Extremal problems for convex geometric hypergraphs and ordered hypergraphs

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Abstract

An ordered hypergraph is a hypergraph whose vertex set is linearly ordered, and a convex geometric hypergraph is a hypergraph whose vertex set is cyclically ordered. Extremal problems for ordered and convex geometric graphs have a rich history with applications to a variety of problems in combinatorial geometry. In this paper, we consider analogous extremal problems for uniform hypergraphs, and determine the order of magnitude of the extremal function for various ordered and convex geometric paths and matchings. Our results generalize earlier works of Braß-Károlyi-Valtr, Capoyleas-Pach and Aronov-Dujmovič-Morin-Ooms-da Silveira. We also provide a new variation of the Erdős-Ko-Rado theorem in the ordered setting.

1 Introduction

In this paper, we study extremal problems for ordered and convex geometric uniform hypergraphs. Our focus is on determining the maximum number of edges in ordered or convex geometric hypergraphs that contain no path or matching of a specified size. We study extremal problems simultaneously in the ordered and convex geometric settings to compare and contrast their behaviors.

An *ordered graph* is a graph together with a linear ordering of its vertex set. Extremal problems for ordered graphs have a long history (see [15, 19, 12]). Given ordered graphs F and G, say that F is an *ordered subgraph* of G if there is an order preserving map $f : V(F) \to V(G)$ such that $f(e) \in E(G)$ for every $e \in E(F)$. Let $\mathrm{ex}_{\to}(n, F)$ denote the maximum number of edges in an n-vertex ordered graph that does not contain the ordered graph F as an ordered subgraph. This extremal problem can also be phrased in terms of pattern-avoiding matrices (see [12, 14] for more background). An ordered graph has *interval chromatic number* two if it is bipartite with bipartition $A \cup B$ and A precedes B in the ordering of the vertices. Most of the theory is concerned with such graphs. A central open problem in the area was posed by Pach and Tardos [15].

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Conjecture A. *Let F be an ordered acyclic graph with interval chromatic number two. Then* $\mathrm{ex}_{\to}(n, F) = O(n \cdot \mathrm{polylog}\, n)$.

In support of Conjecture A, Korándi, Tardos, Tomon and Weidert [12] proved for a wide class of forests F that $\mathrm{ex}_{\to}(n, F) = n^{1+o(1)}$. Some of our results are concerned with ordered graphs F as in Conjecture A, though we only consider cases when $\mathrm{ex}_{\to}(n, F)$ has order of magnitude n or $n \log n$ and we sometimes prove sharper bounds than those requested by the conjecture.

A *convex geometric (cg) graph* or *cgg* is a graph together with a cyclic ordering of its vertex set. Given a cgg F, let $\mathrm{ex}_{\circlearrowright}(n, F)$ denote the maximum number of edges in an n-vertex cgg that does not contain F as a cgg (defined analogously to the linearly ordered case). Extremal problems for geometric graphs have a long history, beginning with theorems on disjoint line segments [11, 18, 13], to more recent results on crossing matchings [3, 5]. In the vein of Conjecture A, Braß [2] asked for the determination of all acyclic graphs F such that $\mathrm{ex}_{\circlearrowright}(n, F)$ is linear in n, and this problem remains open (recently it was solved for trees [9]).

An *ordered* (*convex geometric*) r-graph is an r-uniform hypergraph whose vertex set is linearly (cyclically) ordered. We denote by $\mathrm{ex}_{\to}(n, F)$ ($\mathrm{ex}_{\circlearrowright}(n, F)$) the maximum number of edges in an n-vertex ordered (cg) r-graph that does not contain F, and let $\mathrm{ex}(n, F)$ denote the usual (unordered) extremal function. In all our results, F will be an ordered or cg r-graph. We have chosen to omit the parameter r in the notation $\mathrm{ex}_{\to}(n, F)$ and $\mathrm{ex}_{\circlearrowright}(n, F)$ but will indicate this parameter in the notation for F (as a superscript).

Although the theory of cg (hyper)graphs can be studied independently of geometric context, extremal problems for both cg graphs and hypergraphs are frequently motivated by problems in discrete geometry [4, 16, 2, 1]. In the opposite direction, recently we [8] determined $\mathrm{ex}_{\circlearrowright}(n, F)$ for a particular cg r-graph F and this gives the current best bound for the extremal problem for tight paths in uniform hypergraphs. This shows that the solution to extremal hypergraph problems in the convex geometric setting can have applications in more general contexts.

2 Results

Let P_3^2 be the linearly ordered path with three edges with ordered vertex set $1 < 2 < 3 < 4$ and edge set $\{13, 32, 24\}$. In the convex geometric setting we use P_3^2 to denote the unique cg graph isomorphic to the path with three edges where the edges 13 and 24 cross. We then have

$$\mathrm{ex}_{\to}(n, P_3^2) = 2n - 3 = \mathrm{ex}_{\circlearrowright}(n, P_3^2) \qquad \text{for } n \geq 3 \tag{1}$$

where the former is a folklore result and the latter is due to Braß, Károlyi and Valtr [3]. To our knowledge, (1) are the only known nontrivial exact results for connected ordered or convex geometric graphs that have crossings in their embedding. These two simple exact results therefore provide a good launchpad for further investigation in the hypergraph case. This is the direction we take, extending (1) to longer paths and to the hypergraph setting. In the process, we will also discover some subtle differences between the ordered and convex geometric cases which are not visible in (1).

There are many ways to extend the definition of a path to hypergraphs and we choose one of the most natural ones, namely tight paths. There are also many possibilities for the ordering of the vertices of the path and again we make a rather natural choice, namely crossing paths which are defined below (a similar notion was studied by Capoyleas and Pach [5] who considered the corresponding question for matchings in a cg graph).

Notation. For a positive integer n, we write $[n] := \{1, \dots, n\}$. An r-uniform *tight k-path* is an r-graph with vertex set $\{v_1, \dots, v_{k+r-1}\}$ and edge set $\{\{v_i, v_{i+1}, \dots, v_{i+r-1}\} : i \in [k]\}$. We will often denote a tight k-path just by listing its vertices in the order $v_1 \dots v_{k+r-1}$. We let $<$ denote the underlying ordering of the vertices of an ordered hypergraph. In the case of convex geometric hypergraphs, we slightly abuse the same notation so that $u_1 < u_2 < \cdots < u_\ell$ is shorthand for $u_1 < u_2 < \cdots < u_\ell < u_1$ which means that moving clockwise in the cyclic ordering of the vertices from u_1 we first encounter u_2, then u_3, and so on until we finally encounter u_ℓ and then u_1 again. In other words, $u_1, \dots, u_\ell$ is a cyclic interval where the vertices are listed in clockwise order. Moreover, given disjoint sets X, Y, we write $X < Y$ to denote the fact all elements of X appear in the linear or cyclic ordering followed by all elements of Y; in particular we do not have $x < y < x' < y'$ for $x, x' \in X$ and $y, y' \in Y$. When needed, we use the notation $\mathbf{\Omega}_n$ to denote the vertex set of a generic n-vertex convex geometric hypergraph, with the clockwise ordering of the vertices. We will use the notation $e(H) = |E(H)|$ where H is any hypergraph.

