Extremal problems for convex geometric hypergraphs and ordered hypergraphs

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Abstract

An ordered hypergraph is a hypergraph whose vertex set is linearly ordered, and a convex geometric hypergraph is a hypergraph whose vertex set is cyclically ordered. Extremal problems for ordered and convex geometric graphs have a rich history with applications to a variety of problems in combinatorial geometry. In this paper, we consider analogous extremal problems for uniform hypergraphs, and determine the order of magnitude of the extremal function for various ordered and convex geometric paths and matchings. Our results generalize earlier works of Braß-Károlyi-Valtr, Capoyleas-Pach and Aronov-Dujmovič-Morin-Ooms-da Silveira. We also provide a new generalization of the Erdős-Ko-Rado theorem in the ordered setting.

1   Introduction

An *ordered graph* is a graph together with a linear ordering of its vertex set. Extremal problems for ordered graphs have a long history, and were studied extensively in papers by Pach and Tardos [17], Tardos [21] and Korándi, Tardos, Tomon and Weidert [14]. Let $\text{ex}_\to(n, F)$ denote the maximum number of edges in an n-vertex ordered graph that does not contain the ordered graph F. This extremal problem is phrased in [14] in terms of pattern-avoiding matrices. Marcus and Tardos [16] showed that if the forbidden pattern is a permutation matrix, then the answer is in fact linear in n, and thereby solved the Stanley-Wilf Conjecture, as well as a number of other well-known open problems. A central open problem in the area was posed by Pach and Tardos [17], in the form of the following conjecture. An ordered graph has interval chromatic number two if it is bipartite with bipartition $A \cup B$ and A precedes B in the ordering of the vertices.

Conjecture A. *Let F be an ordered acyclic graph with interval chromatic number two. Then* $\text{ex}_\to(n, F) = O(n \cdot \text{polylog}\, n)$.

In support of Conjecture A, Korándi, Tardos, Tomon and Weidert [14] proved for a wide class of

forests F that $\mathrm{ex}_{\rightarrow}(n, F) = n^{1+o(1)}$. This conjecture is related to a question of Braß in the context of convex geometric graphs.

A *convex geometric (cg) graph* is a graph together with a cyclic ordering of its vertex set. Given a convex geometric graph F, let $\mathrm{ex}_{\circlearrowleft}(n, F)$ denote the maximum number of edges in an n-vertex convex geometric graph that does not contain F. Extremal problems for geometric graphs have a fairly long history, going back to theorems on disjoint line segments [13, 20, 15], and more recent results on crossing matchings [3, 5]. Motivated by the famous Erdős unit distance problem, the first author [8] showed that the maximum number of unit distances between points of a convex n-gon is $O(n \log n)$. In the vein of Conjecture A, Braß [2] asked for the determination of all acyclic graphs F such that $\mathrm{ex}_{\circlearrowleft}(n, F)$ is linear in n, and this problem remains open (recently it was solved for trees [10]).

In this paper, we study extremal problems for ordered and convex geometric uniform hypergraphs. An *ordered (convex geometric) r-graph* is an r-uniform hypergraph whose vertex set is linearly (cyclically) ordered. Although the theory of cg (hyper)graphs can be studied independently of any geometric context, extremal problems for both cg graphs and hypergraphs are frequently motivated by problems in discrete geometry [4, 18, 2, 1]. Instances of the extremal problem for two disjoint triangles in the convex geometric setting are connected to the well-known triangle-removal problem [12]. In [9] we show that certain types of paths in the convex geometric setting give the current best bounds for the notorious extremal problem for tight paths in uniform hypergraphs.

One of the goals of this paper is to study extremal problems simultaneously in the ordered and cg settings and compare and contrast their behaviors.

2   Results

We denote by $\mathrm{ex}_{\rightarrow}(n, F)$ ($\mathrm{ex}_{\circlearrowleft}(n, F)$) the maximum number of edges in an n-vertex ordered (cg) r-graph that does not contain F, and let $\mathrm{ex}(n, F)$ denote the usual (unordered) extremal function. Let P be the linearly ordered path with three edges with ordered vertex set $1 < 2 < 3 < 4$ and edge set $\{13, 32, 24\}$. In the convex geometric setting we use P to denote the unique cg graph isomorphic to the path with three edges where the edges 13 and 24 cross. We then have

$$\mathrm{ex}_{\rightarrow}(n, P) = 2n - 3 = \mathrm{ex}_{\circlearrowleft}(n, P) \qquad \text{for } n \geq 3 \tag{1}$$

where the former is a folklore result and the latter is due to Braß, Károlyi and Valtr [3]. To our knowledge, (1) are the only known nontrivial exact results for connected ordered or convex geometric graphs that have crossings in their embedding. These two simple exact results therefore provide a good launchpad for further investigation in the hypergraph case. This is the direction we take, extending (1) to longer paths and to the hypergraph setting. In the process, we will also discover some subtle differences between the ordered and convex geometric cases which are not visible in (1).

There are many ways to extend the definition of a path to hypergraphs and we choose one of the most natural ones, namely tight paths. There are also many possibilities for the ordering of the vertices of the path and again we make a rather natural choice, namely crossing paths which are defined below (a similar notion was studied by Capoyleas and Pach [5] who considered the corresponding question for matchings in a cg graph).

A *tight k-path* is an r-graph whose edges have the form $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$ for $0 \le i < k$. Typically, we list the vertices $v_0 v_1 \ldots v_{k+r-2}$ in a tight k-path. We let $<$ denote the underlying ordering of the vertices of an ordered hypergraph. In the case of convex geometric hypergraphs, we slightly abuse the same notation so that $u_1 < u_2 < \cdots < u_\ell$ is shorthand for $u_1 < u_2 < \cdots < u_\ell < u_1$ which means that moving clockwise in the cyclic ordering of the vertices from u_1 we first encounter u_2, then u_3, and so on until we finally encounter u_ℓ and then u_1 again. In other words, $u_1, \ldots, u_\ell$ is a cyclic interval where the vertices are listed in clockwise order. When needed, we use the notation Ω_n to denote the vertex set of a generic n-vertex convex geometric hypergraph, with the clockwise ordering of the vertices.

Definition 1 (Crossing paths in ordered and convex geometric hypergraphs). *An r-uniform crossing k-path P_k^r in an ordered or convex geometric hypergraph is a tight k-path $v_0 v_1 \ldots v_{r+k-2}$ with the ordering*

(i) $v_0 < v_1 < v_2 < \cdots < v_{r-1}$,

(ii) $v_j < v_{j+r} < v_{j+2r} < \cdots < v_{j+1}$ *for* $j < r-1$ *and*

(iii) $v_0 < v_{r-1} < v_{2r-1} < v_{3r-1} < \cdots < v_{\lfloor \frac{r+k-2}{r} \rfloor r - 1}$.

