path P_s has ordered vertex set $v_1 < \cdots < v_s$ and edge set $\{v_i v_j : |i - j| \leq \ell\}$. In particular, $P_s^1 = P_s$. The Ramsey number $r(P_s^\ell, P_n^\ell)$ is the minimum N such that every red/blue coloring of $\binom{[N]}{2}$ results in a red copy of P_s^ℓ or a blue copy of P_n^ℓ.

In [69] it was shown that the problem of determining $r(P_n^\ell, P_n^\ell)$ is closely related to the hypergraph Ramsey function $r_3(s, P_n)$. Conlon-Fox-Lee-Sudakov [14] asked whether $r(P_n^\ell, P_n^\ell)$ is polynomial in n for every fixed $\ell \geq 1$. Actually, the problem in [14] is about the Ramsey number of ordered graphs with bandwidth at most ℓ but P_n^ℓ contains all such graphs so an upper bound for P_n^ℓ provides an upper bound for the bandwidth problem. This question was answered by Balko-Cibulka-Král-Kynčl [3]. Later a better bound was proved by Mubayi [69] for $\ell = 2$.

Theorem 7.17 (Balko-Cibulka-Král-Kynčl [3] ($\ell \geq 3$), Mubayi [69] ($\ell = 2$))**.** *There is an absolute constant $c > 0$ and for every $\ell > 0$ there exists $c = c_\ell$ such that*

$$r(P_n^\ell, P_n^\ell) < \begin{cases} c\, n^{19.487} & \text{for } \ell = 2 \\ c_\ell\, n^{128\ell} & \text{for } \ell \geq 3. \end{cases} \tag{10}$$

The main open problem here is to improve the exponents above. To our knowledge, there are no nontrivial lower bounds published for this problem.

Problem 7.18 (Balko-Cibulka-Král-Kynčl [3])**.** *Determine the growth rate of $r(P_n^\ell, P_n^\ell)$ for every fixed $\ell \geq 2$.*

8 The Erdős-Szekeres problem and an induced Ramsey question

Let $ES_d(n)$ denote the minimum N, such that any set of N points in $\mathbb{R}^d$ in general position (no $d+1$ members on a common hyperplane) contains n points in convex position. In the plane, classical results of Erdős and Szekeres [42, 43] imply that $2^{n-2} + 1 \leq ES_2(n) \leq 4^n$, and very recently, the second author improved this result substantially.

Theorem 8.1 (Suk [83])**.** *$ES_2(n) = 2^{n+o(n)}$.*

The Erdős-Szekeres convex polygon problem is not very well understood in higher dimensions. Projections into lower-dimensional spaces can be used to bound these functions, since most generic projections preserve general position, and the preimage of a set in convex position must itself be in convex position. Hence $ES_d(n) \leq ES_2(n) = 2^{n+o(n)}$. However, the best known lower bound, due to Károlyi and Valtr [53], is only on the order of $2^{cn^{1/(d-1)}}$. An old conjecture of Füredi (see Chapter 3 in [61]) says that this lower bound is essentially the truth.

Conjecture 8.2. *For $d \geq 3$, $ES_d(n) = 2^{\Theta(n^{1/(d-1)})}$.*

It was observed by Motzkin [65] that any set of $d+3$ points in $\mathbb{R}^d$ in general position contains either 0, 2, or 4 $(d+2)$-tuples not in convex position. In fact, one of the earliest applications of Ramsey numbers was discovered by Erdős and Szekeres [42], who showed that $ES_2(n) \leq r_4(5, n)$ by this observation. By defining a hypergraph H whose vertices are N points in $\mathbb{R}^d$ in general position, and edges are $(d+2)$-tuples not in convex position, then an independent set in H would correspond to a set of points in convex position. This leads us to the following combinatorial parameter.

Definition 8.3. Let $f_k(n)$ be the maximum N such that there is a k-graph on $[N]$ with independence number at most n and every $k+1$ points spanning $0, 2$ or 4 edges.

The geometric construction of Károlyi and Valtr [53] implies that $f_k(n) \geq ES_{k-2}(n) - 1 \geq 2^{cn^{1/(k-3)}}$, for fixed $k \geq 5$. One might be tempted to prove Conjecture 8.2 by establishing a similar upper bound for $f_k(n)$. However, our next result shows that this is not possible.

