Cliques with many colors in triple systems

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Abstract
Erdős and Hajnal constructed a 4-coloring of the triples of an \(N\)-element set such that every \(n\)-element subset contains 2 triples with distinct colors, and \(N\) is double exponential in \(n\). Conlon, Fox and Rödl asked whether there is some integer \(q \geq 3\) and a \(q\)-coloring of the triples of an \(N\)-element set such that every \(n\)-element subset has 3 triples with distinct colors, and \(N\) is double exponential in \(n\). We make the first nontrivial progress on this problem by providing a \(q\)-coloring with this property for all \(q \geq 9\), where \(N\) is exponential in \(n^{2+cq}\) and \(c > 0\) is an absolute constant.

1 Introduction
The Ramsey number \(r_k(n; q)\) is the minimum integer \(N\) such that for any \(q\)-coloring of the \(k\)-tuples of an \(N\)-element set \(V\), there is a subset \(A \subset V\) such that all of the \(k\)-tuples of \(A\) have the same color. Estimating \(r_3(n; 2)\) is one of the most central problems in combinatorics. The best known bounds, due to Erdős, Hajnal and Rado [5, 4], state that there are positive constants \(c\) and \(c'\) such that
\[
2^{cn^2} < r_3(n; 2) < 2^{c'n^2}. \tag{1}
\]
Erdős conjectured that the upper bound is closer to the truth, namely, \(r_3(n; 2)\) grows double exponentially in \(\Theta(n)\), and he even offered a $500 reward for a proof. His conjecture is supported by the fact that a double exponential growth rate is known when we have 4 colors [3, 4], that is, for fixed \(q \geq 4\)
\[
r_3(n; q) = 2^{2^{\Theta(n)}}. \tag{2}
\]
In this paper, we study the following generalization of \(r_3(n; q)\). For integers \(n > q \geq t \geq 2\), let \(f(n; q, t)\) denote the maximum integer \(N\) such that there is a \(q\)-coloring of the triples of an \(N\)-element set \(V\) with the property that every subset of \(V\) of size \(n\) induces at least \(t\) distinct colors. Thus when \(t = 2\), we have
\[f(n; q, 2) = r_3(n; q) - 1,\]
and for \(q \geq t \geq 3\), we have \(f(n; q, t) < r_3(n; q)\). When \(t = 3\), Conlon, Fox, and Rödl raised the following problem [2].

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**Problem 1.1** (Conlon-Fox-Rödl). Is there an integer $q \geq 3$ and a positive constant $c$ such that $f(n; q, 3) > 2^{2^cn}$ holds for all $n > 2$?

A simple application of the Probabilistic Method (see [1]) shows that $f(n; q, 3) > 2^{cn^2}$, where $c = c(q)$. Our main result is the following.

**Theorem 1.2.** There is an absolute constant $c > 0$ such that for all integers $n > q \geq 9$,

$$f(n; q, 3) \geq 2^{n^{2+c}}.$$  

For larger values of $t$, we show the following.

**Theorem 1.3.** Given integers $q \geq t \geq 2$, there is an $n_0 = n_0(q, t)$ such that for all integers $n > n_0$,

$$f(n; q, t) \geq 2^{n^{\log(q/(t-1))/4}}.$$  

Both proofs are based on a stepping-up argument introduced by Erdős and Hajnal [3]. We start with the proof of Theorem 1.3 in the next section, as it is a direct application of the stepping-up method. The proof of Theorem 1.2 combines a more general stepping-up argument with induction, and is given in Section 3. Throughout this paper, all logarithms are in base 2.

## 2 Forcing many colors

In this section, we prove Theorem 1.3. We will need the following lemma.

**Lemma 2.1.** Given integers $q \geq t \geq 2$, there is an integer $m_0$ such that the following holds. For every $m \geq m_0$, there is a $q$-coloring $φ$ of the pairs of $U = \{0, 1, \ldots, \left[\frac{q}{(t-1)^{m/4}}\right] - 1\}$ such that every subset of size $m$ induces at least $t$ distinct colors.

