Cliques with many colors in triple systems

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Abstract

Erdős and Hajnal constructed a 4-coloring of the triples of an N-element set such that every n-element subset contains 2 triples with distinct colors, and N is double exponential in n. Conlon, Fox and Rödl asked whether there is some integer $q \geq 3$ and a q-coloring of the triples of an N-element set such that every n-element subset has 3 triples with distinct colors, and N is double exponential in n. We make the first nontrivial progress on this problem by providing a q-coloring with this property for all $q \geq 9$, where N is exponential in n^{2+cq} and $c > 0$ is an absolute constant.

1 Introduction

The Ramsey number $r_k(n;q)$ is the minimum integer N such that for any q-coloring of the k-tuples of an N-element set V, there is a subset $A \subset V$ such that all of the k-tuples of A have the same color. Estimating $r_3(n;2)$ is one of the most central problems in combinatorics. The best known bounds, due to Erdős, Hajnal and Rado [5, 4], state that there are positive constants c and c' such that

$$2^{cn^2} < r_3(n;2) < 2^{2^{c'n}}. \tag{1}$$

Erdős conjectured that the upper bound is closer to the truth, namely, $r_3(n;2)$ grows double exponentially in $\Theta(n)$, and he even offered a \$500 reward for a proof. His conjecture is supported by the fact that a double exponential growth rate is known when we have 4 colors [3, 4], that is, for fixed $q \geq 4$

$$r_3(n;q) = 2^{2^{\Theta(n)}}. \tag{2}$$

In this paper, we study the following generalization of $r_3(n;q)$. For integers $n > q \geq t \geq 2$, let $f(n;q,t)$ denote the maximum integer N such that there is a q-coloring of the triples of an N-element set V with the property that every subset of V of size n induces at least t distinct colors. Thus when $t = 2$, we have

$$f(n;q,2) = r_3(n;q) - 1,$$

and for $q \geq t \geq 3$, we have $f(n;q,t) < r_3(n;q)$. When $t = 3$, Conlon, Fox, and Rödl raised the following problem [2].

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Problem 1.1 (Conlon-Fox-Rödl)**.** *Is there an integer $q \geq 3$ and a positive constant c such that $f(n; q, 3) > 2^{2^{cn}}$ holds for all $n > 2$?*

A simple application of the Probabilistic Method (see [1]) shows that $f(n; q, 3) > 2^{cn^2}$, where $c = c(q)$. Our main result is the following.

Theorem 1.2. *There is an absolute constant $c > 0$ such that for all integers $n > q \geq 9$,*

$$f(n; q, 3) \geq 2^{n^{2+c\cdot q}}.$$

For larger values of t, we show the following.

Theorem 1.3. *Given integers $q \geq t \geq 2$, there is an $n_0 = n_0(q, t)$ such that for all integers $n > n_0$,*

$$f(n; q, t) \geq 2^{n^{\log(q/(t-1))}/4}.$$

Both proofs are based on a stepping-up argument introduced by Erdős and Hajnal [3]. We start with the proof of Theorem 1.3 in the next section, as it is a direct application of the stepping-up method. The proof of Theorem 1.2 combines a more general stepping-up argument with induction, and is given in Section 3. Throughout this paper, all logarithms are in base 2.

2 Forcing many colors

In this section, we prove Theorem 1.3. We will need the following lemma.

Lemma 2.1. *Given integers $q \geq t \geq 2$, there is an integer m_0 such that the following holds. For every $m \geq m_0$, there is a q-coloring ϕ of the pairs of $U = \{0, 1, \ldots, \lfloor (q/(t-1))^{m/4} \rfloor - 1\}$ such that every subset of size m induces at least t distinct colors.*