Theorem 8.4. *For each $n \geq k \geq 5$ there exists $c_k > 0$ such that $f_k(n) > 2^{c_k n^{k-4}}$.*

Proof. Let $k \geq 5$ and $N = 2^{cn^{k-4}}$ where $c = c_k > 0$ is sufficiently small and to be chosen later. We are to produce a k-graph H on N vertices with $\alpha(H) \leq n$ and every $k+1$ vertices of H span 0, 2, or 4 edges. Let $\phi : \binom{[N]}{k-3} \to \binom{[k-1]}{2}$ be a random $\binom{k-1}{2}$-coloring, where each color appears on each edge independently with probability $1/\binom{k-1}{2}$. For $e = (v_1, \ldots, v_{k-1})$, where $v_1 < v_2 < \cdots < v_{k-1}$, define the function $rk_e : \binom{e}{k-3} \to \binom{[k-1]}{2}$ as follows: for all $\{i,j\} \in \binom{[k-1]}{2}$, let

$$rk_e(e \setminus \{v_i, v_j\}) = \{i, j\}.$$

Define the $(k-1)$-graph

$$G = G_\phi := \left\{ e \in \binom{[N]}{k-1} : \phi(e \setminus \{u,v\}) = rk_e(e \setminus \{u,v\}) \text{ for all } \{u,v\} \in \binom{e}{2} \right\}.$$

For example, if $k = 4$ (which is excluded for the theorem but we allow it to illustrate this construction) then $\phi : [N] \to \{12, 13, 23\}$ and $e = v_1 < v_2 < v_3 \in G$ iff $\phi(v_1) = 23, \phi(v_2) = 13$, and $\phi(v_3) = 12$.

Finally, define the k-graph

$$H = H_\phi := \left\{ e \in \binom{[N]}{k} : |G[e]| \text{ is odd} \right\}.$$

Claim 8.5. *$|H[S]|$ is even for every $S \in \binom{[N]}{k+1}$.*

Proof. Let $S \in \binom{[N]}{k+1}$ and suppose for contradiction that $|H[S]|$ is odd. Then

$$2|G[S]| = \sum_{f \in G[S]} 2 = \sum_{f \in G[S]} \sum_{\substack{e \in \binom{S}{k} \\ e \supset f}} 1 = \sum_{e \in \binom{S}{k}} |G[e]| = \sum_{e \notin H[S]} |G[e]| + \sum_{e \in H[S]} |G[e]|.$$

The first sum on the RHS above is even by definition of H and the second sum is odd by definition of H and the assumption that $|H[S]|$ is odd. This contradiction completes the proof. $\square$

Claim 8.6. *$|G[e]| \leq 2$ for every $e \in \binom{[N]}{k}$.*

Proof. For sake of contradiction, suppose that for $e = (v_1, \ldots, v_k)$, where $v_1 < \cdots < v_k$, we have $|G[e]| \geq 3$. Let $e_p = e \setminus \{v_p\}$ for $p \in [k]$ and suppose that $e_i, e_j, e_l \in G$ with $i < j < l$. Let

$Y = e \setminus \{v_i, v_j, v_l\}$ and $Y' = Y \setminus \{\min Y\}$. Let us first assume that $i > 1$ so that $\min Y = v_1$. In this case,

$$rk_{e_i}(Y' \cup \{v_j\}) = \{1, l-1\}$$

since we obtain $Y' \cup \{v_j\}$ from e_i by removing $\min Y$ and v_l which are the first and $(l-1)$st elements of e_i. Similarly,

$$rk_{e_l}(Y' \cup \{v_j\}) = \{1, i\},$$

since we obtain $Y' \cup \{v_j\}$ from e_l by removing $\min Y$ and v_i which are the first and ith elements of e_i. Because $l > i+1$, we conclude that $rk_{e_i}(Y' \cup \{v_j\}) \neq rk_{e_l}(Y' \cup \{v_j\})$.