**Proof.** Given $q \geq t \geq 2$, let $m_0 = m_0(q, t)$ be a sufficiently large integer that will be determined later. Color the pairs of $U = \{0, 1, \ldots, \left[\frac{q}{(t-1)^{m/4}}\right]\}$ uniformly independently at random with colors $\{α_1, \ldots, α_q\}$. Let $X$ denote the number of subsets $A \subset U$ of size $m$ that have less than $t$ distinct colors among their pairs. Then we have

$$\mathbb{E}[X] \leq \binom{|U|}{m} \left(\frac{q}{t-1}\right)^{\binom{m}{2}} \leq \left(\frac{q}{t-1}\right)^{m^2/4} \left(\frac{t-1}{q}\right)^{m^2/2} = q^{t-1} \left(\frac{q}{t-1}\right)^{-m^2/4}.$$  

By setting $m_0 = m_0(q, t)$ sufficiently large, we have for all $m \geq m_0$, $\mathbb{E}[X] < 1$. Hence, there is a $q$-coloring $φ : \binom{U}{2} \to \{α_1, \ldots, α_q\}$ such that every subset $A \subset U$ of size $m$ has at least $t$ distinct colors among its pairs. \hfill $\Box$

**Proof of Theorem 1.3.** Given $q \geq t \geq 2$, let $n_0 = n_0(q, t)$ be a sufficiently large integer that will be determined later. Set $M = \left[\frac{(q(t-1))^{m/4}}{t}\right], U = \{0, 1, \ldots, M-1\}$, and let $φ : \binom{U}{2} \to \{α_1, \ldots, α_q\}$ be a $q$-coloring of the pairs of $U$ with the properties described in Lemma 2.1. Set $V = \{0, 1, \ldots, 2^M - 1\}$. In what follows, we will use $φ$ to define a $q$-coloring $χ : \binom{V}{3} \to \{α_1, \ldots, α_q\}$ of the triples of $V$ with the desired properties.

For each $v \in V$, write $v = \sum_{i=0}^{M-1} v(i)2^i$ with $v(i) \in \{0, 1\}$ for each $i$. For $u \neq v$, let $δ(u, v) \in U$ denote the largest $i$ for which $u(i) \neq v(i)$. Notice that we have the following stepping-up properties (see [6])
Property I: For every triple $u < v < w$, $\delta(u, v) \neq \delta(v, w)$.

Property II: For $v_1 < \cdots < v_r$, $\delta(v_1, v_r) = \max_{1 \leq j \leq r-1} \delta(v_j, v_{j+1})$.

Using $\phi : \binom{U}{2} \to \{\alpha_1, \ldots, \alpha_q\}$, we define $\chi : \binom{V}{3} \to \{\alpha_1, \ldots, \alpha_q\}$ as follows. For vertices $v_1 < v_2 < v_3$ in $V$ and $\delta_i = \delta(v_i, v_{i+1})$, we define $\chi(v_1, v_2, v_3) = \alpha_j$ if and only if $\phi(\delta_1, \delta_2) = \alpha_j$. We now need the following lemma.

Lemma 2.2. For $m \geq 2$ let $n = 2^m$. Then for any set of $n$ vertices $v_1, \ldots, v_n \in V$, there is a subset $S \subset \binom{\delta(v_i, v_{i+1})}{1 \leq i \leq n-1}$ with at least $m$ distinct elements such that for each pair $(\delta_r, \delta_s) \in \binom{B}{2}$, there is a triple $v_i < v_j < v_k$ in $\{v_1, \ldots, v_n\}$ such that $\chi(v_i, v_j, v_k) = \phi(\delta_r, \delta_s)$.

