The Erdős-Szekeres problem and an induced Ramsey question

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Abstract

Motivated by the Erdős-Szekeres convex polytope conjecture in \( \mathbb{R}^d \), we initiate the study of the following induced Ramsey problem for hypergraphs. Given integers \( n > k \geq 5 \), what is the minimum integer \( g_k(n) \) such that any \( k \)-uniform hypergraph on \( g_k(n) \) vertices with the property that any set of \( k+1 \) vertices induces 0, 2, or 4 edges, contains an independent set of size \( n \). Our main result shows that \( g_k(n) > 2cn^{k-4} \), where \( c = c(k) \).

1 Introduction

Given a finite point set \( P \) in \( d \)-dimensional Euclidean space \( \mathbb{R}^d \), we say that \( P \) is in general position if no \( d+1 \) members lie on a common hyperplane. Let \( ES_d(n) \) denote the minimum integer \( N \), such that any set of \( N \) points in \( \mathbb{R}^d \) in general position contains \( n \) members in convex position, that is, \( n \) points that form the vertex set of a convex polytope. In their classic 1935 paper, Erdős and Szekeres [1] proved that in the plane, \( ES_2(n) \leq 4^n \). In 1960, they [2] showed that \( ES_2(n) \geq 2^{n-2}+1 \) and conjectured this to be sharp for every integer \( n \geq 3 \). Their conjecture has been verified for \( n \leq 6 \) [1, 8], and determining the exact value of \( ES_2(n) \) for \( n \geq 7 \) is one of the longest-standing open problems in Ramsey theory/discrete geometry. Recently [9], the second author asymptotically verified the Erdős-Szekeres conjecture by showing that \( ES_2(n) = 2^{n+o(n)} \).

In higher dimensions, \( d \geq 3 \), much less is known about \( ES_d(n) \). In [3], Károlyi showed that projections into lower-dimensional spaces can be used to bound these functions, since most generic projections preserve general position, and the preimage of a set in convex position must itself be in convex position. Hence, \( ES_d(n) \leq ES_2(n) = 2^{n+o(n)} \). However, the best known lower bound for \( ES_d(n) \) is only on the order of \( 2^{n^{1/(d-1)}} \), due to Károlyi and Valtr [4]. An old conjecture of Füredi (see Chapter 3 in [5]) says that this lower bound is essentially the truth.

Conjecture 1.1. For \( d \geq 3 \), \( ES_d(n) = 2^{\Theta(n^{1/(d-1)})} \).

It was observed by Motzkin [6] that any set of \( d+3 \) points in \( \mathbb{R}^d \) in general position contains either 0, 2, or 4 \((d+2)\)-tuples not in convex position. By defining a hypergraph \( H \) whose vertices are \( N \)

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points in $\mathbb{R}^d$ in general position, and edges are $(d + 2)$-tuples not in convex position, then every set of $d + 3$ vertices induces $0$, $2$, or $4$ edges. Moreover, by Carathéodory’s theorem (see Theorem 1.2.3 in [5]), an independent set in $H$ would correspond to a set of points in convex position. This leads us to the following combinatorial parameter.

Let $g_k(n)$ be the minimum integer $N$ such that any $k$-uniform hypergraph on $N$ vertices with the property that every set of $k + 1$ vertices induces $0$, $2$, or $4$ edges, contains an independent set of size $n$. For $k \geq 5$, the geometric construction of Károlyi and Valtr [4] mentioned earlier implies that

$$g_k(n) \geq ES_{k-2}(n) \geq 2cn^{1/(k-3)},$$

where $c = c(k)$. One might be tempted to prove Conjecture 1.1 by establishing a similar upper bound for $g_k(n)$. However, our main result shows that this is not possible.

**Theorem 1.2.** For each $k \geq 5$ there exists $c = c(k) > 0$ such that for any $n \geq k$ we have $g_k(n) > 2^{cn^{k-4}}$.

In the other direction, we can bound $g_k(n)$ from above as follows. For $n \geq k \geq 5$ and $t < k$, let $h_k(t, n)$ be the minimum integer $N$ such that any $k$-uniform hypergraph on $N$ vertices with the property that any set of $k + 1$ vertices induces at most $t$ edges, contains an independent set of size $n$. In [7], the authors proved the following.

**Theorem 1.3 ([7]).** For $k \geq 5$ and $t < k$, there is a positive constant $c' = c'(k, t)$ such that

$$h_k(t, n) \leq \text{twr}_t(c'n^{k-t} \log n),$$

where $\text{twr}$ is defined recursively as $\text{twr}_1(x) = x$ and $\text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}$.

Hence, we have the following corollary.

**Corollary 1.4.** For $k \geq 5$, there is a constant $c' = c'(k)$ such that

$$g_k(n) \leq h_k(4, n) \leq 2^{2^{c' n^{k-4} \log n}}.$$

It is an interesting open problem to improve either the upper or lower bounds for $g_k(n)$.