An ordered P_5^2 (Figure 1) and a convex geometric P_7^2 and P_5^3 (Figure 2) are shown below.
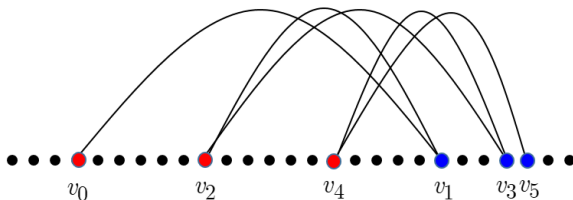

Figure 1: Ordered P_5^2

Our first result generalizes $\mathrm{ex}_\rightarrow(n, P_3^2) = 2n - 3$ to larger k and r.

Theorem 2.1. *Fix $k \ge 1$, $r \ge 2$ and let $n \ge r + k$. Then*

$$\mathrm{ex}_\rightarrow(n, P_k^r) = \begin{cases} \binom{n}{r} - \binom{n-k+1}{r} & \text{for } k \le r+1 \\ \Theta(n^{r-1} \log n) & \text{for } k \ge r+2. \end{cases}$$
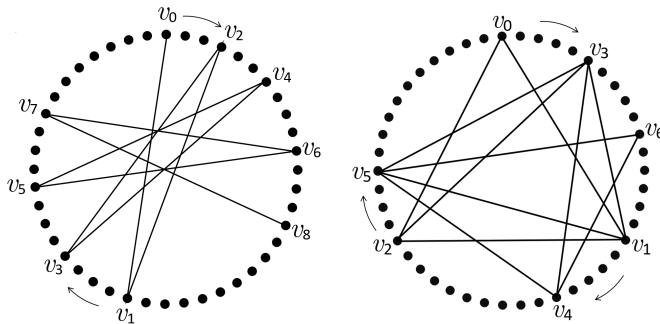
Figure 2: Convex Geometric P_7^2 and P_5^3

Our second theorem generalizes the Braß, Károlyi and Valtr [3] result $\mathrm{ex}_\circlearrowleft(n, P_3^2) = 2n - 3$ to larger k and r.

Theorem 2.2. *Fix $k \geq 1$, $r \geq 2$ and let $n \geq 2r + 1$. Then*

$$\mathrm{ex}_\circlearrowleft(n, P_k^r) = \begin{cases} \Theta(n^{r-1}) & \text{for } 3 \leq k \leq 2r - 1 \\ \binom{n}{r} - \binom{n-r}{r} & \text{for } k = r + 1 \\ \Theta(n^{r-1} \log n) & \text{for } k \geq 2r. \end{cases}$$

For short paths we have the following better bounds, which improve the previous results on this problem by Aronov et. al. [1] when $k = 2$.

Theorem 2.3. *For fixed $2 \leq k \leq r$,*

$$(1 + o(1))\frac{k-1}{3\ln 2r}\binom{n}{r-1} < \mathrm{ex}_\circlearrowleft(n, P_k^r) \leq \frac{(k-1)(r-1)}{r}\binom{n}{r-1}. \tag{2}$$

Furthermore, when $k \in \{2, r\}$, the following sharper bounds hold:

$$\mathrm{ex}_\circlearrowleft(n, P_2^r) \leq \frac{1}{2}\binom{n}{r-1} \tag{3}$$

$$\mathrm{ex}_\circlearrowleft(n, P_r^r) \geq (1 - o(1))(r-2)\binom{n}{r-1}. \tag{4}$$

The lower bound in (4) is close to the upper bound in (2), since the upper bound is $(r-2+1/r)\binom{n}{r-1}$. We remark that it remains open to prove or disprove that for every $r \geq 2$, there exists c_r such that $c_r \to 0$ as $r \to \infty$ and

$$\mathrm{ex}_\circlearrowleft(n, P_2^r) \leq c_r\binom{n}{r-1} + o(n^{r-1}).$$

Theorems 2.1 and 2.2 reveal a discrepancy between the ordered setting and the convex geometric setting: in the convex geometric setting, crossing paths of length up to $2r - 1$ have extremal function

of order n^{r-1}, whereas this phenomenon only occurs for crossing paths of length up to $r+1$ in the ordered setting. In fact, we know that $\mathrm{ex}_\circlearrowright(n, P_k^r) = \mathrm{ex}_\rightarrow(n, P_k^r)$ iff $k \in \{1, r+1\}$.

2.1 Crossing matchings

Let M_k^2 denote the cgg consisting of k pairwise crossing line segments. In other words, there is a labelling of the vertices such that the edges of the matching are $v_i v_{k+i}$ for $1 \leq i \leq k$, and $v_1 < v_2 < \cdots < v_{2k}$.

Capoyleas and Pach [5] proved the following theorem which extended a result of Ruzsa (he proved the case $k = 3$) and settled a question of Gärtner and conjecture of Perles:

Theorem 2.4 (Capoyleas-Pach [5]). *For all $n \geq 2k - 1$, $\mathrm{ex}_\circlearrowright(n, M_k^2) = 2(k-1)n - \binom{2k-1}{2}$.*

As mentioned earlier, a related open problem of Braß [2] is to determine all acyclic graphs F such that $\mathrm{ex}_\circlearrowright(n, F) = O(n)$.

For $r \geq 2$, an r-uniform *crossing k-matching M_k^r* has vertex set $v_1, v_2, \ldots, v_{rk}$ on a convex n-gon in clockwise order and consists of the edges $\{v_i, v_{i+k}, \ldots, v_{i+(r-1)k}\}$ for $1 \leq i \leq k$. Note that crossing paths have the property that if we take every rth edge of the path, we obtain a crossing matching.

One can similarly define a *crossing k-matching M_k^r* in ordered r-graphs: it has vertex set $v_1, v_2, \cdots, v_{rk}$ with $v_1 < v_2 < \ldots < v_{rk}$ and consists of the edges $\{v_i, v_{i+k}, \ldots, v_{i+(r-1)k}\}$ for $1 \leq i \leq k$. However, if we consider a cg r-graph G_1 and an ordered r-graph G_2 with the same set of vertices and the same set of edges (only the ordering in G_1 is linear and in G_2 is circular), then with our definitions a set F of edges is a crossing matching in G_1 if and only if it is a crossing matching in G_2. It follows that

$$\mathrm{ex}_\circlearrowright(n, M_k^r) = \mathrm{ex}_\rightarrow(n, M_k^r) \qquad \text{for all } k, r, n.$$

Aronov, Dujmovič, Morin, Ooms and da Silveira [1] considered the case $k = 2$, $r = 3$ and determined the order of magnitude in those cases; our result below provides better bounds. The $k = 2$ case of Theorem 2.5 could be viewed as an ordered version of the Erdős-Ko-Rado Theorem.