Proof. We proceed by induction on $m$. The base case $m = 2$ follows from Property I. For the inductive step, assume that the statement holds for all $m' < m$. Let $v_1, \ldots, v_n \in V$ such that $v_1 < \cdots < v_n$ and $n = 2^m$. Let $\delta_i = \delta(v_i, v_{i+1})$, for $i = 1, \ldots, n-1$. Set $\delta_w = \max\{\delta_i : 1 \leq i \leq n-1\}$ and notice that, by Properties I and II above, $\delta_w > \delta_i$ for all $i \neq w$. Set $S = \{v_1, \ldots, v_w\}$ and $T = \{v_{w+1}, \ldots, v_n\}$. Then either $|S|$ or $|T|$ has size at least $2^{m-1}$. Without loss of generality, we can assume that $|S| \geq 2^{m-1}$ since a symmetric argument would follow otherwise. By the induction hypothesis, there is a subset $B_0 \subset \{\delta_1, \ldots, \delta_{w-1}\} \subset U$ with at least $m-1$ distinct elements and for each pair $(\delta_r, \delta_s) \in \binom{B_0}{2}$, there is a triple $v_i < v_j < v_k$ in $S$ such that $\chi(v_i, v_j, v_k) = \phi(\delta_r, \delta_s)$. Hence $B \subset U$ has the desired properties, and this completes the proof of the claim. $\square$

Set $n_0 = \lceil 2^{m_0} \rceil$ where $m_0$ is defined in Lemma 2.1. Then for all $n > n_0$ we have $m > m_0$. Thus, by Lemma 2.1 and Lemma 2.2, any set of $n$ vertices in $V$ induces at least $t$ distinct colors with respect to $\chi$. Since $|V| = 2^{q/(t-1)^m/4}$ and $n = 2^m$, we have $|V| = 2^{\log(q/(t-1))/4}$. $\square$

3 Forcing three colors

In this section, we prove Theorem 1.2. We will need the following lemma.

Lemma 3.1. Let $r > 3$ and set $V_3 = \{0, 1, \ldots, \lfloor 2^{r/2} \rfloor - 1\}$. Then there is a 3-coloring $\phi_3 : \binom{V_3}{3} \to \{\beta_1, \beta_2, \beta_3\}$ of the triples of $V_3$ such that every subset of size $r$ induces at least three distinct colors.

We omit the proof of Lemma 3.1 as it follows by the same probabilistic argument used for Lemma 2.1. Hence, Lemma 3.1 implies that $f(n; 3, 3) \geq 2^{n^2/24}$. Together with the following recursive formula, Theorem 1.2 quickly follows.

Theorem 3.2. For integers $n > q \geq 9$, we have

$$f(n; q, 3) \geq (f([n/\log n], q - 6, 3))^{n^{1/4}/2}.$$
We will also need the following lemma, whose proof is also omitted since it follows from the same probabilistic argument as in Lemma 2.1.

**Lemma 3.3.** Let $s > 3$ and set $V_2 = \{0, 1, \ldots, \lfloor 2^{s/4} \rfloor \}$. Then there is a 3-coloring $\phi_2 : \binom{V_2}{2} \to \{\alpha_1, \alpha_2, \alpha_3\}$ of the pairs of $V_2$ such that every subset of size $s$ induces at least three distinct colors.

**Proof of Theorem 3.2.** Given $n > q \geq 9$, let $r = \lfloor n/\log n \rfloor$ and $s = \lfloor \log n \rfloor$. Set $N_2 = \lfloor 2^{s/4} \rfloor$, $N_3 = f(r; q - 6, 3)$, and

$$V_2 = \{0, 1, \ldots, N_2 - 1\} \quad \text{and} \quad V_3 = \{0, 1, \ldots, N_3 - 1\}.$$

Using Lemma 3.3, we obtain $\phi_2 : \binom{V_2}{2} \to \{\alpha_1, \alpha_2, \alpha_3\}$ such that every subset of $V_2$ of size $s$ induces at least three colors. Likewise, by definition of $f(r, q - 6, 3)$, we obtain $\phi_3 : \binom{V_3}{3} \to \{\beta_1, \ldots, \beta_{q-6}\}$ such that every subset of $V_3$ of size $r$ induces at least three distinct colors. We now apply the following more general stepping-up procedure.