Definition 1 (Crossing paths in ordered and convex geometric hypergraphs)**.** *An r-uniform crossing k-path P_k^r is an ordered or convex geometric tight k-path with vertex set $V = \{v_1, \dots, v_{k+r-1}\}$ and edge set $\{\{v_i, \dots, v_{i+r-1}\} : i \in [k]\}$ and the following ordering of V. First partition V into $V_1 \cup \dots \cup V_r$ such that for $j \in [r]$,*

$$V_j = \{v_i : i \equiv j \ mod \ r\} = \{v_j, v_{j+r}, \dots, v_{j+tr}\} \qquad \text{where} \qquad t = 1 + \left\lfloor \tfrac{k-1-j}{r} \right\rfloor.$$

Next order the sets $\{V_j : j \in [r]\}$ as $V_1 < \cdots < V_r$. Finally, for $j \in [r]$, order the elements in each V_j as

$$v_j < v_{j+r} < \cdots < v_{j+tr}.$$

Example. If $r = 3$ and $k = 7$, then P_k^r has vertex set $V = \{v_1, v_2, \dots, v_9\}$

$$V_1 = \{v_1, v_4, v_7\}, \quad V_2 = \{v_2, v_5, v_8\}, \quad V_3 = \{v_3, v_6, v_9\}$$

and ordering

$$v_1 < v_4 < v_7 \quad < \quad v_2 < v_5 < v_8 \quad < \quad v_3 < v_6 < v_9.$$

Another way to view the ordering is to write the vertices in rows of size r and then the ordering of the vertices is given by reading the columns from left to right:

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9. \end{array}$$

This path P_7^3 is illustrated in Figure 1 on the right, and P_8^2 is drawn on the left:

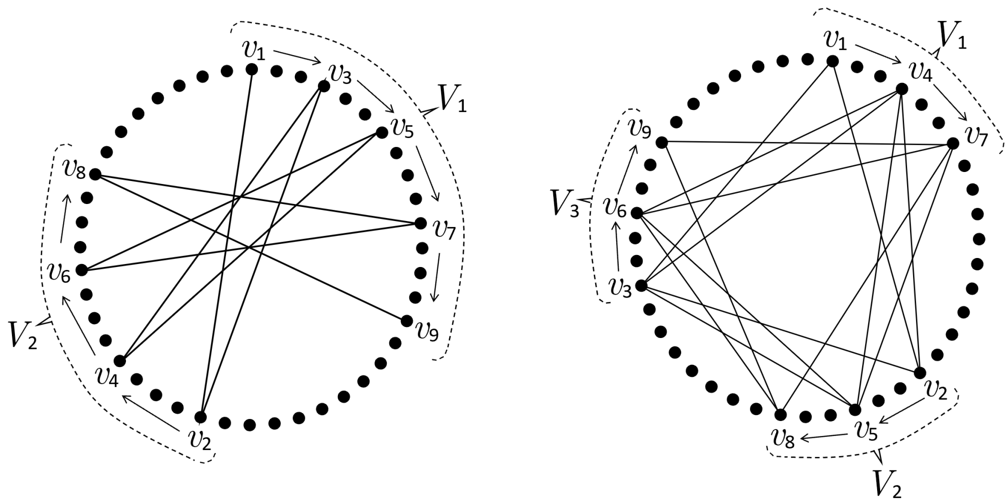

Figure 1: Convex geometric paths P_8^2 and P_7^3

Our first result generalizes $\mathrm{ex}_{\rightarrow}(n, P_3^2) = 2n - 3$ to larger k and r.

Theorem 2.1. *Fix $k \geq 1$, $r \geq 2$ and let $n \geq r + k - 1$. Then*

$$\mathrm{ex}_{\rightarrow}(n, P_k^r) = \begin{cases} \binom{n}{r} - \binom{n-k+1}{r} & \text{for } k \leq r+1 \\ \Theta(n^{r-1} \log n) & \text{for } k \geq r+2 \end{cases}$$

where the asymptotics are taken as k, r are fixed and $n \to \infty$.

Our second theorem generalizes the result of Braß, Károlyi and Valtr [3] that $\mathrm{ex}_{\circlearrowright}(n, P_3^2) = 2n - 3$ to larger k and r.

Theorem 2.2. *Fix $k \geq 1$, $r \geq 2$ and let $n \geq 2r + 1$. Then*

$$\mathrm{ex}_{\circlearrowright}(n, P_k^r) = \begin{cases} \Theta(n^{r-1}) & \text{for } 2 \leq k \leq 2r-1 \\ \binom{n}{r} - \binom{n-r}{r} & \text{for } k = r+1 \\ \Theta(n^{r-1} \log n) & \text{for } k \geq 2r \end{cases}$$

where the asymptotics are taken as k, r are fixed and $n \to \infty$.

For short paths we have the following better bounds, which improve the previous results on this problem by Aronov et. al. [1] when $k = 2$.

Theorem 2.3. *For fixed $2 \leq k \leq r$ and $n \to \infty$*

$$(1+o(1))\frac{k-1}{3\ln 2r}\binom{n}{r-1} < \mathrm{ex}_{\circlearrowright}(n, P_k^r) \leq \frac{(k-1)(r-1)}{r}\binom{n}{r-1}. \tag{2}$$

Furthermore, when $k \in \{2, r\}$, the following sharper bounds hold:

$$\mathrm{ex}_{\circlearrowright}(n, P_2^r) \leq \frac{1}{2}\binom{n}{r-1} \tag{3}$$

$$\mathrm{ex}_{\circlearrowright}(n, P_r^r) \geq (1-o(1))(r-2)\binom{n}{r-1}. \tag{4}$$

The lower bound in (4) is close to the upper bound in (2), since the upper bound is $(r-2+1/r)\binom{n}{r-1}$. We remark that it remains open to prove or disprove that for every $r \geq 2$, there exists c_r such that $c_r \to 0$ as $r \to \infty$ and

$$\mathrm{ex}_{\circlearrowright}(n, P_2^r) \leq c_r\binom{n}{r-1} + o(n^{r-1}).$$

Theorems 2.1 and 2.2 reveal a discrepancy between the ordered setting and the convex geometric setting: in the convex geometric setting, crossing paths of length up to $2r-1$ have extremal function of order n^{r-1}, whereas in the ordered setting this phenomenon only occurs for crossing paths of length up to $r+1$. In fact, we know that $\mathrm{ex}_{\circlearrowright}(n, P_k^r) = \mathrm{ex}_{\to}(n, P_k^r)$ iff $k \in \{1, r+1\}$.

2.1 Crossing matchings

Let M_k^2 denote the cgg consisting of k pairwise crossing line segments. In other words, there is a labelling of the vertices such that the edges of the matching are $v_i v_{k+i}$ for $1 \leq i \leq k$, and $v_1 < v_2 < \cdots < v_{2k}$.

Capoyleas and Pach [5] proved the following theorem which extended a result of Ruzsa (he proved the case $k = 3$) and settled a question of Gärtner and a conjecture of Perles [17]:

Theorem 2.4 (Capoyleas-Pach [5])**.** *For all $n \geq 2k-1$, $\mathrm{ex}_{\circlearrowright}(n, M_k^2) = 2(k-1)n - \binom{2k-1}{2}$.*

As mentioned earlier, a related open problem of Braß [2] is to determine all acyclic graphs F such that $\mathrm{ex}_{\circlearrowright}(n, F) = O(n)$.

For $r \geq 2$, an r-uniform *crossing k-matching M_k^r* has vertex set $v_1, v_2, \dots, v_{rk}$ on a convex n-gon in clockwise order and consists of the edges $\{v_i, v_{i+k}, \dots, v_{i+(r-1)k}\}$ for $1 \leq i \leq k$. Note that crossing paths have the property that if we take every rth edge of the path, we obtain a crossing matching.

One can similarly define a *crossing k-matching M_k^r* in ordered r-graphs: it has vertex set $v_1, v_2, \cdots, v_{rk}$

with $v_1 < v_2 < \ldots < v_{rk}$ and consists of the edges $\{v_i, v_{i+k}, \ldots, v_{i+(r-1)k}\}$ for $1 \le i \le k$. If we consider a cg r-graph G_1 and an ordered r-graph G_2 with the same set of vertices and the same set of edges (only the ordering in G_1 is linear and in G_2 is circular), then with our definitions a set F of edges is a crossing matching in G_1 if and only if it is a crossing matching in G_2. It follows that

$$\mathrm{ex}_{\circlearrowright}(n, M_k^r) = \mathrm{ex}_{\to}(n, M_k^r) \qquad \text{for all } k, r, n.$$

Aronov, Dujmovič, Morin, Ooms and da Silveira [1] considered the case $k = 2$, $r = 3$ and determined the order of magnitude in this case; our result below provides better bounds. Equation (5) could be viewed as an ordered variation of the Erdős-Ko-Rado Theorem since the forbidden configuration consists of two disjoint ordered r-sets with a particular interlacing structure.

