On the order of the classical Erdős-Rogers functions

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Abstract

For an integer $n \geq 1$, the Erdős-Rogers function $f_{s,s+1}(n)$ is the maximum integer m such that every n-vertex K_{s+1}-free graph has a K_s-free induced subgraph with m vertices. It is known that for all $s \geq 3$, $f_{s,s+1}(n) = \Omega(\sqrt{n \log n}/\sqrt{\log \log n})$ as $n \to \infty$. In this paper, we show that for all $s \geq 3$, there exists a constant $c_s > 0$ such that

$$f_{s,s+1}(n) \leq c_s \sqrt{n}\, \log n.$$

This improves previous bounds of order $\sqrt{n}\,(\log n)^{4s^2}$ by Dudek, Retter and Rödl and answers a question of Warnke.

1 Introduction

Given integers $t > s \geq 2$, the *Erdős-Rogers function* $f_{s,t}(n)$ is the maximum integer m such that every n-vertex K_t-free graph has a K_s-free induced subgraph with m vertices. These quantities in the case $t = s+1$ were studied by Erdős and Rogers [10] more than sixty years ago while addressing a question of Hajnal, and are generalizations of the classical Ramsey numbers $r(s,t)$. The study of $f_{s,t}(n)$ for $t > s+1$ has received considerable attention in the recent literature [7, 8, 12, 14, 16, 25, 26], but there is no pair (s,t) with $t > s+1 \geq 4$ for which it is known that $f_{s,t}(n) = n^{\alpha+o(1)}$ for some $\alpha = \alpha(s,t)$.

In this paper we focus on the classical case $f_{s,s+1}(n)$. The determination of $f_{2,3}(n)$ is almost equivalent to determining the triangle-complete graph Ramsey numbers, since $r(3, f_{2,3}(n)) \leq n < r(3, f_{2,3}(n)+1)$. These quantities $r(3,t)$ are known to within a constant factor [1, 5, 4, 11, 15, 23], and from this one deduces $f_{2,3}(n)$ has order of magnitude $\sqrt{n \log n}$ as $n \to \infty$. As observed by Dudek and the first author, the arguments for lower bounds for $f_{2,3}(n)$ generalize to $f_{s,s+1}(n)$ for $s \geq 3$ as follows. Shearer [24] showed that any n-vertex K_{s+1}-free graph of maximum degree d has an independent set of size $\Omega(n \log d / d \log \log d)$ as $d \to \infty$, and the neighborhood of a vertex of degree d is a K_s-free induced subgraph. Therefore for all $s \geq 3$,

$$f_{s,s+1}(n) \geq \max\left\{d, \Omega\left(\frac{n \log d}{d \log \log d}\right)\right\} = \Omega\Big(\frac{\sqrt{n \log n}}{\sqrt{\log \log n}}\Big). \tag{1}$$

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In a breakthrough paper, building on earlier works of Dudek and Rödl, Wolfovitz [28] proved $f_{3,4}(n) = O(\sqrt{n}(\log n)^{120})$, thereby showing $f_{3,4}(n) = \sqrt{n}^{1+o(1)}$. Soon after, Dudek, Retter, Rödl [8] improved this to $f_{3,4}(n) = O(\sqrt{n}\,(\log n)^{32})$ and showed more generally that $f_{s,s+1}(n) = O(\sqrt{n}\,(\log n)^{4s^2})$. Warnke [27] asked whether there exists a constant $c > 0$ such that for all $s \geq 3$, one has $f_{s,s+1}(n) \leq \sqrt{n}(\log n)^c$. In this short paper, we answer this question and significantly improve the afore-mentioned bounds on the Erdős-Rogers functions $f_{s,s+1}(n)$ as follows:

Theorem 1. *For each fixed $s \geq 3$, there exists a constant $c_s > 0$ such that*

$$f_{s,s+1}(n) \quad \leq \quad c_s\sqrt{n}\,\log n. \tag{2}$$

We did not expend too much effort in optimizing the constant c_s in Theorem 1; from the proof one may obtain $c_s \leq 2^{100s}$, which incidentally shows $f_{s,s+1}(n) = n^{1/2+o(1)}$ for $s = o(\log n)$. Dudek, Retter and Rödl [8] asked whether for $s \geq 3$, $f_{s,s+1}(n)/f_{s-1,s}(n)$ is unbounded as $n \to \infty$. This would not be the case for $s \geq 4$ if our upper bound is tight up to a constant factor. We tentatively make the following conjecture:

Conjecture 1. *For all fixed $s \geq 3$, as $n \to \infty$,*

$$f_{s,s+1}(n) = \sqrt{n}(\log n)^{1-o(1)}.$$

This would be in contrast to $f_{2,3}(n)$, which has order of magnitude $\sqrt{n\log n}$ as $n \to \infty$. The key barrier in proving lower bounds on say $C_p(n)$ is showing that in a K_4-free graph one can typically find induced triangle-free subgraphs with substantially more vertices than the independence number of the graph. In particular, a result of Shearer [24] shows every n-vertex K_4-free graph of maximum degree d has an independent set of size $\Omega(n\log d/d\log\log d)$ as $d \to \infty$, so we propose the following:

Conjecture 2. *There exists $\delta > 0$ and d_0 such that if G is an n-vertex K_4-free graph of maximum degree $d > d_0$, then G contains a triangle-free induced subgraph H with*

$$|V(H)| \geq \frac{n\,(\log d)^{1+\delta}}{d}.$$

Conjecture 2 would imply $f_{3,4}(n) = \Omega(\sqrt{n}(\log n)^{1/2+\delta/2})$. We note that the graph constructed in this paper shows that δ cannot be larger than $1/3$ (see the concluding remarks for more details).

The proof of Theorem 1 uses the framework of Wolfovitz [28] and Dudek, Retter, Rödl [8], but requires substantial new ideas, including the construction of Mattheus and the second author [18] based on Hermitian unitals. It also requires some additional technical steps in the probabilistic analysis to obtain more careful control of dependencies in order to use the local lemma (see section 4.3). We remark that we avoid the use of the method of containers as in [18] or [14], which tends to incur the loss of further logarithmic factors.