Set $N = N_3^{N_2}$ and $V = \{0, 1, \ldots, N - 1\}$. For each $v \in V$, write $v = \sum_{i=0}^{N_2-1} v(i)(N_3)^i$ with $v(i) \in V_3$ for each $i$. For $u, v \in V$ with $u < v$, let $\delta(u, v) \in V_2$ denote the largest $i$ for which $u(i) \neq v(i)$. Notice that we no longer have Property I from the previous stepping-up procedure, but we do have the following properties.

**Property II:** For $v_1 < \cdots < v_r$, $\delta(v_1, v_r) = \max_{1 \leq j \leq r-1} \delta(v_j, v_{j+1}).$

**Property III:** For $v_1 < v_2 < v_3$ such that $\delta(v_1, v_2) = \delta(v_2, v_3) = i$, $v_1(i) < v_2(i) < v_3(i)$.

Using $\phi_2$ and $\phi_3$, we define $\chi : \binom{V}{3} \to \{\gamma_1, \ldots, \gamma_q\}$ as follows. For vertices $v_1 < v_2 < v_3$ in $V$, let $\delta_1 = \delta(v_1, v_2)$ and $\delta_2 = \delta(v_2, v_3)$. Then for $i \in \{1, 2, 3\}$,

- set $\chi(v_1, v_2, v_3) = \gamma_i$ if and only if $\delta_1 = \delta_2$ and $\phi_2(\delta_1, \delta_2) = \alpha_i$,
- set $\chi(v_1, v_2, v_3) = \gamma_{3+i}$ if and only if $\delta_1 < \delta_2$ and $\phi_2(\delta_1, \delta_2) = \alpha_i$,

and for $i \in \{1, \ldots, q-6\}$,

- set $\chi(v_1, v_2, v_3) = \gamma_{6+i}$ if and only if $\delta_1 = \delta_2 = j$ and $\phi_3(v_1(j), v_2(j), v_3(j)) = \beta_i$.

Notice that $n \geq \max\{s \cdot r, 2^s\}$. We claim that any set of $n$ vertices $v_1, \ldots, v_n \in V$ induces at least 3 distinct colors with respect to $\chi$. For sake of contradiction, let $A = \{v_1, \ldots, v_n\} \subset V$ such that $v_1 < \cdots < v_n$ and $\chi(v_i, v_j, v_k) \in \{\gamma_x, \gamma_y\}$ for all triples $(v_i, v_j, v_k) \in \binom{A}{3}$. Set $\delta_i = \delta(v_i, v_{i+1})$ for $i = 1, \ldots, n-1$. The proof now falls into the following cases.

**Case 1.** Suppose $\gamma_x, \gamma_y \in \{\gamma_1, \gamma_2, \gamma_3\}$. Then we have $\delta_1 > \delta_2 > \cdots > \delta_{n-1}$. However, $\delta_i \in U = \{0, 1, \ldots, \lfloor 2^{s/4} \rfloor - 1\}$ and $n = 2^s$ which is a contradiction. A similar argument follows if $\gamma_x, \gamma_y \in \{\gamma_4, \gamma_5, \gamma_6\}$.

**Case 2.** Suppose $\gamma_x, \gamma_y \in \{\gamma_7, \ldots, \gamma_{q-6}\}$. Then we must have $\delta_1 = \cdots = \delta_{n-1} = i$ and $v_1(i) < \cdots < v_{n-1}(i)$. Since $n \geq r$, by definition of $\chi$ and $\phi_3$, the set $\{v_1, \ldots, v_n\}$ induces at least three distinct colors, contradiction.

