When are off-diagonal hypergraph Ramsey numbers polynomial?

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Abstract

A natural open problem in Ramsey theory is to determine those 3-graphs H for which the off-diagonal Ramsey number $r(H, K_n^{(3)})$ grows polynomially with n. We make substantial progress on this question by showing that if H is tightly connected or has at most two tight components, then $r(H, K_n^{(3)})$ grows polynomially if and only if H is not contained in an iterated blowup of an edge.

1 Introduction

Given a k-uniform hypergraph H (henceforth, *k-graph*), the *off-diagonal Ramsey number* $r(H, K_n^{(k)})$ is the smallest natural number N such that every red/blue-coloring of the edges of $K_N^{(k)}$, the complete k-graph with N vertices, contains either a red copy of H or a blue copy of $K_n^{(k)}$. For graphs, the $k = 2$ case, we know that $r(H, K_n)$ always grows polynomially with n and the main problem is to determine the growth rate more precisely. This problem remains open even when H is a clique, where the correct polynomial dependency is only understood for K_3 and K_4 — for K_3 the off-diagonal Ramsey number was famously determined up to a constant factor by Ajtai, Komlós and Szemerédi [1] and Kim [12], while for K_4 a recent result of Mattheus and Verstraëte [13] shows that $r(K_4, K_n) = n^{3+o(1)}$. When H is a cycle, even less is known and determining the polynomial order of $r(C_4, K_n)$ in n is a major Erdős problem (see [3, 4] for the best bounds on this problem and [9, 16] for recent progress on other cycle-complete Ramsey numbers).

For 3-graphs H, $r(H, K_n^{(3)})$ does not always grow polynomially. Indeed, it is already the case [8] that $r(K_4^{(3)}, K_n^{(3)}) \geq 2^{\Omega(n \log n)}$. Much of the recent work on off-diagonal hypergraph Ramsey numbers has focused on extending this lower bound to other hypergraphs, so that it is now known that $r(H, K_n^{(3)}) \geq 2^{\Omega(n \log n)}$ for $K_4^{(3)} \setminus e$ and, more generally, all links of odd cycles [11] and for all tight cycles of length not divisible by three [6].

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Here we look in a different direction, our concern being with the problem of classifying those H for which $r(H, K_n^{(3)})$ grows polynomially in n. Although this problem has been studied since at least work of Erdős and Hajnal [10] in the early 1970s, it seems to have been first raised explicitly by the first author [5] at an AIM workshop in 2015. We propose a full classification, as follows.

Conjecture 1.1. *For a 3-graph H, there exists a constant c depending only on H such that $r(H, K_n^{(3)}) \leq n^c$ for all n if and only if H is a subgraph of an iterated blowup of an edge.*

To clarify what we mean by an iterated blowup, we first note that a *blowup of an edge* is simply a complete tripartite 3-graph. An *iterated blowup of an edge* is then any graph which is either a blowup of an edge or formed iteratively by placing another iterated blowup in one or more of the parts in a blowup of an edge. In what follows, we say that a 3-graph is *iterated tripartite* if it is contained in an iterated blowup of an edge.

The direction of Conjecture 1.1 saying that there exists a constant c such that $r(H, K_n^{(3)}) \leq n^c$ if H is iterated tripartite was already known to Erdős and Hajnal [10] (see also [8, 11]). Their concern was with a slightly different Ramsey-type question. For natural numbers $4 \leq s < n$ and $2 \leq t \leq \binom{s}{3}$, they were interested in the Ramsey function $r_3(s, t; n)$, the smallest natural number N such that every red/blue-coloring of the edges of $K_N^{(3)}$ contains either a blue copy of $K_n^{(3)}$ or a set of s vertices with at least t red edges. Regarding the behavior of this function, they conjectured that there should be a polynomial-to-exponential transition for the growth rate of $r_3(s, t; n)$ at $t = t(s)$, the maximum number of edges in an iterated tripartite 3-graph with s vertices. Their conjecture may be seen as a toy model for our Conjecture 1.1.

In their paper, Erdős and Hajnal [10] proved one direction of their conjecture, showing that $r_3(s, t; n)$ grows polynomially in n for $t \leq t(s)$. Their proof, as indicated above, further shows one direction of our Conjecture 1.1. However, it remains open to show that $r_3(s, t; n)$ grows exponentially in a power of n for $t > t(s)$. Some partial results, saying, for instance, that the conjecture holds when s is a power of 3, were proven by Conlon, Fox and Sudakov [8], while the analogous problem in higher uniformities was solved completely by Mubayi and Razborov [15]. In particular, we see that Conjecture 1.1 holds if $|H|$ is a power of 3 and the number of edges in H is larger than that in any iterated blowup with $|H|$ vertices.

In light of these results, it is perhaps surprising that Conjecture 1.1 has not been stated before. One reason for this was the common belief in the community that $r(H, K_n^{(3)})$ should also be polynomial for linear H, where a hypergraph is said to be *linear* if any two edges in the hypergraph share at most one vertex. However, it was recently shown [6] that this belief is mistaken and that there are linear hypergraphs for which $r(H, K_n^{(3)})$ grows superpolynomially. Conjecture 1.1 is stronger again, suggesting that linearity is a red herring in this context.

Towards Conjecture 1.1, we prove two results. We say that a 3-graph H is *tightly connected* if, for any two edges e and f of H there exists a sequence of edges $e = e_0, e_1, \ldots, e_t = f$ such that e_{i-1} and e_i share two vertices for all $i = 1, \ldots, t$. Our first result says that Conjecture 1.1 holds for tightly connected hypergraphs, even giving a lower bound in this case which is exponential in a power of n.