Notation. For a graph G, we write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G. For a set $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G *induced* by X, namely the graph with vertex set X and edge set $\{e \in E(G) : e \subseteq X\}$.

2 Tools from probability

We refer to the book by Alon and Spencer [2] for a reference on probabilistic methods in combinatorics. We make use of three standard tools in probabilistic combinatorics; the first is the well-known *Chernoff Bound*:

Proposition 1. (Chernoff Bound) *Let Z be a binomial random variable with mean μ. Then for any real $\epsilon \in [0,1]$,*

$$\begin{aligned} \Pr(Z > (1+\epsilon)\mu) &\le \exp\Big(-\frac{\epsilon^2\mu}{4}\Big) \qquad \text{and} \\ \Pr(Z < (1-\epsilon)\mu) &\le \exp\Big(-\frac{\epsilon^2\mu}{2}\Big) \end{aligned} \tag{3}$$

The next proposition is derived in a standard way from *Janson's inequality* [2, 13], and we give a proof in the appendix. The technical details serve mainly to obtain an explicit constant in the upper bound in Theorem 1 that is exponential in s. Let χ be a coloring of an n-element set Y with s colors, with color classes $Y_1, Y_2, \dots, Y_s$. Then the *random s-partite graph $G_{n,\rho}(\chi)$* samples edges independently with probability ρ from the complete multipartite graph with parts $Y_1, Y_2, \dots, Y_s$.

Proposition 2. (via Janson's inequality) *Let $s \ge 3$, $n \ge 2^{40s}$ and $\rho = (8s/n)^{2/s}$, and let χ be an s-coloring of an n-element set whose color classes have size at least $n/2s$ each. Then*

$$\Pr(K_s \not\subseteq G_{n,\rho}(\chi)) \le \exp(-2^{2s-4}n). \tag{4}$$

We shall finally require the *Lovász local lemma* [2, 9] in the following form. We write $\overline{A}$ for the complement of an event A in a probability space.

Proposition 3. (Lovász local lemma) *Let $A_1, A_2, \dots, A_n$ be events in some probability space and suppose that for every $i \in [n]$, there is a set $J_i \subset \{1, 2, \dots, n\}$, such that A_i is mutually independent of $\{A_j : j \notin J_i \cup \{i\}\}$. Suppose there exist real numbers $\gamma_i \in [0,1)$ such that for every $i \in [n]$,*

$$\Pr(A_i) \le \gamma_i \prod_{j \in J_i} (1-\gamma_j). \tag{5}$$

Then

$$\Pr\Big(\bigcap_{i=1}^n \overline{A_i}\Big) > \prod_{i=1}^n (1-\gamma_i) > 0. \tag{6}$$

3 Hermitian unitals, s-fans and intersection graphs

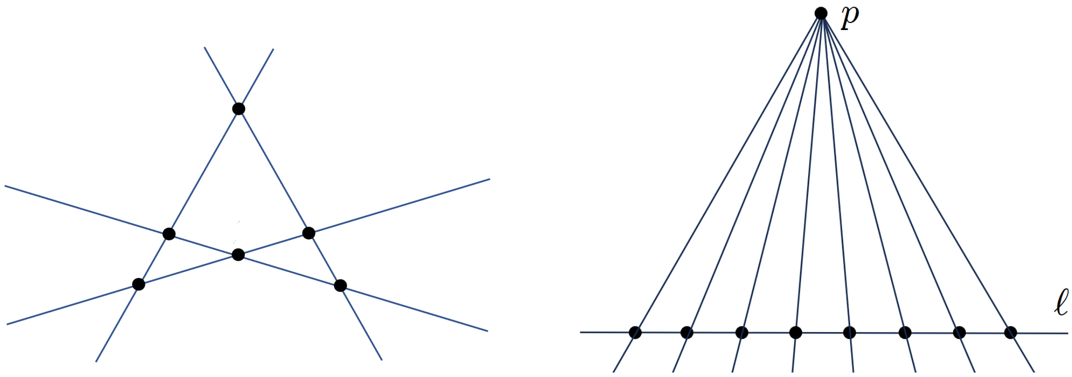
The proof of Theorem 1 appeals to a construction from projective geometry, which was used in [18] to obtain nearly optimal asymptotic bounds on the Ramsey number $r(4,t)$. We very briefly describe the geometry here in elementary terms, while further geometric background is given in Barwick and Ebert [3], Brouwer and van Maldeghem [6] and Piper [22].

3.1 Hermitian unitals in brief. For a partial linear space $\mathcal{H}$, we let $P(\mathcal{H})$ denote the set of points of $\mathcal{H}$ and $L(\mathcal{H})$ denote the set of lines of $\mathcal{H}$. A *unital* in the projective plane $\mathrm{PG}(2,q^2)$ is a set $\mathcal{U}$ of q^3+1 points such that every line of $\mathrm{PG}(2,q^2)$ intersects $\mathcal{U}$ in 1 or $q+1$ points – the latter are referred to as *secants*. A *classical* or *Hermitian unital* $\mathcal{H}_q$ is a partial linear space described in homogeneous co-ordinates as the following set of one-dimensional subspaces of $\mathbb{F}_{q^2}^3$:

$$P(\mathcal{H}_q) = \{\langle x, y, z\rangle \subset \mathbb{F}_{q^2}^3 : x^{q+1} + y^{q+1} + z^{q+1} = 0\}.$$

Here arithmetic is in the finite field $\mathbb{F}_{q^2}$, and $\langle x, y, z\rangle$ is the one-dimensional subspace of $\mathbb{F}_{q^2}^3$ generated by (x, y, z). Then $L(\mathcal{H}_q)$ consists of the intersections of secant lines with $P(\mathcal{H}_q)$, so that there are $q^2(q^2-q+1)$ lines in $\mathcal{H}_q$, each containing exactly $q+1$ points of $\mathcal{H}_q$.