**Problem 1.5.** Determine the tower growth rate for $g_k(n)$.

Actually, this Ramsey function can be generalized further as follows: for every $S \subset \{0, 1, \ldots, k\}$, define $g_k(n, S)$ to be the minimum integer $N$ such that any $N$-vertex $k$-uniform hypergraph with the property that every set of $k + 1$ vertices induces $s$ edges for some $s \in S$, contains an independent set of size $n$. General results for $g_k(n, S)$ may shed light on classical Ramsey problems, but it appears difficult to determine even the tower height for any nontrivial cases.
2 Proof of Theorem 1.2

Let $k \geq 5$ and $N = 2^{c_n^{k-4}}$ where $c = c_k > 0$ is sufficiently small to be chosen later. We are to produce a $k$-uniform hypergraph $H$ on $N$ vertices with $\alpha(H) < n$ and every $k+1$ vertices of $H$ span 0, 2, or 4 edges. Let $\phi: [N]_{k-3} \to (k^{-1})$ be a random $(k^{-1})$-coloring, where each color appears on each $(k-3)$-tuple independently with probability $1/(k^{-1})$. For $f = (v_1, \ldots, v_{k-1}) \in ([N]_{k-1})$, where $v_1 < v_2 < \cdots < v_{k-1}$, define the function $\chi_f: ([N]_{k-3}) \to (k^{-1})$ as follows: for all $\{i, j\} \in (k^{-1})$, let

$$\chi_f(f \setminus \{v_i, v_j\}) = \{i, j\}.$$

We define the $(k-1)$-uniform hypergraph $G$, whose vertex set is $[N]$, such that

$$G = G_\phi := \left\{ f \in ([N]_{k-1}) : \phi(f \setminus \{u, v\}) = \chi_f(f \setminus \{u, v\}) \text{ for all } \{u, v\} \in \binom{f}{2} \right\}.$$

For example, if $k = 4$ (which is excluded for the theorem but we allow it to illustrate this construction) then $\phi: [N] \to \{12, 13, 23\}$ and for $f = (v_1, v_2, v_3)$, where $v_1 < v_2 < v_3$, we have $f \in G$ iff $\phi(v_1) = 23, \phi(v_2) = 13$, and $\phi(v_3) = 12$.

Given a subset $S \subset [N]$, let $G[S]$ be the subhypergraph of $G$ induced by the vertex set $S$. Finally, we define the $k$-uniform hypergraph $H$, whose vertex set is $[N]$, such that

$$H = H_\phi := \left\{ e \in \binom{[N]}{k} : |G[e]| \text{ is odd} \right\}.$$

Claim 2.1. $|H[S]|$ is even for every $S \in \binom{[N]}{k+1}$.

Proof. Let $S \in \binom{[N]}{k+1}$ and suppose for contradiction that $|H[S]|$ is odd. Then

$$2|G[S]| = \sum_{f \in G[S]} \sum_{e \in (k)^{f \setminus 1}} 1 = \sum_{e \in (k)^{H[S]}} |G[e]| = \sum_{e \in H[S]} |G[e]| + \sum_{e \in H[S]} |G[e]|.$$

The first sum on the RHS above is even by definition of $H$ and the second sum is odd by definition of $H$ and the assumption that $|H[S]|$ is odd. This contradiction completes the proof.

Claim 2.2. $|G[e]| \leq 2$ for every $e \in \binom{[N]}{k}$.

Proof. For sake of contradiction, suppose that for $e = (v_1, \ldots, v_k)$, where $v_1 < \cdots < v_k$, we have $|G[e]| \geq 3$. Let $e_p = e \setminus \{v_p\}$ for $p \in [k]$ and suppose that $e_i, e_j, e_l \in G$ with $i < j < l$. In what follows, we will find a set $S$ of size $k-3$, where $S \subset e_i$ and $S \subset e_l$, such that $\chi_{e_i}(S) \neq \chi_{e_l}(S)$. This will give us our contradiction since $e_i, e_l \in G$ implies that $\chi_{e_i}(S) = \phi(S) = \chi_{e_l}(S)$.

Let $Y = e \setminus \{v_i, v_j, v_l\}$ and $Y' = Y \setminus \{\min Y\}$. Let us first assume that $i > 1$ so that $\min Y = v_1$. In this case,

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{1, l - 1\},$$
since we obtain $Y' \cup \{v_j\}$ from $e_i$ by removing $\min Y$ and $v_l$ which are the first and $(l-1)$st elements of $e_i$. Similarly,

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{1, i\},$$

since we obtain $Y' \cup \{v_j\}$ from $e_i$ by removing $\min Y$ and $v_l$ which are the first and $i$th elements of $e_i$. Because $l > i + 1$, we conclude that $\chi_{e_i}(Y' \cup \{v_j\}) \neq \chi_{e_i}(Y' \cup \{v_j\})$ as desired.

