

HYPERGRAPHS WITH MANY EXTREMAL CONFIGURATIONS

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ABSTRACT. For every positive integer t we construct a finite family of triple systems \mathcal{M}_t , determine its Turán number, and show that there are t extremal \mathcal{M}_t -free configurations that are far from each other in edit-distance. We also prove a strong stability theorem: every \mathcal{M}_t -free triple system whose size is close to the maximum size is a subgraph of one of these t extremal configurations after removing a small proportion of vertices. This is the first stability theorem for a hypergraph problem with an arbitrary (finite) number of extremal configurations. Moreover, the extremal hypergraphs have very different shadow sizes (unlike the case of the famous Turán tetrahedron conjecture). Hence a corollary of our main result is that the boundary of the feasible region of \mathcal{M}_t has exactly t global maxima.

§1. INTRODUCTION

1.1. Stability. Let $r \geq 2$ and let \mathcal{F} be a family of r -uniform hypergraphs (henceforth called r -graphs). An r -graph \mathcal{H} is \mathcal{F} -free if it contains no member of \mathcal{F} as a subgraph. For every natural number n the *Turán number* $\text{ex}(n, \mathcal{F})$ of \mathcal{F} is the maximum number of edges in an \mathcal{F} -free r -graph on n vertices. The *Turán density* $\pi(\mathcal{F})$ of \mathcal{F} is defined as $\pi(\mathcal{F}) = \lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{F}) / \binom{n}{r}$, and \mathcal{F} is *nondegenerate* if $\pi(\mathcal{F}) > 0$. By a theorem of Erdős [4], this is equivalent to \mathcal{F} containing an r -graph which is not r -partite.

The study of $\text{ex}(n, \mathcal{F})$ is perhaps the central topic in extremal graph and hypergraph theory. Curiously, unlike the case for graphs, determining $\pi(\mathcal{F})$ for a family \mathcal{F} of hypergraphs is known to be notoriously hard in general. Indeed, the problem of determining $\pi(K_\ell^r)$ raised by Turán [30], where K_ℓ^r is the complete r -graph on ℓ vertices, is still wide open for all $\ell > r \geq 3$. Erdős offered \$500 for the determination of any $\pi(K_\ell^r)$ with $\ell > r \geq 3$ and \$1000 for the determination of all $\pi(K_\ell^r)$ with $\ell > r \geq 3$.

Current research on hypergraph Turán problems is usually of one of the following two somewhat distinct flavours: Either one takes a family \mathcal{F} of very small hypergraphs and tries to determine its Turán density as well as the extremal configurations (e.g. see [3, 7, 8, 24, 26]) or one attempts to show that certain phenomena impossible for graphs are possible for

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r -graphs as soon as $r \geq 3$. For instance, the work of Frankl and Rödl on jumps [9] or Pikhurko's theorem that for $r \geq 3$ there exist uncountably many Turán densities of infinite families of r -graphs [27] belong to this latter category, and so does the present article.

The classical Erdős-Simonovits stability theorem [29] motivated the second author [25] to make the following definition. A family \mathcal{F} of r -graphs is t -stable if for every $m \in \mathbb{N}$ there exist r -graphs $\mathcal{G}_1(m), \dots, \mathcal{G}_t(m)$ on m vertices such that the following holds. For every $\delta > 0$ there exist $\varepsilon > 0$ and n_0 such that for all $n \geq n_0$ if \mathcal{H} is an \mathcal{F} -free r -graph on n vertices with

$$|\mathcal{H}| > (1 - \varepsilon)\text{ex}(n, \mathcal{F}),$$

then \mathcal{H} can be transformed into an isomorphic copy of some $\mathcal{G}_i(n)$ by adding and removing at most $\delta|\mathcal{H}|$ edges. Say \mathcal{F} is *stable* if it is 1-stable. There are many classical examples of stable families of 3-uniform hypergraphs. For instance, Keevash and the second author [14] proved the stability of cancellative hypergraphs; Füredi and Simonovits [10], and, independently, Keevash and Sudakov [15] proved that the Fano plane is stable.

In general, we denote by $\xi(\mathcal{F})$ the minimum integer t such that \mathcal{F} is t -stable, and set $\xi(\mathcal{F}) = \infty$ if there is no such t . Call $\xi(\mathcal{F})$ the *stability number* of \mathcal{F} .