3.2 O'Nan configurations and fans. One of the remarkable features of the Hermitian unital is that it does not contain the so-called *O'Nan configuration*, namely the configuration of four lines and six points in the left figure below:



O'Nan configuration and s-fan

Definition 1. **(s-fan)** *For $s \geq 3$, an* s-fan *is a set of $s+1$ intersecting lines such that s of the lines are concurrent with a point p whereas the remaining line ℓ is not concurrent with that point. The unique point p contained in s lines is the* point of concurrency *of the s-fan.*

An illustration of an s-fan is shown in the right figure above. The fact that $\mathcal{H}_q$ does not contain the figure on the left was first proved by O'Nan [21] (see [18] for a short linear-algebraic proof). The following lemma is a straightforward consequence.

Lemma 1. *Every set of $s+1 \geq 4$ pairwise intersecting lines in $\mathcal{H}_q$ are either all concurrent with a point of $\mathcal{H}_q$, or form an s-fan.*

Proof. Suppose that $S = \{\ell_1, \ldots, \ell_{s+1}\}$ is a set of pairwise intersecting lines in $\mathcal{H}_q$. If no three lines in S are concurrent, then any four lines in S form an O'Nan configuration. So by relabeling if necessary, we may assume that ℓ_1, ℓ_2, ℓ_3 all contain the point p. If p is concurrent with exactly s lines in S, then S forms an s-fan, so p is concurrent with at most $s-1$ lines in S. Let ℓ and ℓ' be two distinct lines in S not concurrent with p. For $i = 1, 2, 3$, let p_i be the point that lies in both

ℓ and ℓ_i. If $\{p_1, p_2, p_3\} \cap \ell' = \emptyset$, then $\ell_1, \ell_2, \ell, \ell'$ form an O'Nan configuration, since every pair of these lines has a distinct point of concurrency. So let us assume by symmetry that $p_1 \in \ell'$. But now $\ell_2, \ell_3, \ell, \ell'$ form an O'Nan configuration, since every pair of these lines has a distinct point of concurrency. Since there is no O'Nan configuration, we conclude that S forms an s-fan or all lines in S are concurrent with the same point. $\square$

3.3 Intersection graphs. The starting point of the proof of Theorem 1 is the following graph:

Definition 2. (Intersection graph) *The* intersection graph G_q *of* $\mathcal{H}_q$ *is defined as follows:*

$$V(G_q) := L(\mathcal{H}_q) \qquad \textit{and} \qquad E(G_q) := \{\{\ell, \ell'\} : \ell \cap \ell' \cap P(\mathcal{H}_q) \neq \emptyset\}.$$

In words, the vertices of G_q are the secants, and two secants are adjacent iff they intersect in a point on the unital. For each $p \in P(\mathcal{H}_q)$, let C_p be the clique in G_q whose vertex set is

$$V(C_p) := \{\ell \in V(G_q) : p \in \ell\}.$$

Since every two lines intersect in at most one point of the unital, the sets $E(C_p)$ over all $p \in P(\mathcal{H}_q)$ partition $E(G_q)$. We can thus view the graph G_q as a union of $q^3 + 1$ edge-disjoint cliques C_p, one for each point $p \in P(\mathcal{H}_q)$. Additionally, each vertex of G_q is contained in exactly $q + 1$ cliques C_p. More importantly, by Lemma 1, each $(s + 1)$-clique in G_q is either an s-fan in $\mathcal{H}_q$ or a set of lines of $\mathcal{H}_q$ all concurrent with the same point in $P(\mathcal{H}_q)$. Translated into the language of graphs, we obtain the following fact:

$$\begin{array}{l}\textit{For each } (s+1)\textit{-clique } K \textit{ in } G_q, \textit{ there exists } C_p \textit{ such} \\ \textit{that } V(K) \subseteq V(C_p) \textit{ or } |V(K) \cap V(C_p)| = s.\end{array} \tag{A}$$

We employ the following definition for convenience:

Definition 3. (Trivial and non-trivial cliques) *An $(s + 1)$-clique K in G_q such that $V(K) \subseteq V(C_p)$ for some $p \in P(\mathcal{H}_q)$ is a* trivial clique*, and otherwise it is a* non-trivial clique*.*

4 Random sampling

The proof of Theorem 1 starts with the intersection graph G_q. The outline of the randomized construction is as follows. This graph has $q^2(q^2 - q + 1)$ vertices, and is a union of $q^3 + 1$ edge-disjoint cliques C_p each of size q^2, such that each vertex of G_q is contained in exactly $q + 1$ of those cliques. The first step is to randomly sample these cliques with probability $\Theta((\log q)/q)$ and remove all edges in non-sampled cliques to obtain a random graph $G \subseteq G_q$ such that the number of remaining cliques C_p is $\Theta(q^2 \log q)$, each vertex of G is in $\Theta(\log q)$ cliques C_p, and the number of non-trivial cliques K_{s+1} on each edge of G is $\Theta(\log q)^s$. We use the Chernoff Bound (Section 4.1) to prove the existence of G. The next step is to destroy all trivial cliques K_{s+1} in G by taking a random s-partition of each clique $C_p \subseteq G$ to obtain a graph G_χ. Finally, the non-trivial $(s + 1)$-cliques in G are destroyed by randomly sampling edges of G with probability $\rho = \Theta((\log q)^{-2/s})$ to obtain a random subgraph G_ρ. Using Janson's inequality (Section 4.3) and the Lovász local lemma (Section 5), we show that the graph H with vertex set $V(G_q)$ and edge set $E(G_\chi) \cap E(G_\rho)$ is a K_{s+1}-free graph with $n = q^2(q^2 - q + 1)$ vertices such that every set of roughly $\Theta(\sqrt{n} \log n)$ vertices induces a graph containing a K_s (in fact, this K_s will lie within one of the C_ps).

4.1 Sampling cliques. For a constant $a \geq 128$ and $q \geq a \log q$, randomly and independently sample cliques C_p with probability $(a \log q)/(q+1)$. Let P be the random set which indexes the sampled cliques and $G = G_{a,q,s} \subseteq G_q$ be the random subgraph consisting of the union of edges in $E(C_p)$ over all $p \in P$. More precisely, $V(G) = V(G_q)$ and $E(G) = \cup_{p \in P} E(C_p)$. We use the Chernoff Bound, Proposition 1, to prove the following:

Lemma 2. *Let $s \geq 3, a \geq 128$ and $q \geq a \log q$. Then with positive probability:*

(i) $a(q^2 \log q)/2 \leq |P| \leq 2aq^2 \log q$.