Theorem 2.5. *For $n > r > 1$,*

$$\mathrm{ex}_{\circlearrowright}(n, M_2^r) = \binom{n}{r} - \binom{n-r}{r}, \tag{5}$$

and for fixed $k, r > 2$ as $n \to \infty$,

$$(1 - o(1))(k-1)r\binom{n}{r-1} \le \mathrm{ex}_{\circlearrowright}(n, M_k^r) \le 2(k-1)(r-1)\binom{n}{r-1}.$$

Note that, unlike the results for paths, there are no extra $\log n$ factors in the formulas for crossing matchings. We were unable to determine the asymptotic behavior of $\mathrm{ex}_{\circlearrowright}(n, M_k^r)$ for any pair (k, r) with $k, r > 2$.

3 Proof of Theorem 2.1

3.1 Upper bound for $k \le r + 1$

Observe that $\mathrm{ex}_{\to}(n, P_2^1) = 1$ for all $n \ge 1$. We then have the following recurrence:

Proposition 3.1. *Let $2 \le k \le r + 1$ and $n \ge r + k$. Then*

$$\mathrm{ex}_{\to}(n, P_k^r) \le \binom{n-2}{r-2} + \mathrm{ex}_{\to}(n-2, P_{k-1}^{r-1}) + \mathrm{ex}_{\to}(n-1, P_k^r). \tag{6}$$

Proof. Let G be an n-vertex ordered r-graph not containing P_k^r with $e(G) = \mathrm{ex}_{\to}(n, P_k^r)$. We may assume $V(G) = [n]$ with the natural ordering. Let $G_1 = \{e \in G : \{1, 2\} \subset e\}$ and $G_2 = \{e \in G : 1 \in e, 2 \notin e, (e - \{1\}) \cup \{2\} \in G\}$. Let G_3 be obtained from $G - E(G_1) - E(G_2)$ by gluing vertex 1 with vertex 2 into a new vertex $2'$.

Since we have deleted the edges of G_1, our G_3 is an r-graph, and since we have deleted the edges of G_2, G_3 has no multiple edges. Thus $e(G) = e(G_1) + e(G_2) + e(G_3)$.

We view G_3 as an ordered r-graph with vertex set $\{2', 3, \ldots, n\}$. If G_3 contains a crossing ordered path P with edges $e'_1, e'_2, \ldots, e'_k$, then only e'_1 may contain $2'$, and all other edges are edges of G. Thus either P itself is in G or the path obtained from P by replacing e'_1 with $(e'_1 - \{2'\}) \cup \{1\}$ or with $e'_1 - \{2'\} \cup \{2\}$ is in G, a contradiction. Thus G_3 contains no P_k^r and hence

$$e(G_3) \leq \mathrm{ex}_{\rightarrow}(n-1, P_k^r).$$

By definition, $e(G_1) \leq \binom{n-2}{r-2}$. We can construct an ordered $(r-1)$-graph H_2 with vertex set $\{3, 4, \ldots, n\}$ from G_2 by deleting from each edge vertex 1. If H_2 contains a crossing ordered path P' with edges $e''_1, e''_2, \ldots, e''_{k-1}$, then the set of edges $\{e_1, \ldots, e_k\}$ where $e_1 = e''_1 \cup \{1\}$ and $e_i = e''_{i-1} \cup \{2\}$ for $i = 2, \ldots, k$ forms a P_k^r in G, a contradiction. Summarizing, we get

$$\begin{aligned} \mathrm{ex}_{\rightarrow}(n, P_k^r) = e(G) &= e(G_1) + e(G_2) + e(G_3) \\ &\leq \binom{n-2}{r-2} + \mathrm{ex}_{\rightarrow}(n-2, P_{k-1}^{r-1}) + \mathrm{ex}_{\rightarrow}(n-1, P_k^r), \end{aligned}$$

as claimed. □

We are now ready to prove the upper bound in Theorem 2.1 for $k \leq r+1$: we are to show that $\mathrm{ex}_{\rightarrow}(n, P_k^r) \leq \binom{n}{r} - \binom{n-k+1}{r}$. We use induction on $k+n$. Since P_1^r is simply an edge, $\mathrm{ex}_{\rightarrow}(n, P_1^r) = 0$ for any n and r, and the theorem holds for $k = 1$. Also $\mathrm{ex}_{\rightarrow}(n, P_k^r) = \binom{n}{r} - 1$ for $n = r + k - 1$.

Suppose now the upper bound in the theorem holds for all (k', n', r') with $k' + n' < k + n$ and we want to prove it for (k, n, r). By the previous paragraph, it is enough to consider the case $k \geq 2$. Then by Proposition 3.1 and the induction assumption,

$$\begin{aligned} \mathrm{ex}_{\rightarrow}(n, P_k^r) &\leq \binom{n-2}{r-2} + \left[\binom{n-2}{r-1} - \binom{n-k}{r-1}\right] + \left[\binom{n-1}{r} - \binom{n-k}{r}\right] \\ &= \left[\binom{n-2}{r-2} + \binom{n-2}{r-1} + \binom{n-1}{r}\right] - \left[\binom{n-k}{r} + \binom{n-k}{r-1}\right] \\ &= \binom{n}{r} - \binom{n-k+1}{r}, \end{aligned}$$

as required. This proves the upper bound in Theorem 2.1 for $k \leq r+1$. □

3.2 Lower bound for $k \leq r+1$

For the lower bound in Theorem 2.1 for $k \leq r+1$, we provide the following construction. For $1 \leq k \leq r$, let $G(n, r, k)$ be the family of r-tuples $(a_1, \ldots, a_r)$ of positive integers such that

(a) $1 \leq a_1 < a_2 < \ldots < a_r \leq n$ and
(b) there is $1 \leq i \leq k-1$ such that $a_{i+1} = a_i + 1$.

Also, let $G(n,r,r+1) = G(n,r,r) \cup \{(a_1,\ldots,a_r) : a_1 < a_2 < \ldots < a_r = n\}$.

Suppose $G(n,r,k)$ has a crossing P_k^r with edges $e_1,\ldots,e_k$. Let $e_1 = (a_1,\ldots,a_r)$ where $1 \leq a_1 < a_2 < \ldots < a_r \leq n$. By the definition of a crossing path, for each $2 \leq j \leq \min\{k,r\}$, e_j has the form

$$e_j = (a_{j,1},\ldots,a_{j,r}) \textit{ where } a_i < a_{j,i} < a_{i+1} \textit{ for } 1 \leq i \leq j-1 \textit{ and } a_{j,i} = a_i \textit{ for } j \leq i \leq r, \tag{7}$$

and when $k = r+1$, e_{r+1} has the form

$$e_{r+1} = (a_{r+1,1},\ldots,a_{r+1,r}) \textit{ where } a_1 < a_{r+1,1} < a_2 < a_{r+1,2} < \ldots < a_r < a_{r+1,r}. \tag{8}$$

By the definition of $G(n,r,k)$, either there is $1 \leq i \leq k-1$ such that $a_{i+1} = a_i + 1$ or $k = r+1$ and $a_r = n$. In the first case, we get a contradiction with (7) for $j = i+1$. In the second case, we get a contradiction with (8) for $k = r+1$.

In order to calculate $|G(n,r,k)|$, consider the following procedure $\Pi(n,r,k)$ of generating all r-tuples of elements of $[n]$ *not* in $G(n,r,k)$: for each r-tuple $(a_1,\ldots,a_r)$ of positive integers such that $1 \leq a_1 < a_2 < \ldots < a_r \leq n-k+1$, increase a_j by $j-1$ if $1 \leq j \leq k$ and by $k-1$ if $k \leq j \leq r$. By definition, the number of outcomes of this procedure is $\binom{n-k+1}{r}$. Also $\Pi(n,r,k)$ never generates a member of $G(n,r,k)$ and generates each other r-subset of $[n]$ exactly once. $\square$

3.3 Upper bound for $k \geq r+2$

An r-graph is *r-partite* if it has a vertex partition (henceforth r-partition) into r sets such that every edge has exactly one vertex in each set. An ordered r-graph has *interval chromatic number* r if it is r-partite with r-partition $A_1,\ldots,A_r$ and A_i precedes A_{i+1} in the ordering of the vertices for all $i \in [r-1]$.