Next we assume that $i = 1$ and $\min Y = v_q$ where $q > 1$. In this case,

$$rk_{e_i}(Y' \cup \{v_j\}) = \{q-1, l-1\},$$

since we obtain $Y' \cup \{v_j\}$ from e_i by removing v_q and v_l which are the $(q-1)$st and $(l-1)$st elements of e_i. Similarly,

$$rk_{e_l}(Y' \cup \{v_j\}) = \{1, q'\} \qquad \text{where} \qquad q' = q \text{ if } q < l \text{ and } q' = q-1 \text{ if } q > l,$$

since we obtain $Y' \cup \{v_j\}$ from e_l by removing $v_i = v_1$ and v_q which are the first and q'th elements of e_i. If $q \neq 2$ then we immediately obtain $rk_{e_i}(Y' \cup \{v_j\}) \neq rk_{e_l}(Y' \cup \{v_j\})$ as desired. On the other hand, if $q = 2$, then $q' = q = 2$ as well and $l \geq 4$, so $l - 1 \neq q'$ and again

$$rk_{e_i}(Y' \cup \{v_j\}) = \{q-1, l-1\} \neq \{1, q'\} = rk_{e_l}(Y' \cup \{v_j\}).$$

In both cases, we obtain a contradiction since $e_i, e_l \in G$ implies that $rk_{e_i}(Y' \cup \{v_j\}) = \phi(Y' \cup \{v_j\}) = rk_{e_l}(Y' \cup \{v_j\})$. □

Let T_3 be the $(k-1)$-uniform hypergraph with vertex set S with $|S| = k+1$ and three edges e_1, e_2, e_3 such that there are three pairwise disjoint pairs $p_1, p_2, p_3 \in \binom{S}{2}$ with $p_i = \{v_i, v_i'\}$ and $e_i = S \setminus p_i$ for $i \in [3]$.

Claim 8.7. *$T_3 \not\subset G$.*

Proof. Suppose for a contradiction that $T_3 \subset G$ using the notation above. Assume without loss of generality that $v_1 = \min \cup_i p_i$ and $v_2 = \min(p_2 \cup p_3)$. Let $Y = S \setminus (p_1 \cup p_3)$ and note that $Y \in \binom{e_1 \cap e_3}{k-3}$. Let $Y_1 \subset Y$ be the set of elements in Y that are smaller than v_1, so we have the ordering

$$Y_1 < v_1 < v_2 < v_3, v_3'.$$

Now $rk_{e_1}(Y)$ is the pair of positions of v_3 and v_3' in e_1. Both of these positions are at least $|Y_1| + 2$ as $Y_1 \cup \{v_2\}$ lies before p_3. On the other hand the smallest element of $rk_{e_3}(Y)$ is $|Y_1| + 1$ which is the position of v_1 in e_3. This shows that $rk_{e_1}(Y) \neq rk_{e_3}(Y)$ which is a contradiction as both must be equal to $\phi(Y)$ as $e_1, e_3 \subset G$. □

We now show that every $(k+1)$-set S spans $0, 2$ or 4 edges of H. By Claim 8.5, $|H[S]|$ is even. Let G' be the graph with vertex set S and edge set $\{S \setminus f : f \in G[S]\}$. So there is a 1-1 correspondence between $G[S]$ and G' via the map $f \to S \setminus f$. If G' has a vertex x of degree at least three, then $|G[S \setminus \{x\}]| \geq 3$ which contradicts Claim 8.6. Therefore G' consists of disjoint paths and cycles.

Next, observe that Claim 8.7 implies that G' does not contain a matching of size three, for the complementary sets of this matching yield a copy of $T_3 \subset G$. This immediately implies that $k = 5$ for otherwise we obtain a 3-matching in G'. Moreover, the only way to avoid a 3-matching when $k = 5$ is for G' to consist of two components each of which contains a two edge path so we may assume that G' is of this form, with paths abc, uvw. If both uvw and abc are triangles, then $|H[S]| = 0$ as any 5-set A in S contains precisely two edges of G' from A to $S \setminus A$ which yields $|G[A]| = 2$ so $A \notin H$. If both abc and uvw are paths with $deg_{G'}(b) = deg_{G'}(v) = 2$ then a similar argument yields $|H[S]| = 4$ (the four edges are $abcuv, abcvw, uvwab, uvwbc$) and if one is a triangle and the other is a path then we have $|H[S]| = 2$. This concludes the proof that $|H[S]| \in \{0, 2, 4\}$ for all $S \in \binom{[N]}{k+1}$.