(ii) *Each vertex of G is contained in at least $(a \log q)/2$ cliques C_p with $p \in P$.*

(iii) *For every $e \in E(G)$, the number of non-trivial $K_{s+1} \subseteq G$ containing e is at most $k = 4(2a \log q)^s$.*

Proof. By the Chernoff bound (Proposition 1) with $\epsilon = 1/2$, the probability that $|P| > 2aq^2 \log q$ or $|P| < (aq^2 \log q)/2$ is at most $2\exp(-(aq^2 \log q)/16) < 1/3$, so (i) fails with probability less than $1/3$. The number of sampled cliques containing a given vertex of G is at most $(a \log q)/2$ with probability at most $\exp(-(a \log q)/8) = q^{-a/8} < q^{-8}$. Since $|V(G)| \leq q^4 < q^8/3$, the probability that (ii) fails is less than $1/3$.

Claim. *If more than k non-trivial cliques in G contain some edge, then some vertex of G is contained in more than $r = 2a \log q$ sampled cliques.*

If this claim is true, then (iii) happens for a given edge with probability at most $\exp(-(a \log q)/4) < q^{-8}$ by the Chernoff Bound (Proposition 1) with $\epsilon = 1$. Since the number of edges of G_q is at most $(q^3+1) \cdot \binom{q^2}{2} \leq q^8/3$, the union bound shows (iii) then fails with probability less than $1/3$. We then conclude (i) – (iii) hold simultaneously with positive probability.

To prove the claim, fix an edge $\{u, v\} \in E(G)$ contained in more than k non-trivial cliques in G, and suppose for a contradiction that no vertex of G is contained in more than r sampled cliques. Let S_x be the set of sampled cliques containing a vertex x. By (A), for each non-trivial clique $K = K_{s+1} \subseteq G_q$ containing $\{u, v\}$, there exists C_p such that $|V(K) \cap V(C_p)| = s$. If $\{u, v\} \in E(C_p)$, let $V(K) \backslash V(C_p) = \{w\}$. Each clique in $S_u \setminus \{C_p\}$ intersects each clique in $S_v \setminus \{C_p\}$ in at most one point, hence the number of points that lie in both a clique of S_u and a clique of S_v is at most r^2. Since w must be such a point, the number of choices of w is at most r^2. Since $|S_w| \leq r$, the number of ways to select the remaining vertices K is at most $\binom{r}{s-2}$. If on the other hand $\{u, v\} \notin E(C_p)$, then there exist at most $2r$ choices for C_p since $|S_u \cup S_v| \leq 2r$. Then there are at most

$$\binom{|S_u|}{s-1} + \binom{|S_v|}{s-1} \leq 2\binom{r}{s-1}$$

choices for the remaining vertices of K. We conclude the number of non-trivial cliques of order $s+1$ in G containing $\{u, v\}$ is at most

$$r^2 \cdot \binom{r}{s-2} + 2r \cdot 2\binom{r}{s-1} \leq 4r^s = k.$$

This contradiction proves the claim. $\square$

Fix an instance of the graph G satisfying (i) – (iii) in Lemma 2, together with the set P indexing the cliques C_p in G. For a positive integer b and $X \subseteq V(G)$, let

$$P_X = P_{X,b} = \{p \in P : |V(C_p) \cap X| \geq b\}.$$

This is the set of $p \in P$ whose corresponding clique C_p in G has at least b vertices in X.

Lemma 3. *Let $b \geq 1$, $a \geq 128$ and $q \geq a \log q$. Then for any $X \subseteq V(G)$,*

$$\sum_{p \in P_X} |V(C_p) \cap X| > \frac{1}{2}(a \log q) \cdot |X| - 2abq^2 \log q. \tag{7}$$

Proof. As $|P| \leq 2aq^2 \log q$ from Lemma 2(i),

$$\sum_{p \in P \setminus P_X} |V(C_p) \cap X| < b|P| \leq 2abq^2 \log q.$$

Each vertex in X is in at least $(a \log q)/2$ cliques $C_p : p \in P$ by Lemma 2(ii). Therefore

$$\sum_{p \in P} |V(C_p) \cap X| = \sum_{\ell \in X} |\{p \in P : \ell \in V(C_p)\}| \geq \frac{1}{2}(a \log q) \cdot |X|.$$

Subtracting the first inequality from the second gives the lemma. $\square$

4.2 Coloring cliques and Sampling edges. Let $G_\chi \subseteq G$ be obtained from G by taking independently for each $p \in P$ a random s-coloring χ_p of $V(C_p)$ and removing all edges of G whose ends have the same color. This removes from G all trivial copies of K_{s+1}, so by (A), for $s \geq 3$:

Each clique $K_{s+1} \subseteq G_\chi$ is non-trivial. (B)

We now define the random graph $G_\rho \subseteq G$. Let $b \geq 1$ and $\rho \in [0, 1]$ satisfy $b \geq 2^{40s}$ and $\rho = (8s/b)^{2/s}$, and define G_ρ to be the random graph obtained by sampling edges of G independently with probability ρ. The aim of this sampling is to destroy all non-trivial cliques, while still preserving cliques K_s within the C_p, in large enough sets of vertices. Unsurprisingly, this makes strong use of Janson's inequality for the probability that a random s-partite graph is K_s-free, in the form of Proposition 2. We define $H = H_s$ to be the intersection of G_ρ and G_χ, namely,

$$V(H) := V(G) \qquad \text{and} \qquad E(H) := E(G_\rho) \cap E(G_\chi).$$

Our next task is to consider copies of $K_s \subseteq H$ whose vertices are contained in some C_p.