Theorem 2.5. *For $n > r > 1$,*

$$\mathrm{ex}_\circlearrowright(n, M_2^r) = \binom{n}{r} - \binom{n-r}{r}$$

and for fixed $k, r > 2$,

$$(1 - o(1))(k-1)r\binom{n}{r-1} \leq \mathrm{ex}_\circlearrowright(n, M_k^r) \leq 2(k-1)(r-1)\binom{n}{r-1}.$$

Note that, unlike the results on the paths, there are no extra $\log n$ factors in the formulas for crossing matchings. We were unable to determine the asymptotic behavior of $\mathrm{ex}_\circlearrowright(n, M_k^r)$ for any

pair (k, r) with $k, r > 2$.

3 Proof of Theorem 2.1

3.1 Upper bound for $k \leq r + 1$

Observe that $\mathrm{ex}_\to(n, P_2^1) = 1$ for all $n \geq 1$. We then have the following recurrence:

Proposition 3.1. *Let* $2 \leq k \leq r + 1$ *and* $n \geq r + k$. *Then*

$$\mathrm{ex}_\to(n, P_k^r) \leq \binom{n-2}{r-2} + \mathrm{ex}_\to(n-2, P_{k-1}^{r-1}) + \mathrm{ex}_\to(n-1, P_k^r). \tag{5}$$

Proof. Let G be an n-vertex ordered r-graph not containing P_k^r with $e(G) = \mathrm{ex}_\to(n, P_k^r)$. We may assume $V(G) = [n]$ with the natural ordering. Let $G_1 = \{e \in G : \{1, 2\} \subset e\}$ and $G_2 = \{e \in G : 1 \in e, 2 \notin e, e - \{1\} \cup \{2\} \in G\}$. Let G_3 be obtained from $G - E(G_1) - E(G_2)$ by gluing vertex 1 with vertex 2 into a new vertex $2'$.

Since we have deleted the edges of G_1, our G_3 is an r-graph, and since we have deleted the edges of G_2, G_3 has no multiple edges. Thus $e(G) = e(G_1) + e(G_2) + e(G_3)$.

We view G_3 as an ordered r-graph with vertex set $\{2', 3, \ldots, n\}$. If G_3 contains a crossing ordered path P with edges $e'_1, e'_2, \ldots, e'_k$, then only e'_1 may contain $2'$, and all other edges are edges of G. Thus either P itself is in G or the path obtained from P by replacing e'_1 with $e'_1 - \{2'\} + \{1\}$ or with $e'_1 - \{2'\} + \{2\}$ is in G, a contradiction. Thus G_3 contains no P_k^r and hence

$$e(G_3) \leq \mathrm{ex}_\to(n-1, P_k^r).$$

By definition, $e(G_1) \leq \binom{n-2}{r-2}$. We can construct an ordered $(r-1)$-graph H_2 with vertex set $\{3, 4, \ldots, n\}$ from G_2 by deleting from each edge vertex 1. If H_2 contains a crossing ordered path P' with edges $e''_1, e''_2, \ldots, e''_{k-1}$, then the set of edges $\{e_1, \ldots, e_k\}$ where $e_1 = e''_1 + \{1\}$ and $e_i = e''_{i-1} + \{2\}$ for $i = 2, \ldots, k$ forms a P_k^r in G, a contradiction. Summarizing, we get

$$\begin{aligned}
\mathrm{ex}_\to(n, P_k^r) = e(G) &= e(G_1) + e(G_2) + e(G_3) \\
&\leq \binom{n-2}{r-2} + \mathrm{ex}_\to(n-2, P_{k-1}^{r-1}) + \mathrm{ex}_\to(n-1, P_k^r),
\end{aligned}$$

as claimed. $\qquad\square$

We are now ready to prove the upper bound in Theorem 2.1 for $k \leq r + 1$: We are to show that $\mathrm{ex}_\to(n, P_k^r) \leq \binom{n}{r} - \binom{n-k+1}{r}$. We use induction on $k + n$. Since P_1^r is simply an edge, $\mathrm{ex}_\to(n, P_1^r) = 0$ for any n and r, and the theorem holds for $k = 1$.

Suppose now the upper bound in the theorem holds for all (k', n', r') with $k' + n' < k + n$ and we

want to prove it for (k, n, r). By the previous paragraph, it is enough to consider the case $k \geq 2$. Then by Proposition 3.1 and the induction assumption,

$$
\begin{aligned}
\mathrm{ex}_{\rightarrow}(n, P_k^r) & \leq \binom{n-2}{r-2} + \left[\binom{n-2}{r-1} - \binom{n-k}{r-1} \right] + \left[\binom{n-1}{r} - \binom{n-k}{r} \right] \\
& = \left[\binom{n-2}{r-2} + \binom{n-2}{r-1} + \binom{n-1}{r} \right] - \left[\binom{n-k}{r} + \binom{n-k}{r-1} \right] \\
& = \binom{n}{r} - \binom{n-k+1}{r},
\end{aligned}
$$

as required. This proves the upper bound in Theorem 2.1 for $k \leq r + 1$. □

3.2 Lower bound for $k \leq r + 1$

For the lower bound in Theorem 2.1 for $k \leq r + 1$, we provide the following construction. For $1 \leq k \leq r$, let $G(n, r, k)$ be the family of r-tuples $(a_1, \ldots, a_r)$ of positive integers such that

> (a) $1 \leq a_1 < a_2 < \ldots < a_r \leq n$ and
> (b)   there is $1 \leq i \leq k - 1$ such that $a_{i+1} = a_i + 1$.

Also, let $G(n, r, r+1) = G(n, r, r) \cup \{(a_1, \ldots, a_r) : a_1 < a_2 < \ldots < a_r = n\}$.

Suppose $G(n, r, k)$ has an ordered crossing P_k^r with edges $e_1, \ldots, e_k$. Let $e_1 = (a_1, \ldots, a_r)$ where $1 \leq a_1 < a_2 < \ldots < a_r \leq n$. By the definition of a crossing ordered path, for each $2 \leq j \leq k$, e_j has the form

$$
e_j = (a_{j,1}, \ldots, a_{j,r}) \text{ where } a_i < a_{j,i} < a_{i+1} \text{ for } 1 \leq i \leq j-1 \text{ and } a_{j,i} = a_i \text{ for } j \leq i \leq r. \quad (6)
$$

By the definition of $G(n, r, k)$, either there is $1 \leq i \leq k - 1$ such that $a_{i+1} = a_i + 1$ or $k = r + 1$ and $a_r = n$. In the first case, we get a contradiction with (6) for $j = i + 1$. In the second case, we get a contradiction with (6) for $j = r + 1$.