Let us now argue that $\alpha(H) \leq n$. Indeed, we will show that this happens with positive probability and conclude that an H with this property exists. For a given k-set, the probability that it is an edge of H is $p < 1$ where p depends only on k. Consequently, the probability that H has an independent set of size n is at most

$$\binom{N}{n}(1-p)^{c'n^{k-3}}$$

for some $c' > 0$. Note that the exponent $k - 3$ above is obtained by taking a partial Steiner $(n, k, k-3)$ system S within a potential independent set of size n and observing that we have independence within the edges of S. A short calculation shows that this probability is less than 1 as long as c is sufficiently small. □

In the other direction, we know that $f_k(n) < r_k(k+1, 5; n) \leq \mathrm{twr}_4(cn^{k-4})$ for $k \geq 6$ (see Section 3).

Problem 8.8. *Determine the tower growth rate for $f_k(n)$.*

9 More off-diagonal problems

In this section we consider k-graph Ramsey numbers of the form $r_k(H, n) := r_k(H, K_n^{(k)})$ where H is a (fixed) k-graph and n grows.

9.1 $K_4^{(3)}$ minus an edge and a generalization

Let $K_4^{(3)} \setminus e$ denote the 3-uniform hypergraph on four vertices, obtained by removing one edge from $K_4^{(3)}$. A simple argument of Erdős and Hajnal [38] implies $r(K_4^{(3)} \setminus e, K_n^{(3)}) < (n!)^2$. This was generalized in [73] as follows. A *k-half-graph*, denote by $B = B^{(k)}$, is a k-uniform hypergraph on $2k-2$ vertices, whose vertex set is of the form $S \cup T$, where $|S| = |T| = k-1$, and whose edges are all k-subsets that contain S, and one k-subset that contains T. So $B^{(3)} = K_4^{(3)} \setminus e$. Write $r_k(B, n) = r(B^{(k)}, K_n^{(k)})$.

Theorem 9.1 (Mubayi-Suk [73]). *For each $k \geq 4$ there exists $c = c_k$ such that*

$$2^{cn} < r_k(B, n) < (n!)^{k-1}.$$

A problem that goes back to the 1972 paper of Erdős and Hajnal (for $k = 3$) is to improve the lower bound above. Indeed, $r_3(B, n) = r_3(4, 3; n)$ and this is therefore a very special case of the Erdős-Hajnal problem discussed earlier.

Problem 9.2. *Show that for each $k \geq 3$ there exists $c = c_k$ such that $r_k(B, n) > 2^{cn \log n}$.*

9.2 Independent neighborhoods

Definition 9.3. A k-uniform triangle $T^{(k)}$ is a set of $k + 1$ edges $b_1, \ldots, b_k, a$ with $b_i \cap b_j = R$ for all $i < j$ where $|R| = k - 1$ and $a = \cup_i (b_i - R)$. In other words, k of the edges share a common $(k - 1)$-set of vertices, and the last edge contains the remaining point in all these previous edges.

When $k = 2$, then $T^{(2)} = K_3$, so in this sense $T^{(k)}$ is a generalization of a graph triangle. We may view a $T^{(k)}$-free k-graph as one in which all neighborhoods are independent sets, where the neighborhood of an $R \in \binom{V(H)}{k-1}$ is $\{x : R \cup \{x\} \in H\}$. As usual, write $r_k(T, n)$ for $r(T^{(k)}, K_n^{(k)})$.

Bohman, Frieze and Mubayi [6] proved that for fixed $k \geq 2$, there are positive constants c_1 and c_2 with

$$c_1 \frac{n^k}{(\log n)^{k/(k-1)}} < r_k(T, n) < c_2 n^k.$$

They conjectured that the upper bound could be improved to $o(n^k)$ and believed that the log factor in the lower bound could also be improved. Results of Kostochka-Mubayi-Verstraëte [56] proved this and then Bohman-Mubayi-Picollelli [9] achieved a matching lower bound by analyzing the hypergraph independent neighborhood process. This may be viewed as a hypergraph generalization of the results of Ajtai-Komlós-Szemerédi [2] for graphs.