Proof. Given $q \geq t \geq 2$, let $m_0 = m_0(q, t)$ be a sufficiently large integer that will be determined later. Color the pairs of $U = \{0, 1, \ldots, \lfloor (q/(t-1))^{m/4} \rfloor$ uniformly independently at random with colors $\{\alpha_1, \ldots, \alpha_q\}$. Let X denote the number of subsets $A \subset U$ of size m that have less than t distinct colors among their pairs. Then we have

$$\mathbb{E}[X] \leq \binom{|U|}{m}\binom{q}{t-1}\left(\frac{t-1}{q}\right)^{\binom{m}{2}} \leq \left(\frac{q}{t-1}\right)^{m^2/4} q^{t-1}\left(\frac{t-1}{q}\right)^{m^2/2} = q^{t-1}\left(\frac{q}{t-1}\right)^{-m^2/4}.$$

By setting $m_0 = m_0(q, t)$ sufficiently large, we have for all $m \geq m_0$, $\mathbb{E}[X] < 1$. Hence, there is a q-coloring $\phi : \binom{U}{2} \to \{\alpha_1, \ldots, \alpha_q\}$ such that every subset $A \subset U$ of size m has at least t distinct colors among its pairs. $\square$

Proof of Theorem 1.3. Given $q \geq t \geq 2$, let $n_0 = n_0(q, t)$ be a sufficiently large integer that will be determined later. Set $M = \lfloor (q/(t-1))^{m/4} \rfloor$, $U = \{0, 1, \ldots, M-1\}$, and let $\phi : \binom{U}{2} \to \{\alpha_1, \ldots, \alpha_q\}$ be a q-coloring of the pairs of U with the properties described in Lemma 2.1. Set $V = \{0, 1, \ldots, 2^M - 1\}$. In what follows, we will use ϕ to define a q-coloring $\chi : \binom{V}{3} \to \{\alpha_1, \ldots, \alpha_q\}$ of the triples of V with the desired properties.

For each $v \in V$, write $v = \sum_{i=0}^{M-1} v(i)2^i$ with $v(i) \in \{0, 1\}$ for each i. For $u \neq v$, let $\delta(u, v) \in U$ denote the largest i for which $u(i) \neq v(i)$. Notice that we have the following stepping-up properties (see [6])

Property I: For every triple $u < v < w$, $\delta(u,v) \neq \delta(v,w)$.

Property II: For $v_1 < \cdots < v_r$, $\delta(v_1, v_r) = \max_{1\leq j\leq r-1} \delta(v_j, v_{j+1})$.

Using $\phi : \binom{U}{2} \to \{\alpha_1, \ldots, \alpha_q\}$, we define $\chi : \binom{V}{3} \to \{\alpha_1, \ldots, \alpha_q\}$ as follows. For vertices $v_1 < v_2 < v_3$ in V and $\delta_i = \delta(v_i, v_{i+1})$, we define $\chi(v_1, v_2, v_3) = \alpha_j$ if and only if $\phi(\delta_1, \delta_2) = \alpha_j$. We now need the following lemma.

Lemma 2.2. *For $m \geq 2$ set $n = 2^m$. Then for any set of n vertices $v_1, \ldots, v_n \in V$, where $v_1 < \cdots < v_n$, there is a subset $B \subset \{\delta(v_i, v_{i+1}) : 1 \leq i \leq n-1\}$ with at least m distinct elements such that for each pair $(\delta_r, \delta_s) \in \binom{B}{2}$, there is a triple $v_i < v_j < v_k$ in $\{v_1, \ldots, v_n\}$ such that $\chi(v_i, v_j, v_k) = \phi(\delta_r, \delta_s)$.*

Proof. We proceed by induction on m. The base case $m = 2$ follows from Property I. For the inductive step, assume that the statement holds for all $m' < m$. Let $v_1, \ldots, v_n \in V$ such that $v_1 < \cdots < v_n$ and $n = 2^m$. Let $\delta_i = \delta(v_i, v_{i+1})$, for $i = 1, \ldots, n-1$. Set $\delta_w = \max\{\delta_i : 1 \leq i \leq n-1\}$ and notice that, by Properties I and II above, $\delta_w > \delta_i$ for all $i \neq w$. Set $S = \{v_1, \ldots, v_w\}$ and $T = \{v_{w+1}, \ldots, v_n\}$. Then either $|S|$ or $|T|$ has size at least 2^{m-1}. Without loss of generality, we can assume that $|S| \geq 2^{m-1}$ since a symmetric argument would follow otherwise. By the induction hypothesis, there is a subset $B_0 \subset \{\delta_1, \ldots, \delta_{w-1}\} \subset U$ with at least $m-1$ distinct elements and for each pair $(\delta_r, \delta_s) \in \binom{B_0}{2}$, there is a triple $v_i < v_j < v_k$ in S such that