In order to calculate $|G(n, r, k)|$, consider the following procedure $\Pi(n, r, k)$ of generating all r-tuples of elements of $[n]$ *not* in $G(n, r, k)$: take an r-tuple $(a_1, \ldots, a_r)$ of positive integers such that $1 \leq a_1 < a_2 < \ldots < a_r \leq n - k + 1$ and then increase a_j by $j - 1$ if $1 \leq j \leq k$ and by $k - 1$ if $k \leq j \leq r$. By definition, the number of outcomes of this procedure is $\binom{n-k+1}{r}$. Also $\Pi(n, r, k)$ never generates a member of $G(n, r, k)$ and generates each other r-subset of $[n]$ exactly once. □

3.3 Upper bound for $k \geq r + 2$

An ordered r-graph has interval chromatic number r if it is r-partite with r-partition $A_1, \ldots, A_r$ and A_i precedes A_{i+1} in the ordering of the vertices for all $i \in [r - 1]$.

Let $z_{\rightarrow}(n, F)$ denote the maximum number of edges in an n-vertex ordered r-graph of interval chromatic number r that does not contain the ordered graph F. Pach and Tardos [17] showed that every n-vertex ordered graph may be written as the union of at most $\lceil \log n \rceil$ edge disjoint subgraphs each of whose components is a graph of interval chromatic number two, and deduced that $\mathrm{ex}_{\rightarrow}(n, F) = O(z_{\rightarrow}(n, F) \log n)$ for every ordered graph F. They also observed that the log factor is not present when $z_{\rightarrow}(n, F) = \Omega(n^c)$ and $c > 1$. Unsurprisingly, this phenomenon also holds for ordered r-graphs when $r > 2$. We will use the following result which is a rephrasing of [11], Theorem 1.1.

Theorem 3.1 ([11], Theorem 1.1). *Fix $r \geq c \geq r - 1 \geq 1$ and an ordered r-graph F with $z_{\rightarrow}(n, F) = \Omega(n^c)$. Then*

$$\mathrm{ex}_{\rightarrow}(n, F) = \begin{cases} O(z_{\rightarrow}(n, F) \log n) & \text{if } c = r - 1 \\ O(z_{\rightarrow}(n, F)) & \text{if } c > r - 1. \end{cases}$$

By Theorem 3.1, the following claim yields $\mathrm{ex}_{\rightarrow}(n, P_k^r) = O(n^{r-1} \log n)$ for all $k \geq 2$, i.e., the upper bound in Theorem 2.1 for $k \geq r + 2$.

Proposition 3.2. *For $k \geq 1$, $r \geq 2$, $z_{\rightarrow}(n, P_k^r) = O(n^{r-1})$.*

Proof. We prove a stronger statement by induction on k: *if H is an ordered n-vertex r-graph of interval chromatic number r with r-partition $X_1, X_2, \ldots, X_r$ of sizes $n_1, n_2, \ldots, n_r$ respectively, and H has no crossing k-path, then $e(H) \leq kP$ where*

$$P = \prod_{i=1}^{r} n_i \cdot \sum_{i=1}^{r} \frac{1}{n_i}.$$

The base case $k = 1$ is trivial. For the induction step, assume the result holds for paths of length at most $k - 1$, and suppose $e(H) > kP$. For each $(r-1)$-set S of vertices mark the edge $S \cup \{w\}$ where w is maximum. Let H' be the r-graph of unmarked edges. Since we marked at most P edges, $e(H') > (k-1)P$. By the induction assumption there exists a $P_{k-1}^r = v_1 v_2 \ldots v_{k+r-2} \subset H'$ and we can extend this to a P_k^r in H using the marked edge obtained from the $(r-1)$-set $\{v_k, \ldots, v_{k+r-2}\}$. This proves the proposition. $\square$

3.4 Lower bound for $k \geq r + 2$

We now turn to the lower bound in Theorem 2.1. Let $G(n, r, r+2)$ be the family of r-tuples $(a_1, \ldots, a_r)$ of positive integers such that

(a) $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ and
(b) $a_2 - a_1 = 2^p$, where $p \leq \log_2(n/4)$ is an integer.

The number of choices of $a_1 \leq n/4$ is $n/4$, then the number of choices of a_2 is $\log_2(n/4)$, and the number of choices of the remaining $(r-2)$-tuple $(a_3, \ldots, a_r)$ is at least $\binom{n/2}{r-2}$. Thus if $r \geq 3$ and $n > 20r$, then

$$|G(n, r, r+2)| \geq \frac{n^{r-1}}{(r-2)!3^r} \log_2 n. \tag{7}$$

Suppose $G(n, r, r+2)$ contains a P^r_{r+2} with vertex set $\{a_1, \ldots, a_{2r+1}\}$ and edge set $\{a_i \ldots a_{i+r-1} : 1 \leq i \leq r+2\}$. By the definition of ordered path, the vertices are in the following order on $[n]$:

$$a_1 < a_{r+1} < a_{2r+1} < a_2 < a_{r+2} < a_3 < a_{r+3} < \ldots < a_r < a_{2r}. \tag{8}$$

Hence the 2nd, $r+1$st and $r+2$nd edges are

$$\{a_{r+1}, a_2, a_3 \ldots, a_r\}, \qquad \{a_{r+1}, a_{r+2} \ldots, a_{2r}\}, \qquad \{a_{2r+1}, a_{r+2}, \ldots, a_{2r}\}.$$

The differences between the second and the first coordinates in these three vectors are

$$d_1 = a_2 - a_{r+1}, \qquad d_2 = a_{r+2} - a_{r+1}, \qquad d_3 = a_{r+2} - a_{2r+1}.$$

By (8), we have $d_1, d_3 < d_2 < d_1 + d_3$ so it is impossible that all the three differences d_1, d_2, d_3 are powers of two. This yields the lower bound in Theorem 2.1 for $k \geq r+2$. □

4 Proof of Theorem 2.2

We begin with the upper bounds when $r + 1 < k \leq 2r - 1$.

Definition 2. *An ordered r-graph F is a* split hypergraph *if there is a partition of $V(F)$ into intervals $X_1 < X_2 < \cdots < X_{r-1}$ and there exists $i \in [r-1]$ such that every edge of F has two vertices in X_i and one vertex in every X_j for $j \neq i$.*

Every r-graph of interval chromatic number r is a split hypergraph (but not vice versa). We write $e(H)$ for the number of edges in a hypergraph H, $v(H) = \left|\bigcup_{e \in H} e\right|$ and $d(H) = e(H)/v(H)^{r-1}$. The function $d(H)$ could be viewed as a normalized average degree of H. We require the following nontrivial result about split hypergraphs.