4.3 The event A_X. In this section, for each set $X \subseteq V(H)$, we define an event A_X such that

$$\overline{A_X} \qquad \Longrightarrow \qquad H[X] \text{ contains a copy of } K_s, \tag{8}$$

and we find an upper bound on $\Pr(A_X)$. For a set $X \subseteq V(H)$ and $p \in P_X$, fix a family $\Pi_p(X)$ of $r_p(X) = \lfloor |V(C_p) \cap X|/b \rfloor$ disjoint subsets of $V(C_p) \cap X$ of size b each. Then C_p is *bad* if for every

$Y \in \Pi_p(X)$, the induced subgraph $H[Y]$ is K_s-free, and we let $A_{X,p}$ be the event that C_p is bad. Finally, define

$$A_X = \bigcap_{p \in P_X} A_{X,p}.$$

We say X is *bad* if all $C_p : p \in P_X$ are bad. In other words, A_X is the event that X is bad. If A_X does not occur, then, as promised in (8), by definition $H[X]$ contains a copy of K_s. It is critical here that instead of defining A_X to be the event that no K_s is contained in $H[V(C_p) \cap X]$, we instead partitioned each $V(C_p) \cap X$ into sets of size b and defined the event A_X based on these partitions. This reduces the dependencies between various events that will be considered later in a local lemma calculation.

Lemma 4. *Let $s \geq 3$, $b \geq 2^{40s}$ and $\rho \in [0,1]$ satisfy $\rho = (8s/b)^{2/s}$. Then for any $X \subseteq V(H)$,*

$$\Pr(A_X) \leq \exp\Big(-\frac{1}{32s} \sum_{p \in P_X} |V(C_p) \cap X|\Big). \tag{9}$$

Proof. Since the colorings χ_p are independent over $p \in P_X$, the events $A_{X,p}$ are independent over $p \in P_X$. For $Y \in \Pi_p(X)$, let A_Y be the event that $H[Y]$ is K_s-free. These events are independent over all Y. By the Chernoff bound (Proposition 1), the probability that χ_p assigns some color fewer than $b/2s$ times to vertices of Y is at most $\exp(-b/8s)$. Fix a coloring χ of Y where every color appears at least $b/2s$ times. Then the graph $H[Y]$ is a random s-partite graph which we denoted in Section 2 by $G_{b,\rho}(\chi)$. By Proposition 2, for any such coloring χ,

$$\Pr(K_s \not\subseteq G_{b,\rho}(\chi)) \leq \exp(-2^{2s-4}b).$$

As there are at most s^b choices of χ, the union bound over s-colorings χ gives

$$\Pr(A_Y) \leq \exp\Big(-\frac{b}{8s}\Big) + s^b \exp(-2^{2s-4}b).$$

Since $s \geq 3$, $2^{2s-4}b \geq 2b \log s \geq 2b$ and therefore

$$\Pr(A_Y) \leq \exp\Big(-\frac{b}{8s}\Big) + \exp(-b) \leq 2\exp\Big(-\frac{b}{8s}\Big) \leq \exp\Big(-\frac{b}{16s}\Big).$$

Here we used $b \geq 16s$. Since $|\Pi_p| = r_p(X)$, we obtain

$$\begin{aligned} \Pr(A_X) &= \prod_{p \in P_X} \Pr(A_{X,p}) \\ &= \prod_{p \in P_X} \prod_{Y \in \Pi_p} \Pr(A_Y) \leq \prod_{p \in P_X} \exp\Big(-\frac{b}{16s} \cdot r_p(X)\Big). \end{aligned}$$

Recall $|V(C_p) \cap X| \geq b$ for $p \in P_X$, so $b \cdot r_p(X) \geq |V(C_p) \cap X|/2$, and therefore

$$\Pr(A_X) \leq \prod_{p \in P_X} \exp\Big(-\frac{|V(C_p) \cap X|}{32s}\Big) = \exp\Big(-\frac{1}{32s} \sum_{p \in P_X} |V(C_p) \cap X|\Big).$$

This proves the lemma. $\square$

5 Proof of Theorem 1

To prove Theorem 1, for each $s \geq 3$ let $G = G_{a,q,s}$, defined in Section 4.1, where $a = 2^{10}s$ and q is a prime power satisfying $q \geq a \log q$. Let $H \subseteq G$ denote the random graph defined in Section 4.2, which is the intersection of the two random graphs G_χ and G_ρ, with parameters

$$b = 2^{40s+2/s} \cdot 2a \log q \qquad \text{and} \qquad \rho = \left(\frac{8s}{b}\right)^{\frac{2}{s}}.$$

For convenience we omit rounding and assume b is an integer. Let $\mathcal{K}$ be the family of non-trivial $(s+1)$-cliques $K \subseteq G$, and let B_K be the event $G[K] \subseteq G_\rho$. Let $\mathcal{X} = \{X \subseteq V(H) : |X| = 8bq^2\}$ and A_X be the event X is bad, as in Section 4.3. Due to (B),

$$\begin{array}{l}\textit{if none of the events } B_K : K \in \mathcal{K} \textit{ or } A_X : X \in \mathcal{X} \textit{ occur,} \\ \textit{then } H \textit{ is } K_{s+1}\textit{-free and } K_s \subseteq H[X] \textit{ for all } X \in \mathcal{X}.\end{array} \tag{C}$$

Specifically, (C) implies every set of $8bq^2$ vertices of H induces a subgraph containing K_s. This shows for any prime power $q \geq a \log q$,

$$f_{s,s+1}(q^2(q^2 - q + 1)) \leq 8bq^2.$$

By Bertrand's postulate, there exists a prime between any positive integer and its double, so letting $q \geq a \log q$ be a prime between $n^{1/4}$ and $2n^{1/4}$, we find for $s \geq 3$ and $n \geq 2$,

$$\begin{aligned} f_{s,s+1}(n) &\leq 8bq^2 \leq 8 \cdot 2^{40s+1/s} \cdot 2^{11}s \cdot q^2 \cdot \log q \\ &\leq 2^{100s} \cdot \sqrt{n} \log n. \end{aligned}$$

It remains to prove (C) holds with positive probability, via the local lemma (Proposition 3).