Let $z_{\rightarrow}(n,F)$ denote the maximum number of edges in an n-vertex ordered r-graph of interval chromatic number r that does not contain the ordered r-graph F. Pach and Tardos [15] showed that every n-vertex ordered graph may be written as the union of at most $\lceil \log_2 n \rceil$ edge disjoint subgraphs each of whose components is a graph of interval chromatic number two, and deduced that $\mathrm{ex}_{\rightarrow}(n,F) = O(z_{\rightarrow}(n,F)\log n)$ for every ordered graph F. They also observed that the log factor is not present when $z_{\rightarrow}(n,F) = \Omega(n^c)$ and $c > 1$. This phenomenon also holds for ordered r-graphs when $r > 2$. We will use the following result which is a rephrasing of Theorem 1.2 in [10].

Theorem 3.1 ([10], Theorem 1.2). *Fix $r \geq c \geq r-1 \geq 1$ and an ordered r-graph F with $z_{\rightarrow}(n,F) = \Omega(n^c)$. Then*

$$\mathrm{ex}_{\rightarrow}(n,F) = \begin{cases} O(z_{\rightarrow}(n,F)\log n) & \textit{if } c = r-1 \\ O(z_{\rightarrow}(n,F)) & \textit{if } c > r-1. \end{cases}$$

By Theorem 3.1, the following claim yields $\mathrm{ex}_{\rightarrow}(n,P_k^r) = O(n^{r-1}\log n)$ for all $k \geq 2$, i.e., the upper bound in Theorem 2.1 for $k \geq r+2$. Given an r-graph H, the *shadow* of H is

$$\partial H = \{S : |S| = r-1, S \subset e \text{ for some } e \in E(H)\}.$$

Proposition 3.2. *For $k \geq 1$, $r \geq 2$, $z_{\rightarrow}(n, P_k^r) = O(n^{r-1})$.*

Proof. We prove a stronger statement by induction on k: *if H is an ordered r-graph of interval chromatic number r with r-partition $X_1, X_2, \dots, X_r$ of sizes $n_1, n_2, \dots, n_r$ respectively, and H has no crossing k-path, then $e(H) \leq kP$ where*

$$P = \sum_{i=1}^{r} \prod_{j \neq i} n_j.$$

The base case $k = 1$ is trivial as $e(H) = 0$. For the induction step, assume the result holds for $k-1$, and suppose $e(H) > kP$. For each $(r-1)$-set $S \in \partial H$ mark the edge $S \cup \{w\}$ where $w = \max\{x : S \cup \{x\} \in E(H)\}$. In other words, if S has no vertex in X_j, then w is the largest vertex in X_j for which $S \cup \{w\} \in E(H)$. Observe that the number of marked edges is at most P.

Let H' be the r-graph of unmarked edges. Then $e(H') \geq e(H) - P > kP - P = (k-1)P$. By the induction assumption there exists a crossing $P_{k-1}^r = v_1 v_2 \dots v_{k+r-2} \subset H'$. Recall that the edges of this P_{k-1}^r are $\{v_i, \dots, v_{i+r-1}\}$ for $i \in [k-1]$. Let $S = \{v_k, \dots, v_{k+r-2}\}$ and suppose that $S \cap X_j = \emptyset$. Since $\{v_{k-1}, \dots, v_{k+r-2}\} = S \cup \{v_{k-1}\}$ is an edge of the P_{k-1}^r, we know that $v_{k-1} \in X_j$ and also that $v_i < v_{k-1}$ if $i < k-1$ and $v_i \in X_j$ by the definition of crossing path. This means that $S \in \partial H$, so it lies in a marked edge $S \cup \{w\}$ where $w \in X_j$. In order to extend this P_{k-1}^r to a P_k^r in H we will use the marked edge $S \cup \{w\}$. By the definition of w, we have $v_{k-1} < w$ and this implies that $v_i < w$ for all $v_i \in X_j$. Consequently, $v_1 \dots v_{k+r-2} w$ is a crossing k-path in H. This proves the proposition. □

3.4 Lower bound for $k \geq r+2$

We now turn to the lower bound in Theorem 2.1. Let $G(n, r, r+2)$ be the family of r-tuples $(a_1, \dots, a_r)$ of positive integers such that

(a) $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ and
(b) $a_2 - a_1 = 2^p$, where $0 \leq p \leq \log_2(n/4)$ is an integer.

For each choice of $a_1 \in \{1, \dots, \lfloor n/4 \rfloor\}$, the number of choices of a_2 is at least $\lfloor \log_2(n/4) \rfloor$, and the number of the choices of the remaining $(r-2)$-tuple $(a_3, \dots, a_r)$ is at least $\binom{n/2}{r-2}$. Thus if $r \geq 3$ then, as $n \to \infty$,

$$|G(n, r, r+2)| \geq \Omega(n^{r-1} \log n). \tag{9}$$

Suppose $G(n, r, r+2)$ contains a P_{r+2}^r with vertex set $\{a_1, \dots, a_{2r+1}\}$ and edge set $\{a_i \dots a_{i+r-1} :$

$1 \le i \le r+2\}$. By the definition of a crossing path, the vertices are in the following order in $[n]$:

$$a_1 < a_{r+1} < a_{2r+1} < a_2 < a_{r+2} < a_3 < a_{r+3} < \ldots < a_r < a_{2r}. \tag{10}$$

Hence the ordered tuples corresponding to the 2nd, $r+1$st and $r+2$nd edges are

$$(a_{r+1}, a_2, a_3 \ldots, a_r), \qquad (a_{r+1}, a_{r+2} \ldots, a_{2r}), \qquad (a_{2r+1}, a_{r+2}, \ldots, a_{2r}).$$

The differences between the second and the first coordinates in these three tuples are

$$d_1 = a_2 - a_{r+1}, \qquad d_2 = a_{r+2} - a_{r+1}, \qquad d_3 = a_{r+2} - a_{2r+1}.$$

By (10), we have $d_1, d_3 < d_2 < d_1 + d_3$ so it is impossible that all the three differences d_1, d_2, d_3 are powers of two. This yields the lower bound in Theorem 2.1 for $k \ge r+2$. □

4 Proof of Theorem 2.2

We begin with the upper bounds when $k \le 2r-1$.

Definition 2. *An ordered r-graph F is a* split hypergraph *if there is a partition of $V(F)$ into intervals $X_1 < X_2 < \cdots < X_{r-1}$ and there exists $i \in [r-1]$ such that every edge of F has two vertices in X_i and one vertex in every X_j for $j \ne i$.*

Every r-graph of interval chromatic number r is a split hypergraph (but not vice versa). We write $v(H) = \left|\bigcup_{e \in H} e\right|$ for the number of vertices in a hypergraph H, and $d(H) = e(H)/v(H)^{r-1}$. The function $d(H)$ could be viewed as a normalized average degree of H. We require the following nontrivial result about split hypergraphs. This result can be considered as an extension of the classical Erdős-Kleitman [7] theorem about large r-partite subgraphs to ordered hypergraphs.

Theorem 4.1 ([10], Theorem 1.2)**.** *For $r \ge 3$ there exists $c = c_r > 0$ such that every ordered r-graph H contains a split subgraph G with $d(G) \ge c\, d(H)$.*

Proposition 4.1. *For $r \ge 3$ there exists $C = C_r > 0$ such that* $\mathrm{ex}_{\circlearrowright}(n, P^r_{2r-1}) \le C\, n^{r-1}$.