**Case 3.** Suppose $\gamma_x \in \{\gamma_1, \gamma_2, \gamma_3\}$ and $\gamma_y \in \{\gamma_4, \gamma_5, \gamma_6\}$. Then in this case, for any triple $v_i < v_j < v_k$, we have $\delta(v_i, v_j) \neq \delta(v_j, v_k)$ and $\phi_2(\delta(v_i, v_j), \delta(v_j, v_k)) = \alpha_z$ for some fixed $z$. Set $\delta_w = \max\{\delta_i : 1 \leq i \leq n-1\}$ and notice that, by Property II above, $\delta_w > \delta_i$ for all $i \neq w$. Therefore, a straightforward adaptation of Lemma 2.2 gives us the following claim.
Claim 3.4. For \( s \geq 2 \), any set of \( 2^s \) vertices \( v_1, \ldots, v_{2^s} \in V \), with the properties described above, there is a subset \( B \subset \{ \delta(v_i, v_{i+1}) : 1 \leq i \leq 2^s - 1 \} \) with at least \( s \) distinct elements such that \( \phi_2(\delta_i, \delta_j) = \alpha_z \) for every pair \( (\delta_i, \delta_j) \in \binom{B}{2} \).

However, this contradicts Lemma 3.3.

Case 4. Suppose \( \gamma_x \in \{ \gamma_1, \ldots, \gamma_6 \} \) and \( \gamma_y \in \{ \gamma_7, \ldots, \gamma_9 \} \). Without loss of generality, we can assume that \( \gamma_x = \gamma_1 \) and \( \gamma_y = \gamma_7 \) since a symmetric argument would follow otherwise. Notice that there is an integer \( w_1 \in \{1, \ldots, r\} \) such that \( \delta(v_1, v_{w_1}) > \delta(v_{w_1}, v_{w_1+1}) \). Indeed, otherwise if \( \delta_1 = \cdots = \delta_r \), by the definition of \( \chi \) and the properties of \( \phi_3 \) described above, the set \( \{ v_1, \ldots, v_r \} \) induces at least three distinct colors with respect to \( \chi \), contradiction.

The same argument shows that there must be an integer \( w_2 \in \{w_1 + 1, \ldots, w_1 + r\} \) such that \( \delta(v_{w_1}, v_{w_2}) > \delta(v_{w_2}, v_{w_2+1}) \). Since \( n \geq s \cdot r \), a repeated application of the argument above shows that there are integers \( w_1 < \cdots < w_{s-1} \), such that

\[
\delta(v_1, v_{w_1}) > \delta(v_{w_1}, v_{w_2}) > \delta(v_{w_2}, v_{w_3}) > \cdots > \delta(v_{w_{s-1}}, v_{w_{s-1}+1}).
\]

By Property II, \( \chi \) colors every triple in \( \{ v_1, v_{w_1}, \ldots, v_{w_{s-1}}, v_{w_{s-1}+1} \} \) with color \( \gamma_1 \). However, this implies that the set

\[
S = \{ \delta(v_1, v_{w_1}), \delta(v_{w_1}, v_{w_2}), \ldots, \delta(v_{w_{s-2}}, v_{w_{s-1}}), \delta(v_{w_{s-1}}, v_{w_{s-1}+1}) \} \subset U,
\]

has the property that \( |S| = s \) and \( \phi_2 : \left( \binom{S}{2} \right) \rightarrow \alpha_1 \), which is a contradiction. Since \( |V| = N^2_3 \),

\[
f(n; q, 3) \geq |V| \geq (f([n/\log n]; q - 6, 3))^{1/4}/2.
\]

This completes the proof of Theorem 3.2. \( \square \)

Combining Theorem 3.2 with the fact that \( f(n; 3, 3) > 2^{n^2/24} \) gives the following.

Theorem 3.5. For fixed \( q \geq 3 \) and for all \( n > 3 \) we have

\[
f(n; q, 3) > 2^{n^{2+1/4} \left( \frac{q-3}{q} \right)^{-o(1)}}.
\]

References