Theorem 9.4 (Kostochka-Mubayi-Verstraëte [56], Bohman-Mubayi-Picollelli [9])**.** *For fixed $k \geq 3$ there are positive constants c_1 and c_2 with*

$$c_1 \frac{n^k}{\log n} < r_k(T, n) < c_2 \frac{n^k}{\log n}.$$

9.3 Cycles and paths

For fixed $s \geq 3$ the graph Ramsey number $r(C_s, n) = r(C_s, K_n)$ has been extensively studied. The case $s = 3$ is one of the oldest questions in Ramsey theory and it is known that $r(C_3, K_n) = \Theta(n^2/\log n)$ (see [2, 51] and [8, 44] for recent improvements). The next case $r(C_4, K_n)$ seems substantially more difficult. An old open problem of Erdős [36] asks whether there is a positive ϵ for which $r(C_4, K_n) = O(n^{2-\epsilon})$. The current best upper bound $r(C_4, K_n) = O(n^2/\log^2 n)$ is an unpublished result of Szemerédi which was reproved in [78] and the current best lower bound is $\Omega(n^{3/2}/\log n)$ from [7]. For longer cycles, the best known bounds can be found in [7, 82], and the order of magnitude of $r(C_s, K_n)$ is not known for any fixed $s \geq 4$.

There are several natural ways to define a cycle in hypergraphs. The two that have been investigated the most are tight cycles and loose cycles.

9.3.1 Tight cycles

For $k \geq 2$ and $s > 3$, the *tight cycle* $TC_s^{(k)}$ is the k-graph with vertex set $\mathbb{Z}_s$ (integers modulo s) and edge set

$$\{\{i, i+1, \ldots, i+k-1\} : i \in \mathbb{Z}_s\}.$$

We can view the vertex set of $TC_s^{(k)}$ as s points on a circle and the edge set as the s subintervals each containing k consecutive vertices.

When $s \equiv 0 \pmod 3$ the tight cycle $TC_s^{(3)}$ is 3-partite, and in this case it is trivial to observe that $r_3(TC_s, n) := r(TC_s^{(3)}, K_n^{(3)})$ grows polynomially in n. The growth rate of this polynomial is not known for any $s > 3$. When $s \not\equiv 0 \pmod 3$ the Ramsey number is exponential in n.

Theorem 9.5 (Mubayi-Rödl [71]). *Fix $s \geq 5$ and $s \not\equiv 0$ (mod 3). There are positive constants c_1 and c_2 such that*

$$2^{c_1 n} < r_3(TC_s, n) < 2^{c_2 n^2 \log n}.$$

Note that when $s = 4$, the cycle $TC_4^{(3)}$ is $K_4^{(3)}$ and in this case the lower bound was proved much earlier by Erdős and Hajnal [38], and in fact has been improved to $2^{c_1 n \log n}$ more recently by Conlon, Fox and Sudakov [19].

As s gets large, the tight cycle $TC_s^{(3)}$ becomes sparser, so one might expect that $r_3(TC_s, n)$ decreases as a function of n (for fixed s). Mubayi and Rödl [71] asked whether for each fixed $s \geq 4$ there exists ϵ_s such that $\epsilon_s \to 0$ as $s \to \infty$ and

$$r_3(TC_s, n) < 2^{n^{1+\epsilon_s}}$$

for all sufficiently large n. This was answered positively by the following recent result.

Theorem 9.6 (Mubayi [70]). *Fix a positive integer $s \not\equiv 0$ (mod 3) such that $s \geq 16$ or $s \in \{8, 11, 14\}$. There is a positive constant c_s such that*

$$r_3(TC_s, n) < 2^{c_s n \log n}.$$

Problem 9.7. *Prove similar bounds for $s \in \{4, 5, 7, 10, 13\}$ and determine whether the* log *factor in the exponent is necessary.*

The problem of determining $r_k(TC_s, n) := r(TC_s^{(k)}, K_t^{(k)})$ for fixed $s > k > 3$ seems harder as k grows. It was shown in [71] that we have a lower bound

$$r_k(TC_s, n) > 2^{c_{k,s} n^{k-2}}$$

The best upper bound that is known (for fixed $s > k$ and all n) is the trivial one $r_k(s, n)$. Consequently, we have

$$2^{c_{k,s} n^{k-2}} < r_k(TC_s, n) < r_k(s, n) < \mathrm{twr}_{k-1}(n^{d_{s,k}}).$$

Closing the gap above seems to be a very interesting open problem. For the case $s = k + 1$, one has a substantially better lower bound as

$$r_k(TC_{k+1}, n) = r_k(k+1, n) > \mathrm{twr}_{k-2}(bn^{\log n})$$

where $b = b_k$.