Dependencies. For the dependencies between the events $B_K : K \in \mathcal{K}$ and $A_X : X \in \mathcal{X}$, we note A_X is determined by the following set of edges of G:

$$\hat{E}(X) = \bigcup_{p \in P_X} \bigcup_{Y \in \Pi_p} E(G[Y]).$$

Since $|Y| = b$ for each $Y \in \Pi_p$, and $r = r_p(X) = |\Pi_p| = \lfloor |V(C_p) \cap X|/b \rfloor$,

$$\begin{aligned} |\hat{E}(X)| &= \sum_{p \in P_X} \sum_{i=1}^{r} \binom{b}{2} \\ &= \sum_{p \in P_X} \left\lfloor \frac{|V(C_p) \cap X|}{b} \right\rfloor \binom{b}{2} \;\leq\; \frac{1}{2} b \sum_{p \in P_X} |V(C_p) \cap X|. \end{aligned} \tag{10}$$

For convenience, let $\hat{E}(K)$ also denote $E(G[K])$ if $K \in \mathcal{K}$. It is important here that since G_ρ samples edges of G independently with probability ρ, for any $\mathcal{J} \subseteq \mathcal{K}$ and $K \in \mathcal{K}$, we observe

the event B_K is mutually independent with $\{B_{K'} : K' \in \mathcal{J}\}$ if

$$\bigcup_{K' \in \mathcal{J}} E(K') \cap E(K) = \emptyset.$$

By Lemma 2(iii), each edge of G is contained in at most $k = 4(2a \log q)^s$ cliques $K \in \mathcal{K}$. Fixing any $K \in \mathcal{K}$, there are at most

$$\kappa = \binom{s+1}{2} \cdot k \leq bk \tag{11}$$

choices of $K' \in \mathcal{K}$ such that $\hat{E}(K') \cap \hat{E}(K) \neq \emptyset$. Hence B_K is mutually independent of any set of events omitting the events $K' \in \mathcal{K}'$ just described. For a given B_K, we make no assumptions on its mutual independence with any subset of the events A_X.

Since each edge of G is in at most k cliques in $\mathcal{K}$, by (10), for each $X \in \mathcal{X}$ there are at most

$$\lambda_X = k \cdot |\hat{E}(X)| \leq \frac{1}{2} bk \cdot \sum_{p \in P_X} |V(C_p) \cap X| \tag{12}$$

choices of $K \in \mathcal{K}$ such that $\hat{E}(K) \cap \hat{E}(X) \neq \emptyset$, and any set of other events A_K are mutually independent with A_X. This is the critical point for which the definition of A_X was carefully chosen: instead of A_X being the event that no K_s is contained in the subgraph induced by $V(C_p) \cap X$, we instead partitioned each C_p into sets of size b and defined the event A_X based on these partitions. We make no assumptions on the independence among the events A_X.

Local lemma inequalities. Let $N = |\mathcal{X}|$. The local lemma (Proposition 3) implies the probability that (C) holds is positive if there are reals $\gamma, \delta \in [0,1)$ such that for all $K \in \mathcal{K}$ and all $X \in \mathcal{X}$,

$$\Pr(B_K) \leq \gamma(1-\gamma)^{\kappa}(1-\delta)^N \qquad \text{and} \qquad \Pr(A_X) \leq \delta(1-\gamma)^{\lambda_X}(1-\delta)^N.$$

We claim that these inequalities are satisfied if we select $\delta = 1/(N+1)$ and $\gamma = 1/64sbk$.

First inequality. We have $\Pr(B_K) = \rho^{\binom{s+1}{2}}$ and $(1-\delta)^N \geq 1/e$. By (11), $(1-\gamma)^{\kappa} \geq 1 - \kappa\gamma > 1/2$, so it is sufficient to show $2e\Pr(B_K)/\gamma \leq 1$ for the first inequality to hold. Since $\rho = (8s/b)^{2/s}$,

$$\rho^{\binom{s+1}{2}} = \left(\frac{8s}{b}\right)^{s+1}.$$

Since $b = 2^{40s}k^{1/s}$, and $s \geq 3$,

$$\begin{aligned}
\frac{2e}{\gamma}\Pr(B_K) = 128esbk \cdot \rho^{\binom{s+1}{2}} &= \frac{128esk(8s)^{s+1}}{b^s} \\
&= \frac{128esk(8s)^{s+1}}{2^{40s^2}k} \\
&< 2es \cdot \left(\frac{8s}{64^{s-1}}\right)^{s+1} < \frac{2es}{8^{s+1}} < 1.
\end{aligned}$$

Here we used $64^{s-1} > 64s$ and $2es < 8^{s+1}$ for $s \geq 3$. This verifies the first inequality.

Second inequality. For the second inequality, we use $1 - \gamma \geq \exp(-2\gamma)$, which is valid since $\gamma \leq 1/2$. Recalling $(1-\delta)^N \geq 1/e$, it is enough to show

$$e \cdot \Pr(A_X) \leq \exp(-\log(N+1) - 2\gamma\lambda_X). \tag{13}$$

By Lemma 4,

$$\Pr(A_X) \leq \exp\Big(-\frac{1}{32s}\sum_{p \in P_X} |V(C_p) \cap X|\Big).$$

Using $|V(G)| = q^2(q^2 - q + 1)$,

$$\log(N+1) = \log\Big[\binom{q^2(q^2-q+1)}{8bq^2} + 1\Big] \le \log\binom{q^4}{8bq^2} - 1 \le 32bq^2 \log q - 1.$$

For (13), it is enough to show

$$\exp\Big(-\frac{1}{32s}\sum_{p\in P_X} |V(C_p)\cap X|\Big) \le \exp(-32bq^2\log q - 2\gamma\lambda_X).$$

Due to (12), it is enough to show

$$\exp\Big(-\frac{1}{32s}\sum_{p\in P_X} |V(C_p)\cap X|\Big) \le \exp\Big(-32bq^2\log q - \frac{1}{64s}\sum_{p\in P_X} |V(C_p)\cap X|\Big).$$

Therefore we require

$$\exp\Big(-\frac{1}{64s}\sum_{p\in P_X} |V(C_p)\cap X|\Big) \le \exp(-32bq^2\log q).$$