The Erdős-Stone-Simonovits theorem [5, 6] and Erdős-Simonovits stability theorem [29] imply that every nondegenerate family of graphs is stable. However, for hypergraphs there are many families (whose Turán densities are unknown) which are conjecturally not stable. Two famous examples are Turán's conjecture on tetrahedra (e.g. see [16, 19, 28]) and the Erdős-Sós conjecture on triple systems with bipartite links (e.g. see [7, 19]). In fact, no Turán density of a nondegenerate family of hypergraphs without the stability property was known (e.g. see [13]) until very recently, when the first two authors constructed a 2-stable family \mathcal{M} of triple systems [19]. Our first main result states that, more generally, for every natural number t there exists a family of triple systems satisfying $\xi(\mathcal{M}_t) = t$.

We identify an r -graph \mathcal{H} with its edge set, use $V(\mathcal{H})$ to denote its vertex set, and denote by $v(\mathcal{H})$ the size of $V(\mathcal{H})$. An r -graph \mathcal{H} is a *blow-up* of an r -graph \mathcal{G} if there exists a map $\psi: V(\mathcal{H}) \rightarrow V(\mathcal{G})$ so that $\psi(E) \in \mathcal{G}$ iff $E \in \mathcal{H}$, and we say \mathcal{H} is \mathcal{G} -colorable if there exists a map $\varphi: V(\mathcal{H}) \rightarrow V(\mathcal{G})$ so that $\varphi(E) \in \mathcal{G}$ for all $E \in \mathcal{H}$. In other words, \mathcal{H} is \mathcal{G} -colorable if and only if \mathcal{H} occurs as a subgraph in some blow-up of \mathcal{G} .

Theorem 1.1. *For every positive integer t there exist constants $0 < n_1 < \dots < n_t$, $0 < \lambda_t < 1/6$, t triple systems $\mathcal{G}_1, \dots, \mathcal{G}_t$ with $v(\mathcal{G}_i) = n_i$ for $i \in [t]$, and a finite family \mathcal{M}_t of triple systems with the following properties.*

- (a) *The inequality $\text{ex}(n, \mathcal{M}_t) \leq \lambda_t n^3$ holds for all positive integers n , and moreover, equality holds whenever n is a multiple of n_i for some $i \in [t]$.*

- (b) For every $\delta > 0$ there exist $\varepsilon > 0$ and N_0 so that the following holds for all $n \geq N_0$. Every \mathcal{M}_t -free triple system \mathcal{H} on n vertices with at least $(\lambda_t - \varepsilon)n^3$ edges can be made \mathcal{G}_i -colorable for some $i \in [t]$ by removing at most δn vertices. Moreover, $\xi(\mathcal{M}_t) = t$.

1.2. Feasible regions. Recall that the *shadow* of an r -graph \mathcal{H} is defined to be the $(r-1)$ -graph

$$\partial\mathcal{H} = \left\{ A \in \binom{V(\mathcal{H})}{r-1} : \text{there is } B \in \mathcal{H} \text{ such that } A \subseteq B \right\}.$$

Call $d(\mathcal{H}) = |\mathcal{H}| / \binom{v(\mathcal{H})}{r}$ the *edge density* and $d(\partial\mathcal{H}) = |\partial\mathcal{H}| / \binom{v(\mathcal{H})}{r-1}$ the *shadow density* of \mathcal{H} .

Given a family \mathcal{F} the *feasible region* $\Omega(\mathcal{F})$ of \mathcal{F} is the set of points $(x, y) \in [0, 1]^2$ such that there exists a sequence of \mathcal{F} -free r -graphs $(\mathcal{H}_k)_{k=1}^\infty$ with $\lim_{k \rightarrow \infty} v(\mathcal{H}_k) = \infty$, $\lim_{k \rightarrow \infty} d(\partial\mathcal{H}_k) = x$, and $\lim_{k \rightarrow \infty} d(\mathcal{H}_k) = y$. The feasible region unifies and generalizes many classical problems such as the Kruskal-Katona theorem [12, 17] and the Turán problem. It was introduced recently in [18] to understand the extremal properties of \mathcal{F} -free hypergraphs beyond just the determination of $\pi(\mathcal{F})$. The general shape of $\Omega(\mathcal{F})$ was analyzed in [18] as follows: For some constant $c(\mathcal{F}) \in [0, 1]$ the projection to the first coordinate,