Proof. Let $c = c_r$ be the constant from Theorem 4.1 and let $C = (2r-1)/c$. Given an n-vertex cg r-graph H with $e(H) > Cn^{r-1}$, we view H as a linearly ordered r-graph (by "opening up" the circular ordering between any two vertices) and apply Theorem 4.1 to obtain an m-vertex split subgraph $G \subset H$ where $e(G) \ge c\, d(H) m^{r-1} > c\, C m^{r-1} = (2r-1)m^{r-1}$. Now, viewing H (as well as G) once again as a cg r-graph, let $X_1 < X_2 < \cdots < X_{r-2} < X$ be cyclic intervals such that every edge of G contains two vertices in X and one vertex in X_i for each $i \in [r-2]$. Our main assertion is the following statement for each $k \in [2r-1]$ that we will prove by induction on k. The case $k = 2r-1$ will complete the proof of the theorem.

If G is a split r-graph with m vertices and parts $X_1 < X_2 < \cdots < X_{r-2} < X$ and $e(G) > km^r$, then G contains a crossing k-path $v_1v_2 \dots v_{k+r-1}$ such that

- $v_i \in X_i$ for $i \not\equiv 0, r-1 \mod r$ and
- $v_i \in X$ for $i \equiv 0, r-1 \mod r$.

This means that when $k = 2r-1$,

$$\{v_{r-1}, v_r, v_{2r-1}, v_{2r}\} \subset X \quad \text{ and } \quad \{v_i, v_{i+r}, v_{i+2r}\} \subset X_i \quad (i \in [r-2]).$$

To prove this assertion we proceed by induction on k, where the base case $k = 1$ is easily verified since $e(G) > 0$. For the induction step, suppose that $1 \le k \le 2r-2$, and we have proved the result for k and wish to prove it for $k+1$. So $e(G) > (k+1)m^{r-1}$.

Case 1. $k \equiv i \not\equiv 0, r-1 \pmod r$ where $0 \le i < r$.

For each $f \in \partial G$ that has no vertex in X_i, delete the edge $f \cup \{v\} \in E(G)$ where v is the largest vertex in X_i in clockwise order for which $f \cup \{v\} \in E(G)$. Let G' be the subgraph that remains after deleting these edges. Then

$$e(G') \ge e(G) - m^{r-1} > (k+1)m^{r-1} - m^{r-1} = km^{r-1},$$

so by induction G' contains a P_k^r with vertices $v_1, \dots, v_k, \dots, v_{k+r-1}$, where $v_i \in X_i$ for $i \not\equiv 0, r-1 \pmod r$ and $v_i \in X$ for $i \equiv 0, r-1 \pmod r$. Our goal is to add a new vertex $v = v_{k+r} \in X_k$ to the end of the path to create a copy of P_{k+1}^r. Let v_{k+r} be the vertex in X_k for which the edge $e_{k+1} = \{v_{k+1}, \dots, v_{k+r}\}$ was deleted in forming G'. Note that v_{k+r} exists as $\{v_k, v_{k+1}, \dots, v_{k+r-1}\} \in E(G)$ and so $\{v_{k+1}, \dots, v_{k+r-1}\} \in \partial G$. Adding vertex v_{k+1} and edge e_{k+1} to our copy of P_k^r yields a copy of P_{k+1}^r since by definition of v_{k+r}, we have $v_k < v_{k+r}$.

Case 2. $k \equiv 0, r-1 \pmod r$.

Since $1 \le k \le 2r-2$, we conclude that $k \in \{r-1, r\}$. If $k = r$, we choose v to be the largest vertex in X in defining G' and apply the following argument similar to Case 1. By induction, G' contains a P_k^r with vertices $v_1, \dots, v_k, \dots, v_{k+r-1}$, where $v_i \in X_i$ for $i \not\equiv 0, r-1 \pmod r$ and $v_i \in X$ for $i \equiv 0, r-1 \pmod r$. Our goal is to add a new vertex $v = v_{k+r}$ to the end of the path where $v_{k+r} = v_{2r} \in X$. Note that we already have the three vertices $v_{r-1} < v_{2r-1} < v_r$ in X and we want to add $v_{2r} \in X$ satisfying $v_{r-1} < v_{2r-1} < v_r < v_{2r}$. But v_r satisfies this property by the way we defined G'. So we may add vertex v_{2r} to our P_k^r to obtain a crossing P_{k+1}^r.

We now assume that $k = r-1$. We modify the definition of G' slightly as follows: for every $f \in \partial G$ which has exactly one vertex in each X_i and in X, if w is the vertex of f in X, then delete $f \cup \{v\} \in E(G)$ where v is the largest such vertex in X satisfying $v < w$.

By induction, G' contains a P_k^r with vertices $v_1, \dots, v_k, \dots, v_{k+r-1}$, where $v_i \in X_i$ for $i \not\equiv 0, r-1$

(mod r) and $v_i \in X$ for $i \equiv 0, r-1 \pmod r$. Our goal is to add a new vertex $v = v_{k+r}$ to the end of the path where $v_{k+r} = v_{2r-1} \in X$.

Note that we already have two vertices $v_{r-1} < v_r$ in X. So we want to add v_{2r-1} satisfying $v_{r-1} < v_{2r-1} < v_r$. Since $\{v_{r-1}, \dots, v_{2r-2}\} \in E(G')$, the $(r-1)$-set $f = \{v_r, \dots, v_{2r-2}\}$ has exactly one vertex $v_r \in X$. Since $f \cup \{v_{r-1}\} = \{v_{r-1}, v_r, \dots, v_{2r-2}\} \in E(G')$, we have $f \in \partial G$ and moreover $f \cup \{v_{r-1}\}$ was not deleted in forming G'. Hence there is a vertex $v \in X$ with $v_{r-1} < v < v_r$ such that the edge $f \cup \{v\} = \{v_r, \dots, v_{2r-2}, v\} \in E(G)$ and the vertex v and edge $f \cup \{v\}$ can be used to extend the P_k^r to a P_{k+1}^r. □

Next we give lower bounds for $k \geq 2r$.

Proposition 4.2. *For $k \geq 2r \geq 4$ we have $\mathrm{ex}_{\circlearrowright}(n, P_k^r) = \Omega(n^{r-1} \log n)$.*

We take the same family $G(n, r, r+2)$ as used for ordered hypergraphs (see Section 3.4), but with the cyclic ordering of the vertex set. When we have a k-edge crossing path $P = w_1 w_2 \dots w_{r+k-1}$, the vertex w_1 does not need to be the leftmost in the first edge $\{w_1, \dots, w_r\}$, so the argument in Section 3.4 does not go through for $k = r+2$. In fact, $G(n, r, r+2)$ does contain P_k^r for $k \leq 2r-1$.

However, suppose $G(n, r, r+2)$ has a crossing $2r$-edge path $P = w_1 \dots w_{3r-1}$, and the ith edge of the path is $A_i = \{w_i, w_{i+1}, \dots, w_{i+r-1}\}$. Suppose vertex w_{r+j} is the leftmost in the set $\{w_r, w_{r+1}, \dots, w_{2r-1}\}$ (here $0 \leq j \leq r-1$). There are two cases.

Case 1: w_{2r+j-1} is to the right of w_{r+j-1}. (This is always the case for $j = 0$). Then writing the edges A_{j+1}, A_{j+r} and A_{j+r+1} as tuples with increasing coordinates, we have

$$A_{j+1} = (w_{j+r}, w_{j+1}, w_{j+2}, \dots, w_{j+r-1}), \quad A_{j+r} = (w_{j+r}, w_{j+r+1} \dots, w_{j+2r-1}),$$

$$\text{and} \quad A_{j+r+1} = (w_{j+2r}, w_{j+r+1}, w_{j+r+2}, \dots, w_{j+2r-1}).$$

The differences between the second and the first coordinates in these three tuples are

$$d_1 = w_{j+1} - w_{j+r}, \qquad d_2 = w_{j+r+1} - w_{j+r}, \qquad d_3 = w_{j+r+1} - w_{j+2r}.$$

Using a similar argument to that used at the end of Section 3.4, we conclude that it is impossible for all the differences d_1, d_2, d_3 to be powers of two.