Applying Lemma 3 and using $a = 2^{10}s$, we find

$$\begin{aligned}
\frac{1}{64s}\sum_{p\in P_X} |V(C_p)\cap X| &\ge \frac{1}{64s}\Big(\frac{1}{2}(a\log q)\cdot|X| - 2abq^2\log q\Big) \\
&= \frac{1}{64s}\Big(\frac{1}{2}(a\log q)\cdot 8bq^2 - 2abq^2\log q\Big) \\
&= \frac{1}{64s}\cdot 2abq^2\log q \quad = \quad 32bq^2\log q.
\end{aligned}$$

This proves the second inequality. □

Concluding remarks

- In this paper, we proved $f_{s,s+1}(n) = O(\sqrt{n}\log n)$ by suitable random sampling of points and lines from Hermitian unitals. A part of the proof essentially involves the union bound over all sets X of lines of size $8bq^2$ – we implicitly assumed in our application of the local lemma in Section 5 that the events A_X depend on all other such events. We first sampled points randomly from the Hermitian unital with probability of order $(\log q)/q$. If we sampled points with a lower probability $o(\log q)/q$, then the union bound no longer works: for large q there are $N \ge \exp(bq^2\log q)$ sets of size $8bq^2$ to consider – see Section 5 – whereas if all the sets $C_p \cap X : p \in P_X$ have size roughly b, then the probability that every C_p is $(s-1)$-colored in a random s-coloring is $\exp(-o(bq^2\log q))$. It may be possible using randomized greedy algorithms akin to the Rödl semirandom method to circumvent this issue and obtain a bound of the form $f_{s,s+1}(n) = o(\sqrt{n}\log n)$, but we did not investigate this technical direction. We conjectured (Conjecture 1) that $f_{s,s+1}(n) = \sqrt{n}(\log n)^{1-o(1)}$ for all $s \ge 3$. Conjecture 2 would imply $f_{3,4}(n) \ge \sqrt{n}(\log n)^{1/2+\delta/2}$ for some $\delta > 0$.

- Using (6) and the proof of Theorem 1, the probability that the n-vertex random graph $H = H_s$ is K_{s+1}-free and yet every set of $8bq^2$ vertices induces a copy of K_s is at least

$$(1-\gamma)^{|\mathcal{K}|}(1-\delta)^{|\mathcal{X}|} = \exp(-O(\delta|\mathcal{X}| + \gamma|\mathcal{K}|)) = \exp(-O(\gamma|\mathcal{K}|)).$$

  Now $|\mathcal{K}| = O(k|E(G)|) = O(kq^6 \log q)$ from Lemma 2(i) and Lemma 2(iii), and $\gamma = O(1/k \log q)$, so we conclude the above quantity is $\exp(-O(q^6))$. The expected number of edges in H is $\Theta(\rho|E(G)|) = \Theta(q^6(\log q)^{1-2/s})$. By the Chernoff Bound, H has $\Theta(q^6(\log q)^{1-2/s})$ edges with probability at least

$$1 - \exp(-\Theta(q^6(\log q)^{1-2/s})).$$

  We conclude that with positive probability, the n-vertex random graph H is a K_{s+1}-free graph with average degree $\Theta(d)$ where $d = \Theta(q^2(\log q)^{1-2/s}) = \Theta(\sqrt{n}(\log n)^{1-2/s})$. Moreover, H has the property that every vertex subset of size at least

$$8bq^2 = \Omega(q^2 \log q) = \Omega\left(\frac{n(\log d)^{1+(s-2)/s}}{d}\right)$$

  induces a copy of K_s. When $s = 3$, this shows that the value of δ in Conjecture 2 cannot be larger than $1/3$. Moreover, H has an independent set (and hence a K_3-free induced subgraph) of size at least $\Omega(n \log d/d \log\log d) = \Omega(\sqrt{n}(\log n)^{2/3-o(1)})$. For $s \geq 4$, H contains a K_s-free induced subgraph with $d = \Omega(\sqrt{n}(\log n)^{1-2/s})$ vertices. It would be interesting for any $s \geq 3$ to exhibit a K_s-free induced subgraph of H with $\sqrt{n}(\log n)^{1-o(1)}$ vertices, if it exists.

- If F is any K_s-free graph, then we can define $f_{F,s}(n)$ to be the largest m such that every n-vertex K_s-free graph has an induced F-free subgraph with m vertices. The proof of Theorem 1 gives

$$f_{F,s}(n) = O(\sqrt{n}\log n).$$

  Indeed, instead of taking a random s-coloring of each sampled clique C_p, one takes a random uniform map $\chi : V(C_p) \to V(F)$, and then place all edges between $\chi^{-1}(u)$ and $\chi^{-1}(v)$ whenever $\{u, v\} \in E(F)$. The analysis is the same as in Theorem 1, apart from a suitable modification of the implicit constants. It would be interesting to determine for $s \geq 3$ whether there exists a K_s-free graph F such that

$$\lim_{n\to\infty} \frac{f_{F,s}(n)}{f_{s-1,s}(n)} = \infty.$$

  In forthcoming work [20], we shall study the quantities $f_{F,s}(n)$.

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7 Appendix : Proof of Proposition 2

To prove Proposition 2, we require some notation and preliminaries. Recall for $s \geq 3$, $n \geq 1$, $\rho \in [0,1]$ and an s-coloring χ of an n-element set with color classes $Y_1, Y_2, \dots, Y_s$, $G_{n,\rho}(\chi)$ is obtained by independently and randomly sampling edges of the complete s-partite graph with parts $Y_1, Y_2, \dots, Y_s$ with probability ρ. The expected number of $K_s \subseteq G_{n,\rho}(\chi)$ is precisely

$$\mu(\chi) = \rho^{\binom{s}{2}} \prod_{i=1}^{s} |Y_i| \tag{14}$$

and the variance is defined by

$$\triangle(\chi) = \sum_{S \subset [s]} \rho^{2\binom{s}{2} - \binom{|S|}{2}} \prod_{i \in S} |Y_i| \prod_{j \notin S} |Y_j|(|Y_j| - 1) = \sum_{S \subset [s]} \rho^{2\binom{s}{2} - \binom{|S|}{2}} \prod_{i=1}^{n} |Y_i| \prod_{j \notin S} (|Y_j| - 1), \tag{15}$$

where the sum is over sets S with $2 \leq |S| \leq s-1$. From Janson's inequality [2, 13] one obtains for any $\rho \in [0,1]$ and $n \geq 1$:

$$\Pr(K_s \not\subseteq G_{n,\rho}(\chi)) \leq \exp\Big(-\frac{1}{2}\mu(\chi)\Big). \tag{16}$$

provided $\triangle(\chi) \leq \mu(\chi)$.