$$\chi(v_i, v_j, v_k) = \phi(\delta_r, \delta_s).$$

Set $B = \{\delta_w\} \cup B_0$, which implies $|B| \geq m$. Then notice that for each pair (δ_w, δ_r), where $\delta_r \in B_0$, by Property I above, we have

$$\chi(v_r, v_{r+1}, v_{w+1}) = \phi(\delta_w, \delta_r).$$

Hence $B \subset U$ has the desired properties, and this completes the proof of the claim. □

Set $n_0 = \lceil 2^{m_0} \rceil$ where m_0 is defined in Lemma 2.1. Then for all $n > n_0$ we have $m > m_0$. Thus, by Lemma 2.1 and Lemma 2.2, any set of n vertices in V induces at least t distinct colors with respect to χ. Since $|V| = 2^{(q/(t-1))^{m/4}}$ and $n = 2^m$, we have $|V| = 2^{n^{\log(q/(t-1))}/4}$. □

3 Forcing three colors

In this section, we prove Theorem 1.2. We will need the following lemma.

Lemma 3.1. *Let $r > 3$ and set $V_3 = \{0, 1, \ldots, \lfloor 2^{r^2/24} \rfloor - 1\}$. Then there is a 3-coloring $\phi_3 : \binom{V_3}{3} \to \{\beta_1, \beta_2, \beta_3\}$ of the triples of V_3 such that every subset of size r induces at least three distinct colors.*

We omit the proof of Lemma 3.1 as it follows by the same probabilistic argument used for Lemma 2.1. Hence, Lemma 3.1 implies that $f(n; 3, 3) \geq 2^{n^2/24}$. Together with the following recursive formula, Theorem 1.2 quickly follows.

Theorem 3.2. *For integers $n > q \geq 9$, we have*

$$f(n; q, 3) \geq (f(\lfloor n/\log n \rfloor, q-6, 3))^{n^{1/4}/2}.$$

We will also need the following lemma, whose proof is also omitted since it follows from the same probabilistic argument as in Lemma 2.1.

Lemma 3.3. *Let $s > 3$ and set $V_2 = \{0, 1, \ldots, \lfloor 2^{s/4} \rfloor\}$. Then there is a 3-coloring $\phi_2 : \binom{V_2}{2} \to \{\alpha_1, \alpha_2, \alpha_3\}$ of the pairs of V_2 such that every subset of size s induces at least three distinct colors.*

Proof of Theorem 3.2. Given $n > q \geq 9$, let $r = \lfloor n/\log n \rfloor$ and $s = \lfloor \log n \rfloor$. Set $N_2 = \lfloor 2^{s/4} \rfloor$, $N_3 = f(r; q-6, 3)$, and

$$V_2 = \{0, 1, \ldots, N_2 - 1\} \qquad \text{and} \qquad V_3 = \{0, 1, \ldots, N_3 - 1\}.$$

Using Lemma 3.3, we obtain $\phi_2 : \binom{V_2}{2} \to \{\alpha_1, \alpha_2, \alpha_3\}$ such that every subset of V_2 of size s induces at least three colors. Likewise, by definition of $f(r, q-6, 3)$, we obtain $\phi_3 : \binom{V_3}{3} \to \{\beta_1, \ldots, \beta_{q-6}\}$ such that every subset of V_3 of size r induces at least three distinct colors. We now apply the following more general stepping-up procedure.