Theorem 1.2. *If H is a 3-graph which is tightly connected and not tripartite, then $r(H, K_n^{(3)}) \geq 2^{\Omega(n^{2/3})}$.*

One objection towards this being strong evidence for Conjecture 1.1 is that a tightly connected hypergraph is almost exactly the opposite of a linear hypergraph. Our second result goes some small way towards overruling this objection. We define a *tight component* of H to be a maximal tightly connected subgraph and observe that the edge set of every 3-graph decomposes into disjoint tight components. We prove that Conjecture 1.1 holds for hypergraphs which have at most two tight components, though our lower bound is considerably weaker in this case.

Theorem 1.3. *If H is a 3-graph with at most two tight components and not iterated tripartite, then $r(H, K_n^{(3)}) \geq 2^{\Omega(\log^2 n)}$.*

The proof of Theorem 1.3 is significantly more technical than that of Theorem 1.2 and may be considered our main result. With that in mind, we will warm up by proving Theorem 1.2 in the next section, before returning to Theorem 1.3 in Section 3. We then conclude with some further remarks and open problems.

2 Tightly connected hypergraphs

In this short section, we prove Theorem 1.2, the statement that if H is tightly connected and not tripartite, then $r(H, K_n^{(3)}) \geq 2^{\Omega(n^{2/3})}$. In fact, we will prove a stronger statement, from which Theorem 1.2 clearly follows.

Theorem 2.1. *For every positive integer N, there is a red/blue edge coloring of $K_N^{(3)}$ vertices such that any red tightly connected subgraph is tripartite and the largest blue clique has order $O((\log N)^{3/2})$.*

Proof of Theorem 2.1. Let $\ell = C \log N$, where $C > 0$ is a sufficiently large absolute constant, and $n = \ell^{3/2}$. Let $r = 0.01\ell$ and let V be an r-trifference code in $\{1, 2, 3\}^\ell$ of size N. What this means is that V is a subset of $\{1, 2, 3\}^\ell$ of size N with the property that, for any triple uvw of elements of V, there are at least r coordinates where $\{u_i, v_i, w_i\} = \{1, 2, 3\}$. Such a set of size N can be found by the first moment method, as it is exponentially unlikely that a random triple uvw is not r-trifferent.

We will define our Ramsey coloring on the complete 3-graph with vertex set V. Let $c(uv)$ denote the set of coordinates in which u, v differ. By the definition of V, $r \leq |c(uv)| \leq \ell$ for all u, v. Define $\phi(uv)$ to be an element of $c(uv)$ picked uniformly at random. The Ramsey coloring of $K_N^{(3)}$ will be χ where $\chi(uvw)$ is red if and only if $\phi(uv) = \phi(vw) = \phi(uw)$. Observe that, by definition, any red triple uvw automatically satisfies $\{u_i, v_i, w_i\} = \{1, 2, 3\}$ for $i = \phi(uv)$.

Red tight components are tripartite. For any red tight component, all pairs in its 2-shadow must share the same ϕ-value i. Then the i-th coordinates of the vertices give a tripartition of the vertices, showing that the component is tripartite.

No blue $K_n^{(3)}$ with positive probability. Fix a set U of n vertices in V; we would like to bound the probability that this n-set forms a blue clique. Let T_{uvw} be the event that the triple uvw is red, that is, that the triangle uvw is monochromatic under ϕ. Each event has probability

$$\Pr[T_{uvw}] = \frac{|c(uv) \cap c(vw) \cap c(uw)|}{|c(uv)| \cdot |c(vw)| \cdot |c(uw)|} = \Theta(\ell^{-2})$$

by our choice of V and $c(\cdot)$, so, since there are $\Theta(n^3)$ total events, the expected number of monochromatic triangles in U is $\mu = \Theta(n^3/\ell^2) = \Theta(\ell^{5/2})$. The Poisson paradigm predicts that

$$\Pr[\bigwedge \overline{T}_{uvw}] \leq e^{-\Theta(\ell^{5/2})}. \tag{1}$$

We first show that this would suffice to prove the theorem. Indeed, since there are at most N^n n-sets and

$$N^n e^{-\Theta(\ell^{5/2})} < 1$$

by our choices of n, ℓ and N, the union bound gives that there is a positive probability no blue n-cliques appear.

Now we prove (1) via Suen's inequality in the form given in Alon–Spencer [2, Theorem 8.7.1], which implies that

$$\Pr[\bigwedge \overline{T_\tau}] \le e^{-\mu+\sum_{\tau\sim\tau'} y(\tau,\tau')}, \tag{2}$$

where τ ranges over all triangles uvw in U and $\tau \sim \tau'$ if these two triangles share at least one edge, with

$$y(\tau,\tau') = (\Pr[T_\tau \wedge T_{\tau'}] + \Pr[T_\tau]\cdot\Pr[T_{\tau'}]) \prod_{\tau''\sim\tau \text{ or } \tau''\sim\tau'} (1-\Pr[T_{\tau''}])^{-1}.$$

Observe that if τ and τ' share an edge, then $\Pr[T_\tau \wedge T_{\tau'}] = O(\ell^{-4})$ and $\Pr[T_\tau]\cdot\Pr[T_{\tau'}] = O(\ell^{-4})$. Moreover, there are $\Theta(n)$ triangles τ'' that share an edge with either τ or τ', so we obtain

$$y(\tau,\tau') = O(\ell^{-4})(1-\Theta(\ell^{-2}))^{-\Theta(n)} = O(\ell^{-4})e^{\Theta(n/\ell^2)} = O(\ell^{-4}),$$

where we used that $n = o(\ell^2)$. Since there are $O(n^4)$ pairs of triangles τ, τ' that share an edge, (2) reduces to

$$\Pr[\bigwedge \overline{T_\tau}] \le e^{-\mu+O(n^4/\ell^4)} = e^{-\Theta(\ell^{5/2})+O(\ell^2)} = e^{-\Theta(\ell^{5/2})},$$

as desired. □

Before moving on to the case with two tight components, we sketch an alternative, but quantitatively weaker, construction for the single component case. Let $\ell = C\log N$ and $n = C^{7/4}\log^2 N$ for C a sufficiently large constant. For each edge uv of the complete graph on $V = [N]$, label it with some $\phi(uv) \in \{1, 2, ..., \ell\}$ chosen independently and uniformly at random. Moreover, assign a string $f(v) \in \{1, 2, 3\}^\ell$ to each vertex v, again chosen independently and uniformly at random. The Ramsey coloring of $K_N^{(3)}$ will be χ where $\chi(uvw)$ is red if and only if