Proof of Proposition 2. For Proposition 2, χ is an s-coloring with color classes $Y_1, Y_2, \dots, Y_s$ satisfying $|Y_i| \geq n/2s$ for all $i \in [s]$. The product in (14) is minimized when $|Y_i| = n/2s$ for all but one value of $i \in [s]$, and the remaining color class has size $n - (s-1)n/2s > n/2$. Therefore

$$\begin{aligned} \mu(\chi) &> \rho^{\binom{s}{2}} \Big(\frac{n}{2s}\Big)^{s-1} \cdot \frac{n}{2} \\ &= \Big(\frac{8s}{n}\Big)^{s-1} \Big(\frac{n}{2s}\Big)^{s-1} \cdot \frac{n}{2} \;=\; 2^{2s-3} \cdot n. \end{aligned}$$

So if $\triangle(\chi) \leq \mu(\chi)$ when $\rho = (8s/n)^{2/s}$ and $n \geq 2^{40s}$, then Proposition 2 follows from (16):

$$\Pr(K_s \not\subseteq G_{n,\rho}(\chi)) \leq \exp\Big(-\frac{1}{2}\mu(\chi)\Big) \leq \exp(-2^{2s-4}n).$$

It remains to prove $\triangle(\chi) \leq \mu(\chi)$ when $\rho = (8s/n)^{2/s}$ and $n \geq 2^{40s}$.

By (14) and (15),

$$\frac{\triangle(\chi)}{\mu(\chi)} = \sum_{S \subset [s]} \rho^{\binom{s}{2} - \binom{|S|}{2}} \prod_{j \notin S} (|Y_j| - 1).$$

The sum is over subsets S of $[s]$ where $2 \leq |S| \leq s-1$. By the inequality of geometric and arithmetic means,

$$\prod_{j \notin S} (|Y_j| - 1) \leq \prod_{j \notin S} |Y_j| \leq \Big(\frac{1}{s - |S|} \sum_{j \notin S} |Y_j|\Big)^{s-|S|} \leq \Big(\frac{n}{s - |S|}\Big)^{s-|S|}.$$

We conclude

$$\frac{\triangle(\chi)}{\mu(\chi)} \leq \sum_{i=2}^{s-1} \rho^{\binom{s}{2} - \binom{i}{2}} \Big(\frac{n}{s-i}\Big)^{s-i} \binom{s}{i}.$$

By definition of ρ,

$$\rho^{\binom{s}{2}-\binom{i}{2}} = \Big(\frac{8s}{n}\Big)^{\frac{(s-i)(s+i-1)}{s}}.$$

Therefore

$$\frac{\triangle(\chi)}{\mu(\chi)} \;\le\; \sum_{i=2}^{s-1}\Big(\frac{8s}{n}\Big)^{\frac{(s-i)(s+i-1)}{s}}\Big(\frac{n}{s-i}\Big)^{s-i}\binom{s}{i} \;=\; \sum_{i=2}^{s-1}\Big(\frac{(8s)^{\frac{s+i-1}{s}}}{(s-i)n^{\frac{i-1}{s}}}\Big)^{s-i}\binom{s}{i}.$$

We break the sum into two pieces. First, for $2 \le i \le \lfloor s/\log(8s)\rfloor \le s/2$,

$$(s-i)n^{\frac{i-1}{s}} \ge \frac{s}{2}\cdot n^{\frac{1}{s}} \ge 2^9 s$$

since $n \ge 2^{40s} \ge 2^{10s}$. Therefore each term in the sum is at most

$$\Big(\frac{8s\cdot(8s)^{\frac{1}{\log(8s)}}}{2^9 s}\Big)^{s-i}\binom{s}{i} \le \Big(\frac{8es}{2^9 s}\Big)^{s-\frac{s}{\log(8s)}}\cdot 2^s \le \Big(\frac{1}{16}\Big)^{\frac{s}{2}}\cdot 2^s = 2^{-s}.$$

Second, for $\lfloor s/\log(8s)\rfloor + 1 \le i \le s-1$, $(i-1)/s \ge 1/2\log(8s)$ and so using $n \ge 2^{40s}$ and $s \ge 3$,

$$(s-i)n^{\frac{i-1}{s}} \ge n^{\frac{1}{2\log(8s)}} \ge 2^{\frac{20s}{\log(8s)}} \ge 128s^3.$$

Therefore each term in the sum is at most

$$\Big(\frac{(8s)^{\frac{s+i-1}{s}}}{128s^3}\Big)^{s-i}\binom{s}{i} \le \Big(\frac{(8s)^2}{128s^3}\Big)^{s-i}\binom{s}{i} = \Big(\frac{1}{2s}\Big)^{s-i}\binom{s}{i}.$$

We conclude

$$\begin{aligned}\frac{\triangle(\chi)}{\mu(\chi)} &\le \sum_{i=2}^{s-1}\Big(\frac{1}{2s}\Big)^{s-i}\binom{s}{i} + \sum_{i=2}^{s-1}2^{-s}\\ &\le \Big(1+\frac{1}{2s}\Big)^{s} - 1 + (s-2)2^{-s} \;\le\; \sqrt{e} - 1 + (s-2)2^{-s}.\end{aligned}$$

Evidently $(s-2)2^{-s} \le 1/8$ for $s \ge 3$, and therefore

$$\sqrt{e} - 1 + (s-2)2^{-s} \le 0.773\ldots < 1.$$

We conclude $\triangle(\chi) < \mu(\chi)$, as required. $\square$

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