Theorem 4.1 ([11], Theorem 1.2). *For $r \geq 3$ there exists $c = c_r > 0$ such that every ordered r-graph H contains a split subgraph G with $d(G) \geq c\,d(H)$.*

Proposition 4.1. *For $r \geq 3$ there exists $C = C_r > 0$ such that, if $r + 1 < k \leq 2r - 1$, then $\mathrm{ex}_\circlearrowleft(n, P^r_k) \leq kC\,n^{r-1}$.*

Proof. Let $c = c_r$ be the constant from Theorem 4.1 and let $C = 1/c$. Given a convex geometric r-graph H with $e(H) > k\,Cn^{r-1}$, we view H as a linearly ordered r-graph (by "opening up" the

circular ordering between any two vertices) and apply Theorem 4.1 to obtain a split subgraph $G \subset H$ where $e(G) > km^{r-1}$ where $m = v(G)$. Now, viewing H once again as a convex geometric r-graph, let $X_0 < X_1 < \cdots < X_{r-3} < X$ be cyclic intervals such that every edge of G contains two vertices in X and one vertex in each $X_i : 0 \leq i \leq r - 3$. Our main assertion is the following:

For $k \in [2r - 1]$, G contains a crossing k-path $v_0 v_1 \ldots v_{k+r-2}$ such that

$\bullet$ $v_i \in X_i$ for $i \not\equiv -1, -2 \mod r$ and

$\bullet$ $v_i \in X$ for $i \equiv -1, -2 \mod r$.

To prove this assertion we proceed by induction on k, where the base case $k = 1$ is trivial. For the induction step, suppose that $1 \leq k \leq 2r - 2$, and we have proved the result for k and we wish to prove it for $k + 1$. Suppose that $k \equiv i \not\equiv 0, -1 \pmod{r}$ where $0 \leq i < r$. For each $f \in \partial G$ that has no vertex in X_{i-1}, delete the edge $f \cup v \in G$ where v is the largest vertex in X_{i-1} in clockwise order. Let G' be the subgraph that remains after deleting these edges. Then

$$e(G') \geq e(G) - m^{r-1} > (k+1)m^{r-1} - m^{r-1} = km^{r-1},$$

so by induction G' contains a P_k^r with vertices $v_0, v_1, \ldots, v_{k-1}, \ldots, v_{k+r-2}$, where $v_i \in X_i$ for $i \not\equiv -1, -2 \pmod{r}$ and $v_i \in X$ for $i \equiv -1, -2 \pmod{r}$. Our goal is to add a new vertex v to the end of the path where $v \in X_{i-1}$. Let $v = v_{k+r-1}$ be the vertex in X_{i-1} for which the edge $e_k = v_k v_{k+1} \ldots v_{k+r-1}$ was deleted in forming G'. Note that v exists as $v_{k-1} v_k \ldots v_{k+r-2} \in E(G)$ and so $v_k \ldots v_{k+r-2} \in \partial G$. Adding vertex v and edge e_k to our copy of P_k^r yields a copy of P_{k+1}^r as required.

Next suppose that $i \equiv 0, -1 \pmod{r}$. Proceed exactly as before except we modify the definition of G' slightly as follows: for every $f \in \partial G$ which has exactly one vertex in each X_i and in X, if w is the vertex of f in X, then delete $f \cup v \in G$ where v is the largest such vertex in X satisfying $v < w$.

By induction, G' contains a P_k^r with vertices $v_0, v_1, \ldots, v_{k-1}, \ldots, v_{k+r-2}$, where $v_i \in X_i$ for $i \not\equiv -1, -2 \pmod{r}$ and $v_i \in X$ for $i \equiv -1, -2 \pmod{r}$. Our goal is to add a new vertex v to the end of the path where $v \in X$ so we may assume that $k \in \{r - 1, r\}$, and we are trying to find vertex v which we will label as $v_{k+r-1} \in \{v_{2r-2}, v_{2r-1}\}$ as above with $v \in X$. Note that we already have the two vertices $v_{r-2} < v_{r-1}$ in X. So we either want to add v_{2r-2} satisfying $v_{r-2} < v_{2r-2} < v_{r-1}$ or we want to add v_{2r-1} satisfying $v_{r-2} < v_{2r-2} < v_{r-1} < v_{2r-1}$. Suppose that $k = r - 1$ so that we are in the first case. Since $v_{r-2} \ldots v_{2r-3} \in E(G')$, the $(r-1)$-set $f = v_{r-1} \ldots v_{2r-3}$ has exactly one vertex $v_{r-1} \in X$. Since $f \cup \{v_{r-2}\} = v_{r-2} v_{r-1} \ldots v_{2r-3} \in E(G')$, we have $f \in \partial G$ and moreover v_{r-2} was not deleted from $f \cup \{v_{r-2}\}$ if forming G'. Hence there is a vertex $v \in X$ with $v_{r-2} < v < v_{r-1}$ such that the edge $f \cup \{v\} = v_{r-1} \ldots v_{2r-3} v \in E(G)$ and the vertex v and edge $f \cup \{v\}$ can be used to extend the P_k^r to a P_{k+1}^r. For the case $k = r$, we choose v to be the largest vertex in X in defining G' and apply an identical argument to that when $i \not\equiv -1, -2 \pmod{r}$. □

Next we give lower bounds for $k \geq 2r$.

Proposition 4.2. *For $k \geq 2r \geq 4$ we have $\mathrm{ex}_{\circlearrowleft}(n, P_k^r) = \Omega(n^{r-1} \log n)$.*

We take the same family $G(n, r, r+2)$ as used for ordered hypergraphs (see Section 3.4), but with the cyclic ordering of the vertex set. When we have a k-edge crossing path $P = w_1 w_2 \ldots w_{r+k-1}$, the vertex w_1 does not need to be the leftmost in the first edge $w_1 \ldots w_r$, so the argument in Section 3.4 does not go through for $k = r+2$. In fact, $G(n, r, r+2)$ does contain P_k^r for $k \leq 2r - 1$.

However, suppose $G(n, r, r+2)$ has a crossing $2r$-edge path $P = w_1 \ldots w_{3r-1}$, and the ith edge of the path is $A_i = w_i w_{i+1} \ldots w_{i+r-1}$. Suppose vertex w_{r+j} is the leftmost in the set $\{w_r, w_{r+1}, \ldots, w_{2r-1}\}$. Then writing the edges A_{j+1}, A_{j+r} and A_{j+r+1} as vectors with increasing coordinates, we have

$$A_{j+1} = \{w_{j+r}, w_{j+1}, w_{j+2}, \ldots, w_{j+r-1}\}, \quad A_{j+r} = \{w_{j+r}, w_{j+r+1} \ldots, w_{j+2r-1}\},$$

$$\text{and} \quad A_{j+r+1} = \{w_{j+2r}, w_{j+r+1}, w_{j+r+2}, \ldots, w_{j+2r-1}\}.$$

The differences between the second and the first coordinates in these three vectors are

$$d_1 = w_{j+1} - w_{j+r}, \qquad d_2 = w_{j+r+1} - w_{j+r}, \qquad d_3 = w_{j+r+1} - w_{j+2r}.$$

As at the end of Section 3.4, it is impossible that all the differences d_1, d_2, d_3 are powers of two. □