Problem 9.8. *For fixed $s > k+1 > 4$ (in particular for $s = k+2$), determine whether $r_k(TC_s, n)$ is at least a tower function in a power of n where the tower height grows with k.*

9.3.2 Loose cycles

For $s \geq 3$, the loose cycle $LC_s^{(k)}$ is the k-graph with vertex set $\mathbb{Z}_{(k-1)s}$ and edge set $\{e_1, e_2, \ldots, e_s\}$ where $e_i = \{i(k-1) - k + 2, \ldots, i(k-1) + 1\}$. In other words, consecutive edges intersect in exactly one vertex and nonconsecutive edges are pairwise disjoint. As usual, write $r_k(LC_s, n)$ for $r(LC_s^{(k)}, K_n^{(k)})$. Since loose cycles are k-partite, it is easy to see that $r_k(LC_s, n)$ has polynomial growth rate for fixed k, s so the question here is to determine the correct power of n.

Theorem 9.9 (Kostochka-Mubayi-Verstraëte [55])**.** *There exists $c_1, c_2 > 0$ such that for all $n \geq 1$,*

$$\frac{c_1 n^{3/2}}{(\log n)^{3/4}} \leq r_3(LC_3, n) \leq c_2 n^{3/2}.$$

For $k \geq 3$, we also have $r_k(LC_3, n) = n^{3/2+o(1)}$.

Analogous to the basic result $r(3, n) = O(n^2/\log n)$ due to Ajtai, Komlós and Szemerédi [2], the authors conjectured something similar for hypergraphs.

Conjecture 9.10 ([55])**.** *For all fixed $k \geq 3$, we have $r_k(LC_3, n) = o(n^{3/2})$.*

Define the 3-graph $F = \{abc, abd, cde\}$. Cooper and Mubayi [23] proved the following weaker version of Conjecture 9.10 in the case $k = 3$:

$$r_3(\{LC_3, F, K_4^{(3)} - e\}, n) = O\left(\frac{n^{3/2}}{(\log n)^{1/2}}\right). \tag{11}$$

Notice that the three forbidden 3-graphs in (11) are all types of triangles, comprising three edges that cyclically share a vertex. Conjecture 9.10 asks that we forbid only one of these three triangles, the loose triangle.

For longer cycles, the following general lower bounds were proved in [55] which improve the bounds given by the standard probabilistic deletion method.

Theorem 9.11 ([55])**.** *For fixed $s, k \geq 3$,*

$$r_k(LC_s, n) > n^{1+1/(3s-1)+o(1)}$$

Further, there exists $c = c_k$ such that

$$r_k(LC_5, n) > c\left(\frac{n}{\log n}\right)^{5/4}.$$

Conjecture 9.12 ([55])**.** *For each $k \geq 3$, there exists $c = c_k$ such that $r_k(LC_5, n) < c\, n^{5/4}$.*

Méroueh [62] recently proved that $r_3(LC_5, n) < cn^{4/3}$ and more generally that

$$r_3(LC_s, n) < c_s n^{1+\frac{1}{\lfloor (s+1)/2 \rfloor}}.$$

He also proved that for odd $s \geq 5$ and $k \geq 4$, $r_k(LC_s, n) < c_{k,s} n^{1+1/\lfloor s/2 \rfloor}$ which slightly improved the exponent $1 + 1/(\lfloor s/2 \rfloor - 1)$ proved by Collier-Cartaino, Graber and Jiang [12] for all $k \geq 3$ and $s \geq 4$.

We refer the reader to [49, 50] for related results that determine the asymptotics of $r_3(TC_n, TC_n)$ and $r_3(LC_n, LC_n)$.