1. $\phi(uv) = \phi(vw) = \phi(uw) = i$ for some i,
2. $f_i(u), f_i(v)$ and $f_i(w)$ are all distinct for this i

and blue otherwise. If there is any red tightly connected 3-graph in χ, then all edges in its 2-shadow have the same ϕ-value i, so the $f_i(v)$ yield a tripartition. To show that there is no blue K_n with positive probability, we use an application of Janson's inequality to show that, with positive probability, for every subset U of V with n vertices and every $i \in \{1, 2, ..., \ell\}$ there are at least $n/6$ vertex-disjoint triangles in U which are monochromatic of color i under ϕ. If we now bring the randomness of f into play, then each such monochromatic triangle forms a blue edge with probability $7/9$. As the randomness for each coordinate f_i of f is independent, the induced hypergraph on any given n-vertex set U is completely blue with probability at most $(7/9)^{\frac{n}{6}\cdot\ell} = \exp(-\Theta(n\ell))$. Since there are $\binom{N}{n} = \exp(n\log N)$ subsets to consider, the union bound and the fact that ℓ dominates $\log N$ completes the analysis, yielding the bound $r(H, K_n^{(3)}) \ge 2^{\Omega(n^{1/2})}$ for every non-tripartite tightly connected 3-graph H.

3 Two tight components

In this section, we prove Theorem 1.3, which says that if H has at most two tight components and is not iterated tripartite, then $r(H, K_n^{(3)}) \ge 2^{\Omega(\log^2 n)}$. We will once again prove Theorem 1.3 by providing a single construction avoiding all such red H simultaenously.

Theorem 3.1. *For every positive integer N, there is a red/blue edge coloring of $K_N^{(3)}$ such that any red subgraph with at most two tight components is iterated tripartite and the largest blue clique has order $\exp(O(\sqrt{\log N}))$.*

For the proof, we will need to show the existence of an edge coloring of the complete graph with certain powerful properties, encapsulated in the following lemma.

Lemma 3.2. *For positive integers $A \geq 20$, $\ell \geq 1$ and $N = 2^\ell$, there exists an edge coloring $\varphi : E(K_N) \to [\ell A]$ such that:*

1. *For each color, the edges with that color form a vertex-disjoint union of bicliques.*
2. *In every k-vertex subset with $k = \exp(\Omega(\log N / \log A))$, there are $\Omega(k^{2.5})$ rainbow triangles.*

We will postpone the proof of this lemma for now, first showing how to use it to prove Theorem 1.3.

Proof of Theorem 3.1. Replace N by the smallest power of 2 exceeding N and let $\ell = \log_2 N$. Let φ be as in Lemma 3.2 with $A, k = \exp(\Theta(\sqrt{\ell}))$. Let g, f_1, f_2, f_3 be four independent random functions on $\binom{[N]}{2}$, where $g : \binom{[N]}{2} \to \{1, 2, 3\}$ uniformly at random and $f_1, f_2, f_3 : \binom{[N]}{2} \to [\ell A]$ uniformly at random. The Ramsey coloring of the complete 3-graph on $[N]$ will be χ where $\chi(uvw)$ for a triple uvw with $u < v < w$ is red if and only if

1. uvw is a rainbow triangle with respect to φ,
2. $g(uv) = 1$, $g(vw) = 2$ and $g(uw) = 3$,
3. $f_1(uw) = f_1(vw) = \varphi(uv)$,
4. $f_2(uv) = f_2(uw) = \varphi(vw)$,
5. $f_3(uv) = f_3(vw) = \varphi(uw)$.

Otherwise, the triple uvw is colored blue.

No large blue cliques. We upper bound the probability that some subset S of order k forms a blue clique. Provided $k = \exp(\Omega(\ell / \log A))$, by the properties of φ, there are $\Omega(k^{2.5})$ rainbow triangles in S. By greedily picking out triangles, we may find $\Theta(k^{1.5})$ of them that are edge-disjoint. Each such triangle is colored red independently with probability $\Theta((\ell A)^{-6})$. Therefore, the probability that none of them is colored red is $e^{-\Theta(k^{1.5}(\ell A)^{-6})}$. By the union bound, the probability that there is a blue clique can then be bounded by

$$\binom{N}{k} e^{-\Theta(k^{1.5}(\ell A)^{-6})} = e^{\Theta(k \log N) - \Theta(k^{1.5}(\ell A)^{-6})},$$

which is less than 1 if $k = \Omega(\ell^{14} A^{12})$. To balance this with the condition that $k = \exp(\Omega(\ell / \log A))$, it suffices to pick $A, k = \exp(\Theta(\sqrt{\ell}))$, as indicated at the outset.

Union of two red tight components is iterated tripartite. We now show that the union of any two red tight components is iterated tripartite. We start by proving a structural result for each red tight component.