Case 2: w_{2r+j-1} is to the left of w_{r+j} (so $j \geq 1$). Then $w_{2r+j-1} < w_j < w_{r+j}$. We now write down the four edges:

$$A_j = (w_j, w_{j+1}, w_{j+2}, \dots, w_{j+r-1}), \quad A_{j+1} = (w_{j+r}, w_{j+1}, w_{j+2}, \dots, w_{j+r-1}),$$

$$A_{j+r} = (w_{2r+j-1}, w_{j+r}, w_{j+r+1}, \dots, w_{j+2r-2}), \text{ and } A_{j+r+1} = (w_{j+2r-1}, w_{j+2r}, w_{j+r+1}, \dots, w_{j+2r-2}).$$

Now the four differences

$$d_1 = w_{j+1} - w_j,\ d_2 = w_{j+1} - w_{j+r},\ d_3 = w_{r+j} - w_{j+2r-1},\ d_4 = w_{j+2r} - w_{j+2r-1}$$

are powers of 2. Since $w_{j+2r-1} < w_j < w_{j+r} < w_{j+2r} < w_{j+1}$,

$$d_4 > d_3 \text{ and } d_1 > d_2.$$

Then $d_2, d_3 < \max\{d_1, d_4\} < d_2 + d_3$ which could not happen for powers of 2. □

Proof of Theorem 2.2. By Proposition 4.1, there exists $C = C_r$ such that

$$\mathrm{ex}_{\circlearrowright}(n, P^r_{2r-1}) \le C\, n^{r-1}$$

and hence $\mathrm{ex}_{\circlearrowright}(n, P^r_k) = O(n^{r-1})$ for all fixed $k \in [2r-1]$. Since there exists a family H of r-sets without edges covering the same $(r-1)$ subset twice and with size $e(H) = \Omega(n^{r-1})$ (see, e.g., [6]) we have for $k \ge 2$, $r \ge 2$, $\mathrm{ex}(n, P^r_k) = \Omega(n^{r-1})$. Since $\mathrm{ex}_{\circlearrowright}(n, P^r_k) \ge \mathrm{ex}(n, P^r_k)$, we get $\mathrm{ex}_{\circlearrowright}(n, P^r_k) = \Theta(n^{r-1})$ for $2 \le k \le 2r-1$. In the case $k = r+1$, Theorem 2.1 gives

$$\mathrm{ex}_{\circlearrowright}(n, P^r_{r+1}) \le \mathrm{ex}_{\rightarrow}(n, P^r_{r+1}) = \binom{n}{r} - \binom{n-r}{r}.$$

On the other hand, since $P^r_{r+1} \supseteq M^2_r$ and $G(n, r, r+1) \not\supseteq M^2_r$,

$$\mathrm{ex}_{\circlearrowright}(n, P^r_{r+1}) \ge \mathrm{ex}_{\circlearrowright}(n, M^2_r) = \mathrm{ex}_{\rightarrow}(n, M^2_r) \ge |G(n, r, r+1)| = \binom{n}{r} - \binom{n-r}{r},$$

so the second statement in Theorem 2.2 follows. It remains to consider $k \ge 2r$, and here we have

$$\mathrm{ex}_{\circlearrowright}(n, P^r_k) \le \mathrm{ex}_{\rightarrow}(n, P^r_k) = O(n^{r-1} \log n)$$

from Theorem 2.1 and the lower bound in Proposition 4.2. □

5 Proof of Theorem 2.3

5.1 Upper bound in Theorem 2.3 for $r \ge k \ge 2$

Let us first prove the upper bound

$$\mathrm{ex}_{\circlearrowright}(n, P^r_k) \le \frac{(k-1)(r-1)}{r} \binom{n}{r-1} \qquad (2 \le k \le r). \tag{11}$$

Recall that our notation for a crossing k-path P^r_k $(k \le r)$ in a cg r-graph with vertex set $\mathbf{\Omega}_n$ is the following: the vertices $v_1, v_2, \ldots, v_{r+k-1}$ form a tight path with edges $e_i = \{v_i, \ldots, v_{i+r-1}\}$, $i \in [k]$

and the (clockwise) ordering of the vertices in $\mathbf{\Omega}_n$ that belong to the tight path is

$$v_1 < v_{1+r} < v_2 < v_{2+r} < \cdots < v_{k-1} < v_{k-1+r} < v_k < v_{k+1} < \cdots < v_r \ \ (< v_1).$$

We define $T_k(H)$ to be the set of r-tuples $(v_k, \dots, v_{k+r-1}) \in V(H)^r$ for which there is a P_k^r in H with vertices $v_1, \dots, v_{k+r-1}$ as ordered above.

Theorem 5.1. *Let $r \geq 2$ and $1 \leq k \leq r$. Then for any cg r-graph H on $\mathbf{\Omega}_n$,*

$$|T_k(H)| \geq r \cdot e(H) - (r-1)(k-1) \cdot |\partial H|.$$

In particular, if H contains no P_k^r, then

$$e(H) \leq \frac{(k-1)(r-1)}{r}|\partial H| \leq \frac{(k-1)(r-1)}{r}\binom{n}{r-1}.$$

Proof. We proceed by induction on k. For $k = 1$ and each edge $e \in E(H)$, the number of copies of P_1^r with edge set $\{e\}$ is r, since after choosing which vertex of e to label with v_1, the order of the remaining vertices of e is determined (they are cyclically ordered). Therefore $|T_1(H)| \geq re(H)$. Suppose $k \geq 2$ and assume by induction that $|T_{k-1}(H)| \geq re(H) - (r-1)(k-2)|\partial H|$. Let L be the following collection of r-tuples in $T_{k-1}(H)$. The underlying elements of L are edges $e = \{x_k, \dots, x_{k+r-1}\} \in E(H)$ with clockwise ordering

$$x_{1+r} < x_{2+r} < \cdots < x_{k-1+r} < x_k < x_{k+1} < \cdots < x_r \ \ (< x_{1+r})$$

and there exist no vertices x such that $x_k < x < x_{k+1}$ and $e - \{x_k\} \cup \{x\} \in E(H)$.

Observe that $|L| \leq (r-1)|\partial H|$ since each element $e \in L$ as above yields a unique $(r-1)$-set $e - \{x_k\} \in \partial H$. Indeed, if $e - \{x_k\} = e - \{x_k'\}$ with $x_k < x_k'$ then $x_k < x_k' < x_{k+1}$ contradicting the definition of L. Each element of ∂H can be cyclically ordered in $r-1$ ways, giving the inequality $|L| \leq (r-1)|\partial H|$.

Our goal is to prove that $|T_k(H)| \geq |T_{k-1}(H) \backslash L|$ via an injection. Then, using the fact that $|L| \leq (r-1)|\partial H|$ and the induction hypothesis, we have

$$|T_k(H)| \geq |T_{k-1}(H) \backslash L| \geq r \cdot e(H) - (k-2)(r-1) \cdot |\partial H| - |L| \geq r \cdot e(H) - (k-1)(r-1) \cdot |\partial H|.$$

We must give an injection $f : T_{k-1}(H) \backslash L \to T_k(H)$. Suppose that $e \in T_{k-1}(H) \backslash L$ and the elements of e have clockwise ordering

$$x_{1+r} < x_{2+r} < \cdots < x_{k-1+r} < x_k < x_{k+1} < \cdots < x_r.$$

Then there exists a vertex x such that $x_k < x < x_{k+1}$ and $e - \{x_k\} \cup \{x\} \in E(H)$. Let A be the set of all such vertices x. Let y be the closest vertex to x_k among all vertices of A. In other words, $x_k < y < x$ for all $x \in A$. Let $f(e) = e - \{x_k\} \cup \{y\}$. Since $k \leq r$, we have $f(e) \in T_k(H)$ as we obtain a P_k^r that ends in $f(e)$ by taking the copy of P_{k-1}^r that ends in e and just adding the edge $f(e)$. Moreover, f is an injection, as if there is an $e' = e - \{x'_k\} \cup \{y'\}$ such that $f(e') = f(e)$ then we have $y = y'$. Since $x_k \neq x'_k$ we may assume that that $x_k < x'_k < y$. But then y would not have been the closest vertex to x_k in A. This contradiction shows that f is indeed an injection and the proof is complete. □