Proof of Theorem 2.2. Proposition 4.1 yields $C = C_r$ such that

$$\mathrm{ex}_{\circlearrowleft}(n, P_k^r) \leq kC \, n^{r-1}$$

for $k \leq 2r - 1$. Since the family of all r-subsets of $[n]$ containing 1 witnesses that for $k \geq 3$, $r \geq 2$, $\mathrm{ex}(n, P_k^r) = \Omega(n^{r-1})$, and $\mathrm{ex}_{\circlearrowleft}(n, P_k^r) \geq \mathrm{ex}(n, P_k^r)$, we get $\mathrm{ex}_{\circlearrowleft}(n, P_k^r) = \Theta(n^{r-1})$ for $3 \leq k \leq 2r - 1$. In the case $k = r + 1$, Theorem 2.1 gives

$$\mathrm{ex}_{\circlearrowleft}(n, P_{r+1}^r) \leq \mathrm{ex}_{\to}(n, P_{r+1}^r) = \binom{n}{r} - \binom{n-r}{r}.$$

On the other hand, since $P_{r+1}^r \supseteq M_r^2$ and $G(n, r, r+1) \not\supseteq M_r^2$,

$$\mathrm{ex}_{\circlearrowleft}(n, P_{r+1}^r) \geq \mathrm{ex}_{\circlearrowleft}(n, M_r^2) = \mathrm{ex}_{\to}(n, M_r^2) \geq |G(n, r, r+1)| = \binom{n}{r} - \binom{n-r}{r},$$

so the second statement in Theorem 2.2 follows. It remains to consider $k \geq 2r$, and here we have

$$\mathrm{ex}_{\circlearrowleft}(n, P_k^r) \leq \mathrm{ex}_{\to}(n, P_k^r) = O(n^{r-1} \log n)$$

from Theorem 2.1 and a lower bound from Proposition 4.2. □

5   Proof of Theorem 2.3

5.1   Upper bound in Theorem 2.3 for $r \geq k \geq 2$

Let us first prove the upper bound

$$\mathrm{ex}_{\circlearrowright}(n, P_k^r) \leq \frac{(k-1)(r-1)}{r} \binom{n}{r-1} \qquad (2 \leq k \leq r). \tag{9}$$

Recall that our notation for a crossing k-path P_k^r $(k \leq r)$ on a cyclically ordered vertex set $\mathbf{\Omega}_n$ is the following: the vertices $v_1, v_2, \ldots, v_{r+k-1}$ form a tight path with edges $e_i = \{v_i, \ldots, v_{i+r-1}\}$, $i \in [k]$ and the (clockwise) ordering of the vertices on $\mathbf{\Omega}_n$ is

$$v_1 < v_{r+1} < v_2 < v_{r+2} < \cdots < v_{k-1} < v_{r+k-1} < v_k < v_{k+1} < \cdots < v_r \ (< v_1).$$

We define $T_k(H)$ to be the set of $(v_k, \ldots, v_{r+k-1}) \in V(H)^r$ for which there is a P_k^r in H with vertices $v_1, \ldots, v_{r+k-1}$ as ordered above. In other words, $T_k(H)$ is the set of ending edges for a P_k^r in H.

Theorem 5.1. *Let $r \geq 2$ and $1 \leq k \leq r$. Then for any cg r-graph H on $\mathbf{\Omega}_n$,*

$$|T_k(H)| \geq r \cdot e(H) - (r-1)(k-1) \cdot |\partial H|.$$

In particular, if H contains no P_k^r, then

$$e(H) \leq \frac{(k-1)(r-1)}{r} |\partial H| \leq \frac{(k-1)(r-1)}{r} \binom{n}{r-1}.$$

Proof. We proceed by induction on k. For $k = 1$, and each edge $e \in E(H)$, the number of copies of P_1^r with edge set $\{e\}$ is r, since after choosing which vertex of e to label with v_1, the order of the remaining vertices of e is determined (they are cyclically ordered). Therefore $|T_1(H)| \geq re(H)$. Suppose $k \geq 2$ and assume by induction that $|T_{k-1}(H)| \geq re(H) - (r-1)(k-2)|\partial H|$. Let L be the collection of r-sets in $T_{k-1}(H)$ with the following property: The elements of L are

$$e = x_{r+1} < \cdots < x_{r+k-1} < x_k < \cdots < x_r$$

where $e \in E(H)$ and there does not exist any vertex x such that $x_k < x < x_{k+1}$ and $e - \{x_k\} \cup \{x\} \in E(H)$. Observe that $|L| \leq (r-1)|\partial H|$ since for each ordered $(r-1)$ set $e - \{x_k\} \in \partial H$ there must be a unique x_k satisfying $x_{r+k-1} < x_k < x_{k+1}$ such that $e \in L$ (the vertex closest to x_{k+1}). Our goal is to prove that $|T_k(H)| \geq |T_{k-1}(H) \backslash L|$ via an injection. Then, using the fact that $|L| \leq (r-1)|\partial H|$ and the induction hypothesis, we have

$$|T_k(H)| \geq |T_{k-1}(H) \backslash L| \geq r \cdot e(H) - (k-2)(r-1) \cdot |\partial H| - |L| \geq r \cdot e(H) - (k-1)(r-1) \cdot |\partial H|.$$

We must give an injection $f : T_{k-1}(H) \backslash L \to T_k(H)$. Suppose that $e = v_{r+1} < \cdots < v_{r+k-1} < v_k < \cdots < v_r \in T_{k-1}(H) \backslash L$. Then there exists a vertex x such that $v_k < x < v_{k+1}$ and $e - \{v_k\} \cup \{x\} \in E(H)$. Let A be the set of all such vertices x. Consider the vertex $y \in A$ such that $y \leq x$ for all $x \in A$. In other words, y is the closest vertex to v_k among all vertices of A. Let $f(e) = e - \{v_k\} \cup \{y\}$. Since $k \leq r$, we clearly have $f(e) \in T_k(H)$ as we obtain a P_k^r that ends in $f(e)$ by taking the copy of P_{k-1}^r that ends in e and just adding the edge $f(e)$. Moreover, f is an injection, as if there is an $e' = e - \{v_k\} \cup \{y'\}$ such that $f(e') = f(e)$, then, assuming that $v_k < y' < y$, y would not have been the closest vertex to v_k in A. This contradiction shows that f is indeed an injection and the proof is complete. $\qquad\square$

5.2   Lower bound in Theorem 2.3 for $r \geq k \geq 2$

Our next goal is to prove the following lower bound in Theorem 2.3 for $r \geq k \geq 2$:

$$\mathrm{ex}_{\circlearrowleft}(n, P_k^r) \geq (1 + o(1)) \frac{k-1}{3 \ln 2r} \binom{n}{r-1}. \tag{10}$$

A *gap* of an r-element subset R of $\mathbf{\Omega}_n$ is a segment of $\mathbf{\Omega}_n$ between two clockwise consecutive vertices of R. We say R has (k, m)-*gaps* if some $k - 1$ consecutive gaps of R all have length more than m – in other words, there are at least m vertices of $\mathbf{\Omega}_n$ in each gap. For $n > r$, let K_n^r be the family of all r-element subsets of $\mathbf{\Omega}_n$. For $n > r \geq k$, let $H(n, r, k, m)$ be the family of the members of K_n^r that have (k, m)-gaps, and $\overline{H}(n, r, k, m)$ be the family of the members of K_n^r that do not have (k, m)-gaps.