Lemma 3.3. *Every red tight component $\mathcal{C}(V_\mathcal{C}, E_\mathcal{C})$ is tripartite and there is a unique tripartition $V_\mathcal{C}^{(1)} \cup V_\mathcal{C}^{(2)} \cup V_\mathcal{C}^{(3)}$ of the vertices $V_\mathcal{C}$. Moreover, there exist three colors $c_\mathcal{C}^{(1)}$, $c_\mathcal{C}^{(2)}$ and $c_\mathcal{C}^{(3)}$ such that, for any ordering i, j, k of $1, 2, 3$, $\varphi(uv) = c_\mathcal{C}^{(k)}$ for any $u \in V_\mathcal{C}^{(i)}$ and $v \in V_\mathcal{C}^{(j)}$.*

Proof. For any hyperedge $\{x,y,z\}$, define $\Phi(\{x,y,z\})$ to be the set $\{\varphi(xy),\varphi(yz),\varphi(xz)\}$. For any two red hyperedges e,e' sharing two vertices x,y, if $e=\{x,y,z\}$ and $e'=\{x,y,z'\}$, then the relative order of x,y,z and the relative order of x,y,z' have to be the same, showing that $g(xz')=g(xz)$ and $g(yz')=g(yz)$. Therefore,

$$\varphi(xz)=f_{g(xz)}(xy)=f_{g(xz')}(xy)=\varphi(xz')$$

and similarly $\varphi(yz)=\varphi(yz')$. Hence, as $\mathcal{C}$ is tightly connected in red, $\Phi(e)$ is the same for any $e\in E_{\mathcal{C}}$.

Now pick an arbitrary $e\in E_{\mathcal{C}}$ and let $c_{\mathcal{C}}^{(1)},c_{\mathcal{C}}^{(2)},c_{\mathcal{C}}^{(3)}$ be an arbitrary enumeration of the elements in $\Phi(e)$. Moreover, let G be the subgraph of K_N obtained by taking the 1-skeleton of $\mathcal{C}$. By induction, it is easy to see that the edges of color $c_{\mathcal{C}}^{(i)}$ in G are connected for each $i\in[3]$.

Now, for any $v\in V_{\mathcal{C}}$, choose arbitrary vertices u,w such that $\{u,v,w\}\in E_{\mathcal{C}}$ and define $i(v)$ to be such that $\varphi(uw)=c_{\mathcal{C}}^{i(v)}$. We show that $i(v)$ does not depend on the choice of u and w. Suppose instead that there are some other u',w' with $\{u',v,w'\}\in E_{\mathcal{C}}$ such that $\varphi(uw)\neq\varphi(u'w')$. Then we know that one of $\varphi(u'v)$ and $\varphi(vw')$ is equal to $\varphi(uw)$. Thus, in G, u,v,w all belong to the connected subgraph of color $\varphi(uw)$ with u and w neighbors. Therefore, without loss of generality, we can assume that there is a monochromatic odd walk from v to u of color $\varphi(uw)$. Since the edges with color $\varphi(uw)$ form a vertex-disjoint union of bicliques, it must then be the case that $\varphi(uv)=\varphi(uw)$. But this is a contradiction, so $i(v)$ must be independent of the choice of u and w.

We partition $V_{\mathcal{C}}$ into $V_{\mathcal{C}}^{(1)},V_{\mathcal{C}}^{(2)}$ and $V_{\mathcal{C}}^{(3)}$ based on the labels $i(v)$. It is clear that $\mathcal{C}$ is tripartite with respect to this tripartition and, as $\mathcal{C}$ is tightly connected, that this is the only possible tripartition. Finally, if i,j,k is an ordering of $1,2,3$, then it is also clear that the connected subgraph of G of color $c_{\mathcal{C}}^{(k)}$ is a bipartite graph on $V_{\mathcal{C}}^{(i)}\cup V_{\mathcal{C}}^{(j)}$. If $u\in V_{\mathcal{C}}^{(i)}$ and $v\in V_{\mathcal{C}}^{(j)}$, then, by the fact that the edges of color $c_{\mathcal{C}}^{(k)}$ form a vertex-disjoint union of bicliques in K_N, we know that $\varphi(uv)=c_{\mathcal{C}}^{(k)}$, as desired. □

Suppose now that $\mathcal{C}$ and $\mathcal{C}'$ are two red tight components whose union is not tripartite. Without loss of generality, this means that there is some $u\in V_{\mathcal{C}}^{(1)}\cap V_{\mathcal{C}'}^{(1)}$ and $v\in V_{\mathcal{C}}^{(2)}\cap V_{\mathcal{C}'}^{(1)}$. If some $w\in V_{\mathcal{C}}^{(3)}$ is in $V_{\mathcal{C}'}^{(2)}\cup V_{\mathcal{C}'}^{(3)}$, then, since every edge between $V_{\mathcal{C}'}^{(1)}$ and $V_{\mathcal{C}'}^{(i)}$ has the same color under φ for each of $i=2,3$, $c_{\mathcal{C}}^{(2)}=\varphi(uw)=\varphi(vw)=c_{\mathcal{C}}^{(1)}$, which is a contradiction. Therefore, $V_{\mathcal{C}}^{(3)}\cap(V_{\mathcal{C}'}^{(2)}\cup V_{\mathcal{C}'}^{(3)})$ must be empty. If there is some $w\in V_{\mathcal{C}}^{(1)}$ that is in $V_{\mathcal{C}'}^{(2)}\cup V_{\mathcal{C}'}^{(3)}$, say $w\in V_{\mathcal{C}'}^{(2)}$, then, since every edge between $V_{\mathcal{C}}^{(1)}$ and $V_{\mathcal{C}}^{(2)}$ has the same color under φ, $\varphi(uv)=c_{\mathcal{C}}^{(3)}=\varphi(vw)=c_{\mathcal{C}'}^{(3)}$. But then picking any $x\in V_{\mathcal{C}'}^{(2)}$ we have a monochromatic triangle uvx, which contradicts the fact that each color class in φ is bipartite. Therefore, $V_{\mathcal{C}}^{(1)}\cap(V_{\mathcal{C}'}^{(2)}\cup V_{\mathcal{C}'}^{(3)})$ must also be empty. Similarly, $V_{\mathcal{C}}^{(2)}\cap(V_{\mathcal{C}'}^{(2)}\cup V_{\mathcal{C}'}^{(3)})$ must be empty. Hence, $V_{\mathcal{C}'}^{(2)}\cup V_{\mathcal{C}'}^{(3)}$ and $V_{\mathcal{C}}$ are disjoint, so the union of $\mathcal{C}$ and $\mathcal{C}'$ is iterated tripartite, as required. □

It remains to prove Lemma 3.2. We first record a technical result about binary trees. To state this result, we define the *weight* of an internal node in a binary tree to be 2 less than the number of leaves in the subtree consisting of this node and all its descendants.