5.2 Lower bound in Theorem 2.3 for $r \geq k \geq 2$

Our next goal is to prove the following lower bound in Theorem 2.3 for $r \geq k \geq 2$:

$$\mathrm{ex}_{\circlearrowright}(n, P_k^r) \geq (1 + o(1))\frac{k-1}{3\ln 2r}\binom{n}{r-1}. \tag{12}$$

A *segment* of $\mathbf{\Omega}_n$ is a sequence of consecutive vertices in the ordering of $\mathbf{\Omega}_n$. A *gap* of an r-element subset R of $\mathbf{\Omega}_n$ is a segment of $\mathbf{\Omega}_n$ between two clockwise consecutive vertices of R that does not include the two vertices. The *length* of a gap is one more than the number of elements of $\mathbf{\Omega}_n$ in the gap. For $k \geq 2$, we say R has a (k, m)*-gap* if some $k - 1$ consecutive gaps of R all have length more than m — in other words, there are at least m vertices of $\mathbf{\Omega}_n$ in each gap. For example, if $n = 8$, $r = 4$, $R = \{v_1, v_2, v_5, v_8\}$ and the ordering of vertices is

$$\underline{v_1} < \underline{v_2} < v_3 < v_4 < \underline{v_5} < v_6 < v_7 < \underline{v_8},$$

then R has a $(3, 2)$-gap due to the 2 consecutive gaps v_3v_4 and v_6v_7. For $n > r$, let K_n^r be the family of all r-element subsets of $\mathbf{\Omega}_n$. For $n > r \geq k$, let $H(n, r, k, m)$ be the family of the members of K_n^r that have (k, m)-gaps, and $\overline{H}(n, r, k, m)$ be the family of the members of K_n^r that do not have (k, m)-gaps.

For a hypergraph H and $v \in V(H)$, let $H\{v\}$ denote the set of edges of H containing v.

Lemma 5.2. *If*

$$m \geq \frac{(n-1)\ln 2r}{(r-1)(k-1)}, \tag{13}$$

then

$$|H(n, r, k, m)| \leq \frac{1}{2}\binom{n}{r}. \quad \textit{Equivalently, } |\overline{H}(n, r, k, m)| \geq \frac{1}{2}\binom{n}{r}. \tag{14}$$

Proof. Instead of proving (14) directly, it will be easier to prove that

$$\forall\, j \in \mathbf{\Omega}_n, \quad |H(n, r, k, m)\{j\}| \leq \frac{1}{2}|K_n^r\{j\}| = \frac{1}{2}\binom{n-1}{r-1}; \tag{15}$$

and (15) implies (14) because $|H(n, r, k, m)| = \frac{n}{r}|H(n, r, k, m)\{j\}|$ and $\binom{n}{r} = \frac{n}{r}|K_n^r\{j\}|$.

Let $\mathbf{\Omega}_n = [n]$. By symmetry, it is enough to prove (15) for $j = n$. First, we show that

$$|H(n,r,k,m)\{n\}| \le r|K^r_{n-(k-1)m}\{n-(k-1)m\}|. \tag{16}$$

Indeed, from each $F \in H(n,r,k,m)\{n\}$, we can get an $F' \in K^r_{n-(k-1)m}\{n-(k-1)m\}$ by deleting the first m vertices in $k-1$ consecutive gaps of length at least $m+1$, and renumbering the remaining $n-(k-1)m$ vertices so that the vertex n of $\mathbf{\Omega}_n$ will be $n-(k-1)m$. On the other hand, each $F' \in K^r_{n-(k-1)m}\{n-(k-1)m\}$ can be obtained this way from r distinct $F \in H(n,r,k,m)\{n\}$. This proves (16).

Now, using $1-x \le e^{-x}$, (16) and (13) yield

$$|H(n,r,k,m)\{n\}| \le r\binom{n-1-(k-1)m}{r-1} = r\binom{n-1}{r-1}\prod_{i=1}^{r-1}\frac{n-(k-1)m-i}{n-i}$$

$$\le r\binom{n-1}{r-1}\exp\Big(-\frac{(k-1)m(r-1)}{n-1}\Big) \le r\binom{n-1}{r-1}\frac{1}{2r},$$

implying (15). $\square$

We are ready to prove (12). Let

$$t = t(r,k) = \left\lceil \frac{(r-1)(k-1)}{\ln 2r} \right\rceil.$$

Suppose $n > r \ge k \ge 2$. If $r = 2$, then $k = 2$, and the bound is trivial; so let $r \ge 3$. Suppose first that t divides n and let $m = n/t$. Then m satisfies (13). By rotating $\mathbf{\Omega}_n$ we find a subgraph H' of $\overline{H}(n,r,k,m)$ with at least $|\overline{H}(n,r,k,m)|/m$ edges such that every edge of H' adds up to zero modulo m. We claim that

$$H' \text{ does not contain a crossing } P^r_k. \tag{17}$$

Indeed, assume H' contains a crossing P^r_k with the vertices $v_0, v_1, \ldots, v_{k+r-2}$. By the definition of crossing paths, $v_0 < v_r < v_1 < v_{1+r} < \cdots < v_{k-1} < v_{k-1+r} < v_k$. Since the set $\{v_1, v_2, \ldots, v_{r-1}\}$ forms an edge together with both v_0 and v_r, $v_r \equiv v_0 \mod m$. Similarly, $v_{r+i} \equiv v_i \mod m$ for all $i < k$. But this means that the edge $\{v_0, v_1, \ldots, v_{r-1}\}$ has $k-1$ consecutive gaps of length more than m, thus it does not belong to $\overline{H}(n,r,k,m)$. This contradiction proves (17).

Thus if $r \ge 3$, $2 \le k \le r$ are fixed, n is a large number divisible by t and $m = n/t$, then by (17) and (14), H' is a cg r-graph not containing a crossing P^r_k with

$$|H'| \ge \frac{1}{2m}\binom{n}{r} \ge \frac{t}{2r}\binom{n-1}{r-1} \ge \frac{(k-1)(r-1)}{2r\ln 2r}\binom{n-1}{r-1} \ge (1+o(1))\frac{k-1}{3\ln 2r}\binom{n}{r-1}.$$

If n is not divisible by t, then let n' be the largest positive integer divisible by t such that $n' \le n$.

Then

$$\mathrm{ex}_{\circlearrowright}(n, P_k^r) \ge \mathrm{ex}_{\circlearrowright}(n', P_k^r) \ge (1+o(1))\frac{k-1}{3\ln 2r}\binom{n'}{r-1} = (1+o(1))\frac{k-1}{3\ln 2r}\binom{n}{r-1}. \quad \square$$

5.3 The case $k = 2$

Here we prove the upper bound (3), namely:

$$\mathrm{ex}_{\circlearrowright}(n, P_2^r) \le \frac{1}{2}\binom{n}{r-1}.$$

Recall that P_2^r on $\mathbf{\Omega}_n$ has vertex set

$$v_1 < v_{1+r} < v_2 < v_3 < \cdots < v_r \ \ (< v_1),$$

and edges $\{v_1, \ldots, v_r\}$ and $\{v_2, \ldots, v_{r+1}\}$. Consider a P_2^r-free cg r-graph H with vertex set $\mathbf{\Omega}_n$. Label the vertices of an $e \in E(H)$ as

$$1 \le a_1 < a_2 < \cdots < a_r \le n,$$

and define $T_1(e) := e \setminus \{a_1\}$ and $T_2(e) := e \setminus \{a_r\}$. Since H is P_2^r-free, we have $T_\alpha(e) \ne T_\alpha(e')$ for $e \ne e' \in H$ (and $\alpha = 1, 2$). Indeed, if we take (in case of $\alpha = 1$) $v_2, \ldots, v_r = a_2, \ldots, a_r$ and $\{v_1, v_{r+1}\} = \{a_1, a_1'\}$ then we obtain a P_2^r.