$$\text{proj}\Omega(\mathcal{F}) = \{x : \text{there is } y \in [0, 1] \text{ such that } (x, y) \in \Omega(\mathcal{F})\},$$

is the interval $[0, c(\mathcal{F})]$. Moreover, there is a left-continuous almost everywhere differentiable function $g(\mathcal{F}) : \text{proj}\Omega(\mathcal{F}) \rightarrow [0, 1]$ such that

$$\Omega(\mathcal{F}) = \{(x, y) \in [0, c(\mathcal{F})] \times [0, 1] : 0 \leq y \leq g(\mathcal{F})(x)\}.$$

Let us call $g(\mathcal{F})$ the *feasible region function* of \mathcal{F} . There are examples showing that $g(\mathcal{F})$ is not necessarily continuous (see [18, Theorem 1.12]) and the present work is part of an effort to figure out how “exotic” these functions can be.

The stability number of \mathcal{F} can give information about the shape of $\Omega(\mathcal{F})$, more precisely, about the number of global maxima of $g(\mathcal{F})$ (e.g. see Proposition 5.2). The family \mathcal{M} of triple systems from [19] for which $\xi(\mathcal{M}) = 2$ has the following additional property: not only are the two near extremal constructions for \mathcal{M} far from each other in edit-distance, but the same is true of their shadows. As a consequence, in addition to $\xi(\mathcal{M}) = 2$, the function $g(\mathcal{M})$ has exactly two global maxima. The authors raised the question of whether there exists a finite family \mathcal{M}_t of triple systems so that the function $g(\mathcal{M}_t)$ has exactly t global maxima for $t \geq 3$ (see [18, Problem 6.10]). Our second main result asserts that the objects constructed in the course of proving Theorem 1.1 give a positive solution to this problem.

Theorem 1.2. *For every positive integer t there exist constants $0 < n_1 < \dots < n_t$, $0 < \lambda_t < 1/6$, and a finite family \mathcal{M}_t of triple systems such that $\text{proj}\Omega(\mathcal{M}_t) = [0, 1]$, and $g(\mathcal{M}_t, x) \leq 6\lambda_t$ for all $x \in [0, 1]$. Moreover, $g(\mathcal{M}_t, x) = 6\lambda_t$ if and only if $x = 1 - 1/n_i$ for some $i \in [t]$.*

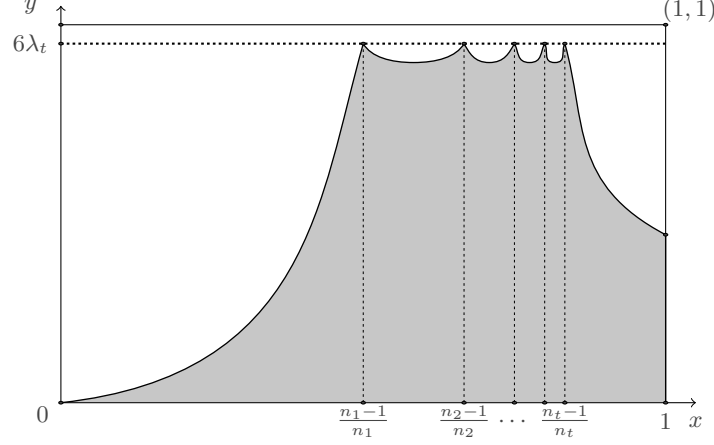


FIGURE 1.1. The function $g(\mathcal{M}_t)$ has exactly t global maxima.

Roughly speaking, the connection between these results is as follows. An r -graph is a *star* if there is a vertex v such that all edges contain v , and an r -graph \mathcal{H} is *semibipartite* if it is \mathcal{S} -colorable for some star \mathcal{S} . Note that this is the same as saying that $V(\mathcal{H})$ has a partition into two parts A and B such that all edges have exactly one vertex in A and $r - 1$ vertices in B . We will see later that our definition of \mathcal{M}_t ensures that every semibipartite 3-graph is \mathcal{M}_t -free. By shrinking A , the shadow density of an n -vertex semibipartite 3-graph \mathcal{H} can be made arbitrarily close to 1 as $n \rightarrow \infty$, so $\text{proj}\Omega(\mathcal{M}_t) = [0, 1]$. The shadows of the triple systems $\mathcal{G}_1, \dots, \mathcal{G}_t$ from Theorem 1.1 are complete graphs and thus their edge densities are the distinct numbers $1 - 1/n_1, \dots, 1 - 1/n_t$. So $g(\mathcal{M}_t, x) = 6\lambda_t$ holds if x is one of those densities and stability allows us to exclude further solutions to this equation.