Lemma 3.4. *There exists a constant C such that, for any binary tree T with k leaves and any collection X of nodes of T with total weight less than $k\log k/C$, the number of triples of leaves whose least common ancestor lies outside X is at least $k^{2.5}$.*

Proof. For any node v, let n_v be the number of leaves in the subtree rooted at v and let m_v be the number of triples of leaves whose least common ancestor is v. Define an *imbalance* in a tree T to be a configuration where u is the parent of v and w, v is the parent of x and y, $n_x>n_w$ and $n_x\geq n_y$. In this case, we say that the imbalance is at u. The *rotation* of the imbalance is the modified tree T' where the subtree rooted at x is swapped with the smaller subtree rooted at w.

Let V_{in} be the set of internal nodes of T. For any function $f : V_{\text{in}} \to [0,1]$ that maps the internal nodes to real numbers in $[0,1]$, we define its *weight* by $\sum_{v\in V_{\text{in}}} f(v)\cdot(n_v-2)$ and its *score* by $\sum_{v\in V_{\text{in}}}(1-f(v))\cdot m_v$. To complete the proof of the lemma, it will suffice to show that there exists a constant C such that any function $f : V_{\text{in}} \to [0,1]$ with weight less than $k\log k/C$ has score at least $k^{2.5}$.

For any $W \geq 0$, let $\nu_T^*(W)$ be the minimum score of a function $f : V_{\text{in}} \to [0,1]$ with weight at most W. If T has an imbalance labeled with u, v, w, x, y as in the definition, let T' be the rotation of the imbalance. Then the only changes in weights or scores occur at u and v: their original weights are $(n_x+n_y+n_w)-2$ and $(n_x+n_y)-2$ and their new weights are $(n_x+n_y+n_w)-2$ and $(n_y+n_w)-2$; meanwhile, m_u, m_v are originally $\frac{1}{2}(n_x+n_y)n_w[(n_x+n_y+n_w)-2]$ and $\frac{1}{2}n_xn_y[(n_x+n_y)-2]$ and in T' the corresponding quantities m'_u, m'_v are $\frac{1}{2}n_x(n_y+n_w)[(n_x+n_y+n_w)-2]$ and $\frac{1}{2}n_yn_w[(n_y+n_w)-2]$.

Let f be the optimizer for $\nu_T^*(W)$. Let $W' = f(u)(n_u-2)+f(v)(n_v-2) \in [0, n_u+n_v-4]$ be the total contribution of u and v to the weight of W. By the optimality of f, we know that $(f(u), f(v))$ maximizes m_ua+m_vb subject to $a, b \in [0,1]$ and $(n_u-2)a+(n_v-2)b \leq W'$. Denote by $M(W')$ the optimum of this linear program. If we set $t = u$ if $m_u/(n_u-2) \geq m_v/(n_v-2)$ and $t = v$ otherwise, then $M(W')$ is a line through the origin with slope $m_t/(n_t-2)$ on $[0, n_t-2]$ and the slope becomes $m_t/(n_t-2)$ on $[n_t-2, n_u+n_v-4]$.

We may similarly define $M'(W')$, where we maximize $m'_ua+m'_vb$ subject to $a, b \in [0,1]$ and $(n'_u-2)a+(n'_v-2)b \leq W'$. Note now that

$$\frac{m'_u}{n'_u-2} = \frac{1}{2}(n_y+n_w)n_x > \frac{1}{2}n_xn_y > \frac{1}{2}n_yn_w = \frac{m'_v}{n'_v-2},$$

so $M'(W')$ is a line through the origin with slope $m'_u/(n'_u-2)$ on $[0, n'_u-2]$ and the slope becomes $m'_v/(n'_v-2)$ on $[n'_u-2, n'_u+n'_v-4]$, after which it remains constant.

Given the definitions of the functions $M(W')$ and $M'(W')$, we make the following claim.

Claim 1. For all $W' \in [0, n_u+n_v-4]$, $M'(W') \geq M(W')$.

Proof. As $M'(W')$ is concave and $M(W')$ is piecewise linear, it suffices to verify this for $W' = 0$, $W' = n_t-2$ and $W' = n_u+n_v+4$. The inequality clearly holds at $W' = 0$. Moreover, since $n_t-2 \leq n_u-2 = n'_u-2$, we know that

$$M'(n_t-2) - M(W') = \frac{m'_u}{n'_u-2}(n_t-2) - \frac{m_t}{n_t-2}(n_t-2) = \left(\frac{m'_u}{n'_u-2} - \frac{m_t}{n_t-2}\right)(n_t-2).$$

Hence, to verify the inequality for $W' = n_t-2$, it suffices to show that $m'_u/(n'_u-2) \geq m_t/(n_t-2)$. Computing directly, we see that

$$\frac{m'_u}{n'_u-2} = \frac{1}{2}(n_y+n_w)n_x > \frac{1}{2}(n_x+n_y)n_w = \frac{m_u}{n_u-2}$$

and

$$\frac{m'_u}{n'_u-2} = \frac{1}{2}(n_y+n_w)n_x > \frac{1}{2}n_xn_y = \frac{m_v}{n_v-2}.$$

Finally, to verify the inequality for $W' = n_u+n_v+4$, note that $M(n_u+n_v-4)$, $M'(n'_u+n'_v-4)$ are equal as they both count the triples of leaves which meet at least two of the subtrees rooted at x, y and w. Since $n'_u+n'_v-4 < n_u+n_v-4$, we have $M'(n_u+n_v-4) = M'(n'_u+n'_v-4) = M(n_u+n_v-4)$, as desired. □

If we now set f' to be the same as f except that $f'(u) = a$, $f'(v) = b$, where (a, b) is the extremizer for $M'(W')$, then both the weight and the score of f' in T' are at most those of f in T, so that $\nu_{T'}^*(W) \leq \nu_T^*(W)$.