We also have $T_1(e) \ne T_2(e')$, otherwise we define $\{v_1, v_{r+1}\} = \{a_1, a_r'\}$ and again obtain a forbidden path. This way we associated two $(r-1)$-sets to each member of H, yielding (3). $\square$

5.4 The case $k = r$

Here we prove (4), namely:

$$\mathrm{ex}_{\circlearrowright}(n, P_r^r) > (1 - o(1))(r-2)\binom{n}{r-1}.$$

Recall that $P_r^r = v_1 v_2 \ldots v_{2r-1}$ with clockwise ordering

$$v_1 < v_{1+r} < v_2 < v_{2+r} < v_3 < \cdots < v_{r-1} < v_{2r-1} < v_r \ \ (< v_1), \tag{18}$$

and edge set $\{e_i = \{v_i, v_{i+1}, \ldots, v_{i+r-1}\} : i \in [r]\}$. Assume the underlying vertex set is $\mathbf{\Omega}_n$. By (18),

for every $1 \le i \le r$, the only vertices in e_i that can be consecutive on $\mathbf{\Omega}_n$ are v_{i+r-1} and v_i. (19)

Assume that the n vertices of $\mathbf{\Omega}_n$ are arranged in clockwise order as $1 < \cdots < n$. Let G be the following cg r-graph with vertex set $\mathbf{\Omega}_n$. An r-set e belongs to $E(G)$ if its elements are ordered as

$$1 \le a_1 < a_2 < \cdots < a_r \le n, \tag{20}$$

and there exists a unique $1 \le t \le r-2$ with $a_t + 1 = a_{t+1}$, and otherwise the a_js are separated. (This also means $\{a_1, a_r\} \neq \{1, n\}$). A quick calculation gives $|E(G)| = (r-2)\binom{n}{r-1} + O(n^{r-2})$.

We claim that G does not contain a P_r^r. Suppose, on the contrary, that $F \subset G$ is a copy of P_r^r as described in (18). Since each member of G (so each member of F as well) contains a unique consecutive pair of $\mathbf{\Omega}_n$, we get that the pairs in (19) should be consecutive. If v_i with $i \in [r]$ is the smallest in $\{v_1, \ldots, v_{2r-1}\}$ then v_{i+r-1} is the largest, so they are separated, they could not form a consecutive pair in e_i when we write e_i in the form $(a_1, \ldots, a_r)$ as in (20). If v_{i+r} with $i \in [r-1]$ is the smallest in $\{v_1, \ldots, v_{2r-1}\}$ then v_{i+r-1} and v_i are the largest, so they could not be consecutive in e_i by the definition of G. $\square$

6 Proof of Theorem 2.5

The upper bound for case $k = 2$ follows from Theorem 2.1 and the cg r-graph $G(n, r, r+1)$ in subsection 3.2 provides the matching lower bound.

We are to show that for $k, r > 2$,

$$(k-1)r\binom{n}{r-1} + O(n^{r-2}) \le \mathrm{ex}_{\circlearrowright}(n, M_k^r) = \mathrm{ex}_{\rightarrow}(n, M_k^r) < 2(k-1)(r-1)\binom{n}{r-1}.$$

A simple construction demonstrating the lower bound in Theorem 2.5 is the following cg r-graph: Fix a $(k-1)$-set $K \subset \mathbf{\Omega}_n$ and let A be the set of r-sets of $\mathbf{\Omega}_n$ that contain at least one vertex from K. Let B be the set of r-sets of $\mathbf{\Omega}_n$ for which some two consecutive vertices have a gap of length at most $k-2$ (this means that there are at most $k-2$ vertices between them in clockwise order, not including endpoints). Note that $|A| = (k-1)\binom{n}{r-1} + O(n^{r-2})$, $|B| = (r-1)(k-1)\binom{n}{r-1} + O(n^{r-2})$ and $|A \cap B| = O(n^{r-2})$. The cg r-graph with vertex set $\mathbf{\Omega}_n$ and edge set $A \cup B$ has $(k-1)r\binom{n}{r-1} + O(n^{r-2})$ edges and it is easy to see that it does not contain M_k^r.

For the upper bound, let H be a cg r-graph with the maximum number of edges on vertex set $\mathbf{\Omega}_n$ with no M_k^r. A *chord* is a line segment joining two vertices of $\mathbf{\Omega}_n$ and its length is one more than the size of the (smallest) gap between the two vertices. For each edge A, choose a shortest chord $ch(A)$, say $v_r v_1$ and view the vertices of A as $v_1, v_2, \ldots, v_r$ in clockwise order. Define the *type of* A to be the vector $\mathbf{t}(A) = (t_1, \ldots, t_{r-1})$ where

$$t_i = v_{i+1} - v_i \text{ for } i = 1, \ldots, r-2 \text{ and } t_{r-1} = n - (t_1 + \ldots + t_{r-2}) = v_1 - v_{r-1}.$$

The coordinates of each vector $\mathbf{t}(A)$ are positive integers, $t_{r-1}(A) \ge 2$, and $t_1(A) + \ldots + t_{r-1}(A) = n$

for each A by definition. The number of such vectors is *exactly* $\binom{n-2}{r-2}$ (because this is equal to the number of ways to mark $r-2$ out of the $n-1$ separators in an ordered set of n dots so that the last separator is not marked). For every given type $\mathbf{t} = (t_1, \dots, t_{r-1})$, the family $H(\mathbf{t})$ of the chords $ch(A)$ of the edges A of type $\mathbf{t}$ does not contain k crossing chords. Thus by Theorem 2.4, $|H(\mathbf{t})| < 2(k-1)n$. Hence, using $r \geq 3$,

$$|E(H)| < 2(k-1)n\binom{n-2}{r-2} = 2(k-1)\frac{(r-1)(n-r+1)}{n-1}\binom{n}{r-1} < 2(k-1)(r-1)\binom{n}{r-1},$$

as claimed. □

7 Concluding remarks

• A hypergraph F is a *forest* if there is an ordering of the edges $e_1, e_2, \dots, e_t$ of F such that for all $i \in \{2, 3, \dots, t\}$, there exists $h < i$ such that $e_i \cap \bigcup_{j<i} e_j \subseteq e_h$. It is not hard to show that $\mathrm{ex}(n, F) = O(n^{r-1})$ for each r-uniform forest F. It is therefore natural to extend the Pach-Tardos Conjecture A to r-graphs as follows:

Conjecture B. *Let $r \geq 2$. Then for any ordered r-uniform forest F with interval chromatic number r, $\mathrm{ex}_{\to}(n, F) = O(n^{r-1} \cdot \mathrm{polylog}\, n)$.*

Theorem 3.1 shows that to prove Conjecture B, it is enough to consider the setting of r-graphs of interval chromatic number r. Theorem 2.1 verifies this conjecture for crossing paths, and also shows that the $\log n$ factor in Theorem 3.1 is necessary. It would be interesting to find other general classes of ordered r-uniform forests for $r \geq 3$ for which Conjecture B can be proved. A related problem is to determine for which ordered forests F we have $\mathrm{ex}_{\to}(n, F) = O(n^{r-1})$? This is a hypergraph generalization of Braß' question [2] which was solved recently for trees [9].

• It appears to be substantially more difficult to determine the exact value of the extremal function for r-uniform crossing k-paths in the convex geometric setting than in the ordered setting. It is possible to show that for $k \leq 2r-1$,

$$c(k, r) = \lim_{n\to\infty} \frac{\mathrm{ex}_{\circlearrowright}(n, P_k^r)}{\binom{n}{r-1}}$$

exists. We do not as yet know the value of $c(k, r)$ for any pair (k, r) with $2 \leq k \leq r$, even though in the ordered setting Theorem 2.1 captures the exact value of the extremal function for all $k \leq r+1$, and $c(r+1, r) = r$.

• One can consider more general orderings of tight paths, namely instead of the vertices whose subscripts are congruent to a modulo r increasing within an interval, we can specify which congruence classes of vertices are increasing within their interval and which are decreasing. Our methods can handle such situations as well.

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