For a hypergraph H and $v \in V(H)$, let $H\{v\}$ denote the set of edges of H containing v.

Lemma 5.2. *If*

$$m \geq \frac{(n-1) \ln 2r}{(r-1)(k-1)}, \tag{11}$$

then

$$|H(n, r, k, m)| \leq \frac{1}{2} \binom{n}{r}. \quad \textit{Equivalently, } |\overline{H}(n, r, k, m)| \geq \frac{1}{2} \binom{n}{r}. \tag{12}$$

Proof. Instead of proving (12) directly, it will be easier to prove that

$$\textit{for every } j \in \mathbf{\Omega}_n, \quad |H(n, r, k, m)\{j\}| \leq \frac{1}{2} |K_n^r \{j\}| = \frac{1}{2} \binom{n-1}{r-1}; \tag{13}$$

and (13) implies (12) because $|H(n, r, k, m)| = \frac{n}{r} |H(n, r, k, m)\{j\}|$ and $\binom{n}{r} = \frac{n}{r} |K_n^r \{j\}|$.

Recall the vertex set of $\mathbf{\Omega}$ is $\{0, 1, 2, \ldots, n - 1\}$. By symmetry, it is enough to prove (13) for $j = n - 1$. First, we show that

$$|H(n, r, k, m)\{n - 1\}| \leq r |K_{n-(k-1)m}^r \{n - 1 - (k-1)m\}|. \tag{14}$$

Indeed, from each $F \in H(n, r, k, m)\{n-1\}$, we can get an $F' \in K^r_{n-(k-1)m}\{n-1-(k-1)m\}$ by deleting the first m vertices in $k-1$ consecutive gaps of length at least $m+1$, and renumbering the remaining $n-(k-1)m$ vertices so that the vertex $n-1$ of $\mathbf{\Omega}$ will be $(n-1)-(k-1)m$. On the other hand, each $F' \in K^r_{n-(k-1)m}\{n-1-(k-1)m\}$ can be obtained this way from r distinct $F \in H(n, r, k, m)\{n-1\}$. This proves (14).

Now, using $1 - x \leq e^{-x}$, (14) and (11) yield

$$|H(n, r, k, m)\{n-1\}| \leq r \binom{n-1-(k-1)m}{r-1} = r \binom{n-1}{r-1} \prod_{i=1}^{r-1} \frac{n-(k-1)m-i}{n-i}$$

$$\leq r \binom{n-1}{r-1} \exp\left(-\frac{(k-1)m(r-1)}{n-1}\right) \leq r \binom{n-1}{r-1} \frac{1}{2r},$$

yielding (13). □

We are ready to prove (10). Let

$$t = t(r, k) = \left\lceil \frac{(r-1)(k-1)}{\ln 2r} \right\rceil.$$

Suppose $n > r \geq k \geq 2$. If $r = 2$, then $k = 2$, and the bound is trivial; so let $r \geq 3$. Suppose first that t divides n and let $m = n/t$. Then m satisfies (11). By rotating $\mathbf{\Omega}$ we find a subgraph H' of $\overline{H}(n, r, k, m)$ with at least $|\overline{H}(n, r, k, m)|/m$ edges such that every edge of H' adds up to zero modulo m. We claim that

$$H' \text{ does not contain crossing } P_k^r. \tag{15}$$

Indeed, assume H' contains a crossing P_k^r with the vertices $v_0, v_1, \ldots, v_{k+r-2}$. By the definition of crossing paths, $v_0 < v_r < v_1 < v_{1+r} < \cdots < v_{k-1} < v_{k-1+r} < v_k$. Since the set $\{v_1, v_2, \ldots, v_{r-1}\}$ forms an edge together with both v_0 and v_r, $v_r \equiv v_0 \mod m$. Similarly, $v_{r+i} \equiv v_i \mod m$ for all $i < k$. But this means that the edge $\{v_0, v_1, \ldots, v_{r-1}\}$ has $k-1$ consecutive gaps of length more than m, thus it does not belong to $\overline{H}(n, r, k, m)$. This contradiction proves (15).

Thus if $r \geq 3$, $2 \leq k \leq r$ are fixed, n is a large number divisible by t and $m = n/t$, then by (15) and (12), H' is a cg r-graph not containing crossing P_k^r with

$$|H'| \geq \frac{1}{2m} \binom{n}{r} \geq \frac{t}{2r} \binom{n-1}{r-1} \geq \frac{(k-1)(r-1)}{2r \ln 2r} \binom{n-1}{r-1} \geq (1+o(1)) \frac{k-1}{3 \ln 2r} \binom{n}{r-1}.$$

If n is not divisible by t, then let n' be the largest positive integer divisible by t such that $n' \leq n$. Then

$$\mathrm{ex}_\circlearrowright(n, P_k^r) \geq \mathrm{ex}_\circlearrowright(n', P_k^r) \geq (1+o(1)) \frac{k-1}{3 \ln 2r} \binom{n'}{r-1} = (1+o(1)) \frac{k-1}{3 \ln 2r} \binom{n}{r-1}. \quad □$$

5.3   The case $k = 2$

Here we prove the upper bound (3), namely:

$$\mathrm{ex}_\circlearrowleft(n, P_2^r) \leq \frac{1}{2}\binom{n}{r-1}.$$

Recall that P_2^r on $\mathbf{\Omega}_n$ has a vertex set

$$v_1 < v_{r+1} < v_2 < v_3 < \cdots < v_r \ \ (< v_1),$$

and edges $\{v_1, \ldots, , v_r\}$ and $\{v_2, \ldots, v_{r+1}\}$. Consider a P_2^r-free cgh H on the vertex set $\mathbf{\Omega}_n$. Label the vertices of an $e \in H$ as

$$1 \leq a_1 < a_2 < \cdots < a_r \leq n,$$

and define $T_1(e) := e \setminus \{a_1\}$ and $T_2(e) := e \setminus \{a_r\}$. Since H is P_2^r-free, we have $T_\alpha(e) \neq T_\alpha(e')$ for $e \neq e' \in H$ (and $\alpha = 1, 2$). Indeed, if we take (in case of $\alpha = 1$) $v_2, \ldots, v_r = a_2, \ldots, a_r$ and $\{v_1, v_{r+1}\} = \{a_1, a_1'\}$ then we obtain a P_2^r.