That is, rotating imbalances does not increase $\nu_T^*(W)$, so, since we are trying to give a lower bound on $\nu_T^*(W)$, we may rotate as often as we please.

To apply this observation, we now show that for any binary tree T it is possible to do a series of rotations so that the resulting binary tree has no remaining imbalances. This can be done by induction on the number of leaves k. When $k \leq 2$, this is trivial, so suppose that it holds for all binary trees with fewer than k leaves. We first rotate the imbalance at the root if there is one. Therefore, we can always assume that there is no imbalance at the root and we can simply apply the induction hypothesis to the subtrees of the children of the root. The only thing we need to make sure of is that rotating the imbalances in the subtrees does not create an imbalance at the root, which is true as the maximum size of the four subtrees that are two below the root never increases.

We may therefore assume that T has no imbalances. In particular, for any node u with children v and w, we must have that n_v, n_w are both in $[n_u/3, 2n_u/3]$. Otherwise, without loss of generality, we may assume that $n_w < n_u/3$ and $n_v > 2n_u/3$. But then there would exist a child x of v such that $n_x > n_u/3 > n_w$, contradicting that there are no imbalances. Therefore, any node u at depth $d < 0.1 \log k$ has weight $\Omega(3^{-d}k)$ and $m_u = \Omega(3^{-3d}k^3)$. Since $\Omega(3^{-3d}k^3) = \omega(k^{2.5})$, if the score of a function $f : V_{\text{in}} \to [0, 1]$ is less than $k^{2.5}$, then any node whose depth is less than $0.1 \log k$ must be mapped to a number which is at least $1/2$ by f. This shows that the weight of f is $\Omega(k \log k)$, as desired. □

Now we are ready for the proof of the key lemma.

Proof of Lemma 3.2. Let $N = 2^\ell$ and let G be a complete graph on vertex set $[N]$. For each $i \in [\ell]$, let $c_i(u)$ be an independent uniform element of $[A]$. Finally, for any $u \neq v \in [N]$, if $t = v_2(u - v)$, where $v_2(x)$ is the 2-adic valuation function whose value is the highest power of 2 dividing x, we color the edge uv with the product color $(t, (-1)^{\lfloor u/2^t \rfloor}(c_t(u) - c_t(v)) \bmod A)$. Note that as $\lfloor u/2^t \rfloor$ and $\lfloor v/2^t \rfloor$ have different parities, the color of uv does not depend on how we order u and v. Moreover, each color class is the union of vertex-disjoint bicliques, as required.

Let k be a positive integer to be determined later and let $S \subseteq [N]$ be any subset of order k. We wish to upper bound the probability that there are too few rainbow triangles on this set of vertices. To this end, consider the following binary tree T induced by S: place S at the root node; consider the first bit where some elements of S do not agree when written in binary and then split $S = S_0 \cup S_1$ based on the value of this bit, where we assume $|S_0| \geq |S_1|$; attach the binary trees induced by S_0 and S_1 to the root node.

For an internal node corresponding to the set S', let S'_0 and S'_1 be the two subsets corresponding to its children with $|S'_0| \geq |S'_1|$. Suppose that the first bit where some elements disagree is the t-th bit. Call S' *good* if there is no color that appears more than $|S'_0|/2$ times among the collection of $c_t(u)$ with $u \in S'_0$ and *bad* otherwise. If $|S'| \geq 3$, then $|S'_0| \geq 2$, so the probability that S' is bad can be bounded above by

$$|A| \cdot 2^{|S'_0|} \cdot |A|^{-(|S'_0|+1)/2} \leq \exp(-\Omega((|S'| - 2)) \log A),$$

where we used that $A \geq 20$. Moreover, the event that S' is bad is independent from all other similar events.

If S' is good, then we know that, for every $u \in S'_0$, there are at least $|S'_0|/2$ nodes $v \in S'_0$ with $c_t(u) \neq c_t(v)$. For any such $u, v \in S'_0$ and any $w \in S'_1$, it is clear that uvw is rainbow. Therefore, we have found at least

$$\frac{1}{2}\frac{|S'_0|^2}{2}|S'_1| \geq \frac{1}{4}\left(\binom{|S'_0|}{2}|S'_1| + \binom{|S'_1|}{2}|S'_0|\right)$$

rainbow triangles with S' as their least common ancestor. Note that this is exactly a quarter of the number of triplets whose least common ancestor is S'.

If we now apply Lemma 3.4 with X the set of bad nodes in T, we see that if the total weight on the nodes in X is less than $k \log k / C$, then the number of triples of leaves whose least common ancestor is outside X, and therefore good, is at least $k^{2.5}$. By the observation above, this would mean that we have at least $k^{2.5}/4$ rainbow triangles in S. We may therefore assume that the total weight on the set of bad vertices is at least $k \log k / C$. Since the weight of the node corresponding to S' is $|S'| - 2$ and the probability that S' is bad can be bounded by $\exp(-\Omega((|S'| - 2)) \log A)$, the probability that all of the vertices in X are bad is at most $\exp(-\Omega(k \log k \log A))$. Since there are at most 2^k possible choices for X (as there are exactly $k-1$ internal nodes), a union bound implies that the probability there are fewer than $k^{2.5}/4$ rainbow triangles in S is at most $\exp(-\Omega(k \log k \log A))$.