9.3.3 Unordered tight-paths

An *unordered* 3-uniform tight-path $\widetilde{P}_s^{(3)} = \widetilde{P}_s$ is the 3-graph with vertex set $\{v_1, \ldots, v_s\}$ and edge set $\{\{v_i, v_{i+1}, v_{i+2}\} : i \in \{1, \ldots, s-2\}\}$. Note that the vertex set $\{v_1, \ldots, v_s\}$ is not ordered. Results of Phelps and Rödl [76] imply that there are c_1 and c_2 such that

$$c_1 n^2 / \log n < r_3(\widetilde{P}_4, n) < c_2 n^2 / \log n.$$

It is easy to prove that for all $s \geq 5$, there is $c = c_s$ such that $r_3(\widetilde{P}_s, n) < c\,n^2$. A matching lower bound for $s \geq 6$ was provided by Cooper and Mubayi with the following construction. Let H be a 3-uniform hypergraph where $V(H) = [n] \times [n]$, and $E(H) = \{\{ab, ac, db\} \in [n] \times [n] : c > b, d > a\}$. It is easy to see that H is $\widetilde{P}_6$-free and $\alpha(H) < 2n$. Thus we have the following.

Theorem 9.13 (Cooper-Mubayi [24])**.** *For $s \geq 6$ there exists $c = c_s$ such that $r_3(\widetilde{P}_s, n) > cn^2$.*

The construction above has many copies of $\widetilde{P}_5$ so this leaves open the case $s = 5$. Using the trivial lower bound $r_3(\widetilde{P}_4, n)$ we thus have

$$c_1 n^2 / \log n < r_3(\widetilde{P}_5, n) < c_2 n^2.$$

Problem 9.14 ([24])**.** *Determine the order of magnitude of $r_3(\widetilde{P}_5, n)$.*

The corresponding problems for k-uniform hypergraphs when $k > 3$ are wide open.

10 Multicolor Ramsey numbers

In this section we briefly survey problems of the form $r_k(F; q)$ which is the minimum N such that every q-coloring of $\binom{[N]}{k}$ yields a monochromatic copy of F. When $F = K_s^{(k)}$ we use the notation $r_k(s; q)$. In this section we typically think of q going to infinity and all other parameters fixed. For graphs, the main open problem in this regime is to improve the classical bounds

$$2^{cq} < r_2(3; q) < 2^{c' q \log q}.$$

For the hypergraph case, there is a similar gap between the upper and lower bounds.

Theorem 10.1 (Erdős-Rado [40], Erdős-Hajnal-Rado [39]). *Let $s > k \geq 2$. There are positive integers $c = c(s,k) \leq 3(s-k)$, $s_0(k)$, and $c' = c'(s,k)$ such that*

$$\text{twr}_k(c'q) < r_k(s;q) < \text{twr}_k(cq \log q)$$

where the lower bound holds for $s \geq s_0(k)$.

It is worth noting that the lower bound in [39] was stated for the case when the number of colors, q, is fixed while k grows and the bound was only for large cliques. But the proof in [39] applies naturally to our case as well, when q grows and the other parameters are fixed. Conlon, Fox and Sudakov [18] proved a lower bound for cliques of smaller sizes, but still only for $s \geq 2k-1$. Duffus, Lefmann and Rödl [29] took another approach, using shift graphs, and proved a lower bound for cliques of all sizes $s > k$, but require q being fixed and $k \gg q$. Using a slight modification of the stepping-up lemma, due to Erdős and Hajnal (see Chapter 4.7 in [47]) the following result was obtained which applies for all $s \geq k$ and large q.

Theorem 10.2 (Axenovich-Gyárfás-Liu-Mubayi [1]). *For any $s > k \geq 2$ and $q > k2^k$ we have*

$$r_k(s;q) > \text{twr}_k\left(\frac{q}{2^k}\right).$$

Another result from [1] is for the case when $F = K_4^{(3)}$ minus an edge (henceforth $K_4^{(3)} - e$) which relates this multicolor hypergraph Ramsey number to $r_2(3;q)$.

Theorem 10.3 ([1]). *For any $q \geq 2$,*

$$r_2(3;q) \leq r_3(K_4^{(3)} - e; 4q) \quad \textit{and} \quad r_3(K_4^{(3)} - e; q) \leq r_2(3;q) + 1.$$

In the case when F is an unordered tight path with three edges we have the following result.

Theorem 10.4 ([1]). *Let $\widetilde{P}_5 = \widetilde{P}_5^{(3)}$ be the 3-uniform unordered tight path on 5 vertices and 3 edges. Then*

$$2q(1-o(1)) \leq r_3(\widetilde{P}_3; q) \leq 2q+3,$$

and the upper bound is sharp when $q = 2^{2m-1} - 1$.