Next, we assume that $i = 1$ and $\min Y = v_q$ where $q > 1$. In this case,

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{q - 1, l - 1\},$$

since we obtain $Y' \cup \{v_j\}$ from $e_i$ by removing $v_q$ and $v_l$ which are the $(q-1)$st and $(l-1)$st elements of $e_i$. Similarly,

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{1, q'\} \quad \text{where} \quad q' = q \text{ if } q < l \text{ and } q' = q - 1 \text{ if } q > l,$$

since we obtain $Y' \cup \{v_j\}$ from $e_i$ by removing $v_i = v_1$ and $v_q$ which are the first and $q'$th elements of $e_i$. If $q \neq 2$, then we immediately obtain $\chi_{e_i}(Y' \cup \{v_j\}) \neq \chi_{e_i}(Y' \cup \{v_j\})$ as desired. On the other hand, if $q = 2$, then $q' = q = 2$ as well and $l \geq 4$, so $l - 1 \neq q'$ and again

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{q - 1, l - 1\} \neq \{1, q'\} = \chi_{e_i}(Y' \cup \{v_j\}).$$

This completes the proof of the claim. \hfill \Box

Let $T_3$ be the $(k-1)$-uniform hypergraph with vertex set $S$ with $|S| = k + 1$ and three edges $e_1, e_2, e_3$ such that there are three pairwise disjoint pairs $p_1, p_2, p_3 \in \binom{S}{2}$ with $p_i = \{v_i, v'_i\}$ and $e_i = S \setminus p_i$ for $i \in \{1, 2, 3\}$.

**Claim 2.3.** $T_3 \not\subset G$.

**Proof.** Suppose for a contradiction that there is a subset $S \subset [N]$ of size $k + 1$ such that $T_3 \subset G[S]$. Using the notation above, assume without loss of generality that $v_1 = \min \cup p_1$ and $v_2 = \min (p_2 \cup p_3)$. Let $Y = S \setminus (p_1 \cup p_3)$ and note that $Y \in \binom{e_i \cap \cup_{i=3}^{k-3}}{k-3}$. Let $Y_1 \subset Y$ be the set of elements in $Y$ that are smaller than $v_1$, so we have the ordering

$$Y_1 < v_1 < v_2 < \{v_3, v'_3\}.$$

Now, $\chi_{e_1}(Y)$ is the pair of positions of $v_3$ and $v'_3$ in $e_1$. Both of these positions are at least $|Y_1| + 2$ as $Y_1 \cup \{v_2\}$ lies before $p_3$. On the other hand, the smallest element of $\chi_{e_3}(Y)$ is $|Y_1| + 1$ which is the position of $v_1$ in $e_3$. This shows that $\chi_{e_1}(Y) \neq \chi_{e_3}(Y)$, which is a contradiction as both must be equal to $\phi(Y)$ as $e_1, e_3 \subset G$. \hfill \Box

We now show that every $(k+1)$-set $S \subset [N]$ spans 0, 2 or 4 edges of $H$. By Claim 2.1, $|H[S]|$ is even. Let $G'$ be the graph with vertex set $S$ and edge set $\{S \setminus f : f \in G[S]\}$. So there is a 1-1 correspondence between $G[S]$ and $G'$ via the map $f \rightarrow S \setminus f$. If $G'$ has a vertex $x$ of degree at least three, then $|G[S \setminus \{x\}]| \geq 3$ which contradicts Claim 2.2. Therefore $G'$ consists of disjoint paths, cycles, and isolated vertices. This implies that a $k$-set $A \subset S$ is an edge in $H$ exactly when $S \setminus A$ is a vertex of degree 1 in $G'$. Next, observe that Claim 2.3 implies that $G'$ does not contain a
matching of size three, for the complementary sets of this matching yield a copy of $T_3 \subseteq G$. Hence, the number of degree 1 vertices in $G'$ is 0, 2, or 4, and therefore $|H[S]| \in \{0, 2, 4\}$ for all $S \in \binom{[N]}{k+1}$.

Let us now argue that $\alpha(H) < n$, which is a straight-forward application of the probabilistic method. Indeed, we will show that this happens with positive probability and conclude that an $H$ with this property exists. For a given $k$-set, the probability that it is an edge of $H$ is $p > 0$, where $p$ depends only on $k$. Consequently, the probability that $H$ has an independent set of size $n$ is at most

$$\binom{N}{n} (1-p)^{c'n^{k-3}}$$

for some $c' > 0$. Note that the exponent $k - 3$ above is obtained by taking a partial Steiner $(n, k, k - 3)$ system $S$ within a potential independent set of size $n$ and observing that we have independence within the edges of $S$. A short calculation shows that this probability is less than 1 as long as $c$ is sufficiently small. This completes the proof of Theorem 1.2.

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References