To ensure, by a union bound, that there are at least $k^{2.5}/4$ rainbow triangles in all vertex subsets S of order k with positive probability, we need that

$$\binom{N}{k} \exp(-\Omega(k \log k \log A)) < 1,$$

or, equivalently, that $k \log k \log A = \Omega(k \log N)$. Therefore, it suffices to take $k = \exp(\Omega(\log N / \log A))$. □

4 Concluding remarks

The main problem left open by this paper is whether Conjecture 1.1 holds in full generality. However, it would already be interesting to prove it for hypergraphs with three tight components or to find a different proof for the two component case that gives a better bound. A particular case of interest is the Fano plane F, which is the unique 3-graph with seven edges on seven vertices in which every pair of vertices is contained in a unique edge. The Fano plane is not iterated tripartite, so, according to Conjecture 1.1, $r(F, K_n^{(3)})$ should not grow polynomially. A proof of this would considerably strengthen our belief in the conjecture, on whose validity the authors do not form a completely united front. Let us also point out that in trying to prove Conjecture 1.1, one cannot hope to prove a statement analogous to Theorem 2.1 and Theorem 3.1 that gives a single Ramsey construction avoiding the entire family of possible H simultaneously. Indeed, if a red/blue edge coloring of $K_N^{(3)}$ has all its red subgraphs iterated tripartite, it is not hard to show by induction that there is a blue clique of size $2^{\log_3 N + O(1)} = \Omega(N^{\log_3 2})$.

Another problem of interest is to determine the growth rate of $r(\mathcal{H}, K_n^{(3)})$, where $\mathcal{H}$ is the family of all non-tripartite tightly connected 3-graphs. Theorem 2.1 gives the lower bound $r(\mathcal{H}, K_n^{(3)}) \geq 2^{\Omega(n^{2/3})}$. As an upper bound, we can show that $r(\mathcal{H}, K_n^{(3)}) \leq 2^{O(n)}$. To see this, for each positive integer N, let $n(N)$ be the largest positive integer such that every red/blue-colored $K_N^{(3)}$ with no red copy of any hypergraph from $\mathcal{H}$ contains a blue $K_{n(N)}^{(3)}$. We claim that $n(N) \geq 1 + n\left(\lceil \frac{N-1}{2} \rceil\right)$. Assuming this claim, a simple induction implies that $n(2^k - 1) \geq k$ for all integers $k \geq 1$ and we are done.

To prove the claim, we pick an arbitrary vertex $v \in V(K_n^{(3)})$ and consider G_v, the red/blue-colored graph on $V \setminus \{v\}$ where the color of each edge wx is the color of the triple vwx. Since, by assumption, all red tight components in our coloring of $K_n^{(3)}$ are tripartite, all red connected components of G_v are bipartite. This is exactly equivalent to saying that the set of red edges in G_v is bipartite. Therefore, there is a subset V' of $V \setminus \{v\}$ such that $G_v[V']$ contains only blue edges and $|V'| \geq \lceil \frac{N-1}{2} \rceil$. Now consider the coloring of the 3-graph on V', which again has no red copy of any hypergraph from $\mathcal{H}$. By the definition of the function n, there exists a subset V'' of V' of size $n\left(\lceil \frac{N-1}{2} \rceil\right)$ so that V'' is completely blue. It is then easy to see that $V'' \cup \{v\}$ is completely blue as well, as the presence of any red edge would contradict the choice of V'. The required bound on $n(N)$ follows.

While we suspect that the correct bound for $r(\mathcal{H}, K_n^{(3)})$ is of the form $2^{\Theta(n)}$, it would be more interesting if it were $2^{O(n^{1-\epsilon})}$ for some $\epsilon > 0$. Such intermediate growth has been demonstrated [7] for the Ramsey numbers $r(K_4^{(3)}, S_n^{(3)})$, where $S_n^{(3)}$ is the 3-graph on $n+1$ vertices consisting of all $\binom{n}{2}$ edges incident to a given vertex, but it would be very interesting to have such an example with $K_n^{(3)}$ instead of $S_n^{(3)}$. A particular family of cases where such intermediate growth has not been ruled out is $r(C_\ell \setminus e, K_n^{(3)})$, where $C_\ell \setminus e$ is the tight cycle of length $\ell \not\equiv 0 \pmod 3$ with a single edge removed. While Theorem 1.2 again gives $r(C_\ell \setminus e, K_n^{(3)}) \geq 2^{\Omega(n^{2/3})}$, we do not currently know of any upper bound better than $r(C_\ell \setminus e, K_n^{(3)}) \leq r(C_\ell, K_n^{(3)}) \leq 2^{O_\ell(n \log n)}$ for ℓ sufficiently large [14].

It might be interesting to look at the analogous conjecture to Conjecture 1.1 for higher uniformities. If Conjecture 1.1 is indeed correct, its generalization should say that for a k-graph H, there exists a constant c depending only on H such that $r(H, K_n^{(k)}) \leq n^c$ for all n if and only if H is contained in an iterated blowup of an edge. One direction of this conjecture is simple, while certain partial results in the other direction again hold. For example, a straightforward extension of either construction in Section 2 implies that the conjecture holds for 2-*tightly connected hypergraphs*, hypergraphs where any two edges e and f are joined by a sequence of edges $e = e_0, e_1, \ldots, e_t = f$ such that e_{i-1} and e_i share at least two vertices for all $i = 1, \ldots, t$. In particular, if we generalize the argument that yields Theorem 1.2, we obtain the following result.

Theorem 4.1. *If $k \geq 3$ and H is a k-graph which is 2-tightly connected and not k-partite, then $r(H, K_n^{(k)}) \geq 2^{\Omega(n^{2/k})}$.*

One might also try to find necessary and sufficient conditions on k-graphs H under which $r(H, K_n^{(k)})$ is upper bounded by a function of the form 2^{n^c} or $2^{2^{n^c}}$ and so on. In this direction, we make the following, likely difficult, conjecture, which would extend Theorem 4.1. Generalizing the definition above, we say that a hypergraph is s-*tightly connected* if any two edges e and f are joined by a sequence of edges $e = e_0, e_1, \ldots, e_t = f$ such that e_{i-1} and e_i share at least s vertices for all $i = 1, \ldots, t$.