We also have $T_1(e) \neq T_2(e')$, otherwise we define $\{v_1, v_{r+1}\} = \{a_1, a_r'\}$ and again obtain a forbidden path. This way we associated two $(r-1)$-sets to each member of H, yielding (3).   □

5.4   The case $k = r$

Here we prove (4), namely:

$$\mathrm{ex}_\circlearrowleft(n, P_r^r) > (1 - o(1))(r - 2)\binom{n}{r-1}.$$

Recall that P_r^r on $\mathbf{\Omega}_n$ has a vertex set

$$v_1 < v_{r+1} < v_2 < v_{r+2} < v_3 < \cdots < v_{r-1} < v_{2r-1} < v_r \ \ (< v_1), \tag{16}$$

and edges $e_1, \ldots, e_r$, where for $i = 1, \ldots, r$, $e_i = \{v_i, v_{i+1}, \ldots, v_{r+i-1}\}$. By (16),

for every $1 \leq i \leq r$, the only vertices in e_i that can be consecutive on $\mathbf{\Omega}_n$ are v_{i+r-1} and v_i. (17)

Recall that the n vertices of $\mathbf{\Omega}_n$ are arranged in clockwise order as $1 < 2 < 3 < \cdots < n$. Let H be the following family of r-sets of $\mathbf{\Omega}_n$. Label the vertices of an $e \in H$ as

$$1 < a_1 < a_2 < \cdots < a_r < n, \tag{18}$$

and put e into H if there exists $2 \leq i \leq r - 1$ with $a_{i-1} + 1 = a_i$. The number of such $e \in H$ is asymptotically $(r - 2)\binom{n}{r-1} + O(n^{r-2})$.

We claim that H does not contain a P_r^r. Suppose, on the contrary, that $F \subset H$ is a copy of P_r^r as it is described in (16). Choose $i \in [r-1]$ such that the largest number in $\{v_1, \ldots, v_{2r-1}\}$ is either v_i or v_{r+i-1}. Consider e_i in the form $(a_1, \ldots, a_r)$ as in (18). Since $e_i = \{v_i, v_{i+1}, \ldots, v_{r+i-1}\}$, by the choice of i, $v_{i+r-1} \in \{a_{r-1}, a_r\}$. This together with (17) contradicts the definition of H. $\quad\square$

6   Proof of Theorem 2.5

We are to show that for $k, r > 2$,

$$(k-1)r \binom{n}{r-1} - O(n^{r-2}) \le \mathrm{ex}_\circlearrowright(n, M_k^r) = \mathrm{ex}_\rightarrow(n, M_k^r) < 2(k-1)(r-1)\binom{n}{r-1}.$$

A simple construction demonstrating the lower bound in Theorem 2.5 is the following cgh : let A be the set of r-gons that contain at least one vertex from a fixed set of $k-1$ vertices of a convex n-gon, and let B be the set of r-gons that have a side of length at most $k-1$. The cgh $A \cup B$ has $(k-1)r\binom{n}{r-1} + O(n^{r-2})$ edges and does not contain M_k^r.

For the upper bound, let H be a largest r-uniform n-vertex family of sets with vertices on a convex polygon of n points with no M_k^r. For each edge A, choose a shortest chord $ch(A)$, say $v_r v_1$ and view the vertices of A as $v_1, v_2, \ldots, v_r$ in clockwise order. Define the *type of A* to be the vector $\mathbf{t}(A) = (t_1, \ldots, t_{r-1})$ where

$$t_i = v_{i+1} - v_i \text{ for } i = 1, \ldots, r-2 \text{ and } t_{r-1} = n - (t_1 + \ldots + t_{r-2}) = v_1 - v_{r-1}.$$

The coordinates of each vector $\mathbf{t}(A)$ are positive integers, $t_{r-1}(A) \ge 2$, and $t_1(A) + \ldots + t_{r-1}(A) = n$ for each A by definition. The number of such vectors is *exactly* $\binom{n-2}{r-2}$ (because this is the number of ways to mark $r-2$ out of the $n-1$ separators in an ordered set of n dots so that the last separator is not marked). For every given type $\mathbf{t} = (t_1, \ldots, t_{r-1})$, the family $H(\mathbf{t})$ of the chords $ch(A)$ of the edges A of type $\mathbf{t}$ does not contain k crossing chords. Thus by Theorem 2.4, $|H(\mathbf{t})| < 2(k-1)n$. Hence, using $r \ge 3$,

$$|H| < 2(k-1)n\binom{n-2}{r-2} = 2(k-1)\frac{(r-1)(n-r+1)}{n-1}\binom{n}{r-1} < 2(k-1)(r-1)\binom{n}{r-1},$$

as claimed. $\quad\square$

7   Concluding remarks

- A hypergraph F is a *forest* if there is an ordering of the edges $e_1, e_2, \ldots, e_t$ of F such that for all $i \in \{2, 3, \ldots, t\}$, there exists $h < i$ such that $e_i \cap \bigcup_{j<i} e_j \subseteq e_h$. It is not hard to show that $\mathrm{ex}(n, F) = O(n^{r-1})$ for each r-uniform forest F. It is therefore natural to extend the Pach-Tardos

Conjecture A to r-graphs as follows:

Conjecture B. *Let $r \geq 2$. Then for any ordered r-uniform forest F with interval chromatic number r, $\mathrm{ex}_\rightarrow(n, F) = O(n^{r-1} \cdot \mathrm{polylog}\, n)$.*

Theorem 3.1 shows that to prove Conjecture B, it is enough to consider the setting of r-graphs of interval chromatic number r. Theorem 2.1 verifies this conjecture for crossing paths, and also shows that the $\log n$ factor in Theorem 3.1 is necessary. It would be interesting to find other general classes of ordered r-uniform forests for $r \geq 3$ for which Conjecture B can be proved. A related problem is to determine for which ordered forests F we have $\mathrm{ex}_\rightarrow(n, F) = O(n^{r-1})$? This is a hypergraph generalization of Braß' question [2] which was solved recently for trees [10].

• It appears to be substantially more difficult to determine the exact value of the extremal function for r-uniform crossing k-paths in the convex geometric setting than in the ordered setting. It is possible to show that for $k \leq 2r - 1$,

$$c(k, r) = \lim_{n \to \infty} \frac{\mathrm{ex}_\circlearrowright(n, P_k^r)}{\binom{n}{r-1}}$$

exists. We do not as yet know the value of $c(k, r)$ for any pair (k, r) with $2 \leq k \leq r$, even though in the ordered setting Theorem 2.1 captures the exact value of the extremal function for all $k \leq r+1$, and $c(r+1, r) = r$.

• One can consider more general orderings of tight paths, namely instead of the vertices whose subscripts are congruent to a modulo r increasing within an interval (conditions (i), (ii), (iii) in Definition 1), we can specify which congruence classes of vertices are increasing within their interval and which are decreasing. Our methods can handle such situations as well.

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