Set $N = N_3^{N_2}$ and $V = \{0, 1, \ldots, N-1\}$. For each $v \in V$, write $v = \sum_{i=0}^{N_2-1} v(i)(N_3)^i$ with $v(i) \in V_3$ for each i. For $u, v \in V$ with $u < v$, let $\delta(u, v) \in V_2$ denote the largest i for which $u(i) \neq v(i)$. Notice that we no longer have Property I from the previous stepping-up procedure, but we do have the following properties.

Property II: For $v_1 < \cdots < v_r$, $\delta(v_1, v_r) = \max_{1 \leq j \leq r-1} \delta(v_j, v_{j+1})$.

Property III: For $v_1 < v_2 < v_3$ such that $\delta(v_1, v_2) = \delta(v_2, v_3) = i$, $v_1(i) < v_2(i) < v_3(i)$.

Using ϕ_2 and ϕ_3, we define $\chi : \binom{V}{3} \to \{\gamma_1, \ldots, \gamma_q\}$ as follows. For vertices $v_1 < v_2 < v_3$ in V, let $\delta_1 = \delta(v_1, v_2)$ and $\delta_2 = \delta(v_2, v_3)$. Then for $i \in \{1, 2, 3\}$,

- set $\chi(v_1, v_2, v_3) = \gamma_i$ if and only if $\delta_1 > \delta_2$ and $\phi_2(\delta_1, \delta_2) = \alpha_i$,
- set $\chi(v_1, v_2, v_3) = \gamma_{3+i}$ if and only if $\delta_1 < \delta_2$ and $\phi_2(\delta_1, \delta_2) = \alpha_i$,

and for $i \in \{1, \ldots, q-6\}$,

- set $\chi(v_1, v_2, v_3) = \gamma_{6+i}$ if and only if $\delta_1 = \delta_2 = j$ and $\phi_3(v_1(j), v_2(j), v_3(j)) = \beta_i$,

Notice that $n \geq \max\{s \cdot r, 2^s\}$. We claim that any set of n vertices $v_1, \ldots, v_n \in V$ induces at least 3 distinct colors with respect to χ. For sake of contradiction, let $A = \{v_1, \ldots, v_n\} \subset V$ such that $v_1 < \cdots < v_n$ and $\chi(v_i, v_j, v_k) \in \{\gamma_x, \gamma_y\}$ for all triples $(v_i, v_j, v_k) \in \binom{A}{3}$. Set $\delta_i = \delta(v_i, v_{i+1})$ for $i = 1, \ldots, n-1$. The proof now falls into the following cases.

Case 1. Suppose $\gamma_x, \gamma_y \in \{\gamma_1, \gamma_2, \gamma_3\}$. Then we have $\delta_1 > \delta_2 > \cdots > \delta_{n-1}$. However, $\delta_i \in U = \{0, 1, \ldots, \lfloor 2^{s/4} \rfloor - 1\}$ and $n = 2^s$ which is a contradiction. A similar argument follows if $\gamma_x, \gamma_y \in \{\gamma_4, \gamma_5, \gamma_6\}$.

Case 2. Suppose $\gamma_x, \gamma_y \in \{\gamma_7, \ldots, \gamma_{q-6}\}$. Then we must have $\delta_1 = \cdots = \delta_{n-1} = i$ and $v_1(i) < \cdots < v_{n-1}(i)$. Since $n \geq r$, by definition of χ and ϕ_3, the set $\{v_1, \ldots, v_n\}$ induces at least three distinct colors, contradiction.

Case 3. Suppose $\gamma_x \in \{\gamma_1, \gamma_2, \gamma_3\}$ and $\gamma_y \in \{\gamma_4, \gamma_5, \gamma_6\}$. Then in this case, for any triple $v_i < v_j < v_k$, we have $\delta(v_i, v_j) \neq \delta(v_j, v_k)$ and $\phi_2(\delta(v_i, v_j), \delta(v_j, v_k)) = \alpha_z$ for some fixed z. Set $\delta_w = \max\{\delta_i : 1 \leq i \leq n-1\}$ and notice that, by Property II above, $\delta_w > \delta_i$ for all $i \neq w$. Therefore, a straight-forward adaptation of Lemma 2.2 gives us the following claim.