Conjecture 4.2. *If $k > s$ and H is a k-graph which is s-tightly connected and not k-partite, then there exists a positive constant c such that $r(H, K_n^{(k)}) \geq t_s(n^c)$, where the tower function $t_i(x)$ is defined by $t_1(x) = x$ and $t_i(x) = 2^{t_{i-1}(x)}$ for all $i \geq 2$.*

One natural way to prove Conjecture 4.2 would be to consider the growth of the Ramsey number $r(\mathcal{H}_s^{(k)}, K_n^{(k)})$, where $\mathcal{H}_s^{(k)}$ is the family of s-tightly connected k-graphs that are not k-partite. We can make an even stronger conjecture than Conjecture 4.2.

Conjecture 4.3. *If $k > s$, then there exists a positive constant c such that $r(\mathcal{H}_s^{(k)}, K_n^{(k)}) \geq t_s(n^c)$.*

Our methods prove this conjecture for $s = 2$ and Conjecture 4.2 follows immediately from Conjecture 4.3 as $r(H, K_n^{(k)}) \geq r(\mathcal{H}_s^{(k)}, K_n^{(k)})$ for any $H \in \mathcal{H}_s^{(k)}$. We note that all the proofs in this paper followed this strategy. However, we observe that Conjecture 4.3, if true in general, would also determine the tower height of diagonal hypergraph Ramsey numbers, perhaps the central open problem in hypergraph Ramsey theory. Indeed, $r(K_n^{(s)}, K_n^{(s)}) \geq r(\mathcal{H}_s^{(k)}, K_n^{(k)})$ for any positive integers s, k, n with $k > s$, so we would have that the tower height of $r(K_n^{(s)}, K_n^{(s)})$ is at least $s - 1$, agreeing with the longstanding upper bound.

To see the inequality $r(K_n^{(s)}, K_n^{(s)}) \geq r(\mathcal{H}_s^{(k)}, K_n^{(k)})$, given a red/blue-coloring of the edges of $K_N^{(k)}$ with $N = r(K_n^{(s)}, K_n^{(s)})$ that avoids any red H in $\mathcal{H}_s^{(k)}$, label the k parts of each of the red s-tight components from 1 to k. Consider the auxiliary coloring of the edges of $K_N^{(s)}$ where an edge e is red if it is contained in a red edge $\tilde{e}$ in the original coloring and e contains precisely the vertices in $\tilde{e}$ from parts $1, 2, \ldots, s$ in the red s-tight component of $\tilde{e}$; e is blue otherwise. Since $|e| = s$, any red $\tilde{e}$ containing e belongs to the same red s-tight component, showing that the color of e does not depend on the choice of $\tilde{e}$. It is clear that each red

$\tilde{e}$ in $K_N^{(k)}$ contains exactly one red e in the auxiliary coloring of $K_N^{(s)}$. By the definition of $r(K_n^{(s)}, K_n^{(s)})$, this auxiliary coloring contains either a red $K_n^{(s)}$, which would give a red element of $\mathcal{H}_s^{(k)}$, or a blue $K_n^{(s)}$, which would give a blue $K_n^{(k)}$ in the original coloring.

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References

[1] M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, *J. Combin. Theory Ser. A* **29** (1980), 354–360.

[2] N. Alon and J. H. Spencer, **The Probabilistic Method**, Fourth edition, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016.

[3] T. Bohman and P. Keevash, The early evolution of the H-free process, *Invent. Math.* **181** (2010), 291–336.

[4] Y. Caro, Y. Li, C. C. Rousseau and Y. Zhang, Asymptotic bounds for some bipartite graph: complete graph Ramsey numbers, *Discrete Math.* **220** (2000), 51–56.

[5] D. Conlon, Problem asked at AIM Workshop on Graph Ramsey Theory, webpage available at http://aimpl.org/graphramsey/1/, 2015.

[6] D. Conlon, J. Fox, B. Gunby, X. He, D. Mubayi, A. Suk and J. Verstraëte, On off-diagonal hypergraph Ramsey numbers, preprint available at arXiv:2404.02021 [math.CO].

[7] D. Conlon, J. Fox, X. He, D. Mubayi, A. Suk and J. Verstraëte, Hypergraph Ramsey numbers of cliques versus stars, *Random Structures Algorithms* **63** (2023), 610–623.

[8] D. Conlon, J. Fox and B. Sudakov, Hypergraph Ramsey numbers, *J. Amer. Math. Soc.* **23** (2010), 247–266.

[9] D. Conlon, S. Mattheus, D. Mubayi and J. Verstraëte, Ramsey numbers and the Zarankiewicz problem, *Bull. London Math. Soc.* **56** (2024), 2014–2023.

[10] P. Erdős and A. Hajnal, On Ramsey like theorems. Problems and results, in Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), 123–140, Inst. Math. Appl., Southend-on-Sea, 1972.

[11] J. Fox and X. He, Independent sets in hypergraphs with a forbidden link, *Proc. London Math. Soc.* **123** (2021), 384–409.

[12] J. H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures Algorithms* **7** (1995), 173–207.

[13] S. Mattheus and J. Verstraëte, The asymptotics of $r(4,t)$, *Annals of Math.* **199** (2024), 919–941.

[14] D. Mubayi, Improved bounds for the Ramsey number of tight cycles versus cliques, *Combin. Probab. Comput.* **25** (2016), 791–796.

[15] D. Mubayi and A. Razborov, Polynomial to exponential transition in Ramsey theory, *Proc. London Math. Soc.* **122** (2021), 69–92.

[16] D. Mubayi and J. Verstraëte, A note on pseudorandom Ramsey graphs, *J. Eur. Math. Soc.* **26** (2024), 153–161.