For the 3-uniform loose triangle, LC_3, Gyárfás and Raeisi [48] proved that

$$q + 5 \leq r_3(LC_3; q) \leq 3q + 1.$$

It seems to be an interesting open problem to determine which bound above is closer to the truth. The lower bound appears more likely to be the answer. There has been some work on determining $r_3(P;q)$ where $P = \{123, 345, 567\}$ is the 3-uniform loose path of length three. In particular, the lower bound $r_3(P;q) \geq q+6$ for all $q \geq 3$ is sharp for all $q \leq 9$ (see [48, 77]). The general upper bound which comes from the Turán number of P is again $3q+1$. This upper bound was improved by Luczak and Polcyn [60] first to $(2+o(1))q$ and more recently to $\lambda q + O(\sqrt{q})$ where the constant $\lambda = 1.97466..$ is the solution to a particular cubic equation.

Further interesting hypergraph Ramsey problems involving more than two colors can be found in the recent paper of Conlon-Fox-Rödl [17].

11 A bipartite hypergraph Ramsey problem of Erdős

We end with an old problem of Erdős that was perhaps posed to gain a better understanding of the growth rate of the diagonal Ramsey numbers.

Definition 11.1. Let $S_{a,b} = (U, V, E)$ be the 3-uniform hypergraph with vertex set $U \cup V$, where $|U| = a$ and $|V| = b$, such that $E(S_{a,b}) = \{(x, y, z) : x \in U \text{ and } y, z \in V\}$. Write $S_n := S_{n,n}$.

An old result due to Erdős (see [34]) says that $r_3(S_n, S_n) = 2^{O(n^2)}$, which is tight up to a constant factor in the exponent by the standard probabilistic method. We were not able to find a published proof of this result and we therefore present a proof below (of a stronger result).

Theorem 11.2. *For every $c > 0$ and sufficiently large n,*

$$r_3(S_n, S_n) < r_3(S_{2^{cn},n}, S_{2^{cn},n}) < 2^{3n^2}.$$

Proof. We begin with the simple observation that $r_3(S_{1,n}, S_{1,n}) < 1 + r_2(n, n) < 4^n$. Indeed, if $r = 1 + r_2(n, n)$ and $\binom{[r]}{3}$ is 2-colored by χ then we have an induced 2-coloring χ' of $\binom{[r-1]}{2}$ where $\chi'(ij) = \chi(ijr)$. Because $r - 1 = r_2(n, n)$ we have a monochromatic n-set under χ' and this yields a monochromatic $S_{1,n}$ under χ with $U = \{r\}$.

Now we use a simple supersaturation trick to prove the result. Suppose that $N = 2^{c'n^2}$ and χ is a 2-coloring of $\binom{[N]}{3}$. For every r-set of $[N]$, where $r = 4^n$, there is a monochromatic copy of $S_{1,n}$ in χ. Hence the number of monochromatic copies of $S_{1,n}$ in χ is at least

$$\frac{\binom{N}{r}}{\binom{N-n-1}{r-n-1}} = \frac{(N)_{n+1}}{(r)_{n+1}}.$$

At least half of these monochromatic copies of $S_{1,n}$ have the same color, say blue. Now, to each of these blue copies of $S_{1,n}$ with parts $|U| = 1$ and $|V| = n$, we associate the n-set V. A short calculation and the fact that n is large shows that

$$\frac{(N)_{n+1}}{(r)_{n+1}} > \left(\frac{2^c N e}{n}\right)^n > 2^{cn}\binom{N}{n}.$$

Consequently, by the pigeonhole principle, there are at least 2^{cn} blue copies of $S_{1,n}$ associated to the same n-set V. These blue copies together form a blue copy of S_n as desired. □

Erdős stated that an important and difficult problem is to decide if his result can be strengthened to imply all triples that meet both U and V.

Definition 11.3. Let $B_n = (U, V, E)$ be the 3-uniform hypergraph with vertex set $U \cup V$, where $|U| = |V| = n$, such that $E(B_n) = \{(x, y, z) : x, y \in U, z \in V \text{ or } x, y \in V, z \in U\}$.

Clearly we have

$$2^{cn^2} < r_3(B_n, B_n) \le r_3(n, n) \le 2^{2^{c'n}}$$

where the lower bound follows from the probabilistic method.

Problem 11.4 (Erdős [34])**.** *Improve the upper or lower bounds for $r_3(B_n, B_n)$.*

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