Claim 3.4. *For $s \geq 2$, any set of 2^s vertices $v_1, \ldots, v_{2^s} \in V$, with the properties described above, there is a subset $B \subset \{\delta(v_i, v_{i+1}) : 1 \leq i \leq 2^s - 1\}$ with at least s distinct elements such that $\phi_2(\delta_i, \delta_j) = \alpha_z$ for every pair $(\delta_i, \delta_j) \in \binom{B}{2}$.*

However, this contradicts Lemma 3.3.

Case 4. Suppose $\gamma_x \in \{\gamma_1, \ldots, \gamma_6\}$ and $\gamma_y \in \{\gamma_7, \ldots, \gamma_q\}$. Without loss of generality, we can assume that $\gamma_x = \gamma_1$ and $\gamma_y = \gamma_7$ since a symmetric argument would follow otherwise. Notice that there is an integer $w_1 \in \{1, \ldots, r\}$ such that $\delta(v_1, v_{w_1}) > \delta(v_{w_1}, v_{w_1+1})$. Indeed, otherwise if $\delta_1 = \cdots = \delta_r$, by the definition of χ and the properties of ϕ_3 described above, the set $\{v_1, \ldots, v_r\}$ induces at least three distinct colors with respect to χ, contradiction.

The same argument shows that there must be an integer $w_2 \in \{w_1 + 1 \ldots, w_1 + r\}$ such that $\delta(v_{w_1}, v_{w_2}) > \delta(v_{w_2}, v_{w_2+1})$. Since $n \geq s \cdot r$, a repeated application of the argument above shows that there are integers $w_1 < \cdots < w_{s-1}$, such that

$$\delta(v_1, v_{w_1}) > \delta(v_{w_1}, v_{w_2}) > \delta(v_{w_2}, v_{w_3}) > \cdots > \delta(v_{w_{s-1}}, v_{w_{s-1}+1}).$$

By Property II, χ colors every triple in $\{v_1, v_{w_1}, \ldots, v_{w_{s-1}}, v_{w_{s-1}+1}\}$ with color γ_1. However, this implies that the set

$$S = \{\delta(v_1, v_{w_1}), \delta(v_{w_1}, v_{w_2}), \ldots, \delta(v_{w_{s-2}}, v_{w_{s-1}}), \delta(v_{w_{s-1}}, v_{w_{s-1}+1})\} \subset U,$$

has the property that $|S| = s$ and $\phi_2 : \binom{S}{2} \to \alpha_1$, which is a contradiction. Since $|V| = N_3^{N_2}$,

$$f(n; q, 3) \geq |V| \geq (f(\lfloor n/\log n \rfloor; q - 6, 3))^{n^{1/4}/2}.$$

This completes the proof of Theorem 3.2. □

Combining Theorem 3.2 with the fact that $f(n; 3, 3) > 2^{n^2/24}$ gives the following.

Theorem 3.5. *For fixed $q \geq 3$ and for all $n > 3$ we have*

$$f(n; q, 3) > 2^{n^{2+\frac{1}{4}\left\lfloor \frac{q-3}{6} \right\rfloor - o(1)}}.$$

References

[1] N. Alon, J. Spencer, The Probabilistic Method, John Wiley & Sons, New York, NY, 1992.

[2] D. Conlon, J. Fox, V. Rödl, Hedgehogs are not colour blind, *J. Combin.* **8** (2017), 475–485.

[3] P. Erdős and A. Hajnal, On Ramsey like theorems, problems and results, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), pp. 123–140, Inst. Math. Appl., Southhend-on-Sea, 1972.

[4] P. Erdős, A. Hajnal, and R. Rado, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196.

[5] P. Erdős and R. Rado, Combinatorial theorems on classifications of subsets of a given set, *Proc. Lond. Math. Soc.* **3** (1952), 417–439.

[6] R. Graham, B. Rothschild, and J. Spencer, **Ramsey Theory**, 2nd ed., Wiley, New York, 1990.