Organization. In Section 2 we present some definitions related to the Lagrangian of hypergraphs and prove a result about the Lagrangian of a class of almost complete 3-graphs. In Section 3 we use the result from Section 2 to define the extremal configurations, which are balanced blow-ups of $\mathcal{G}_1, \dots, \mathcal{G}_t$, define the forbidden family \mathcal{M}_t , and prove the first part of Theorem 1.1. We prove the second part of Theorem 1.1 in Section 4, and Theorem 1.2 in Section 5. Section 6 contains some concluding remarks on generalisations to r -graphs and open problems.

§2. LAGRANGIAN

In this section we present some definitions related to the Lagrangian of a hypergraph, introduced by Frankl and Rödl in [9], and prove a result (Proposition 2.2 below) about certain almost complete triple systems.

Let \mathcal{G} be an r -graph for some $r \geq 2$. The *neighborhood* of a vertex $v \in V(\mathcal{G})$ is defined to be

$$N_{\mathcal{G}}(v) = \{u \in V(\mathcal{G}) \setminus \{v\} : \text{there is } A \in \mathcal{G} \text{ such that } \{u, v\} \subseteq A\},$$

the link of v is

$$L_{\mathcal{G}}(v) = \{A \in \mathcal{G} : v \in A\},$$

and $d_{\mathcal{G}}(v) = |L_{\mathcal{G}}(v)|$ is called the *degree* of v . Denote by $\delta(\mathcal{G}), \Delta(\mathcal{G})$ the minimum and maximum degree of \mathcal{G} , respectively. For a pair of vertices $u, v \in V(\mathcal{G})$ the neighborhood of $\{u, v\}$ is

$$N_{\mathcal{G}}(u, v) = \{w \in V(\mathcal{G}) \setminus \{u, v\} : \exists A \in \mathcal{G} \text{ such that } \{u, v, w\} \subseteq A\},$$

and $d_{\mathcal{G}}(u, v) = |N_{\mathcal{G}}(u, v)|$ is called the *codegree* of $\{u, v\}$. Denote by $\delta_2(\mathcal{G}), \Delta_2(\mathcal{G})$ the minimum and maximum codegree of \mathcal{G} , respectively.

For an r -graph \mathcal{G} on n vertices (let us assume for notational transparency that $V(\mathcal{G}) = [n]$) the multilinear function $L_{\mathcal{G}} : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$L_{\mathcal{G}}(x_1, \dots, x_n) = \sum_{E \in \mathcal{G}} \prod_{i \in E} x_i \quad \text{for all } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Denote by Δ_{n-1} the standard $(n-1)$ -dimensional simplex, i.e.

$$\Delta_{n-1} = \{(x_1, \dots, x_n) \in [0, 1]^n : x_1 + \dots + x_n = 1\}.$$

Since Δ_{n-1} is compact, a theorem of Weierstraß implies that the restriction of $L_{\mathcal{G}}$ to Δ_{n-1} attains a maximum value, called the *Lagrangian* of \mathcal{G} and denoted by $\lambda(\mathcal{G})$.

For a hypergraph \mathcal{G} the maximum number of edges in a blow-up of \mathcal{G} is related to $\lambda(\mathcal{G})$ (e.g. see Frankl and Füredi [8] or Keevash's survey [13, Section 3]).

Lemma 2.1 ([8, 13]). *Let $r \geq 2$ and let \mathcal{G}, \mathcal{H} be two r -graphs. If \mathcal{H} is a blow up of \mathcal{G} , then $|\mathcal{H}| \leq \lambda(\mathcal{G})v(\mathcal{H})^r$.*

Given a 3-graph \mathcal{G} , by plugging $(1/n, \dots, 1/n)$ into $L_{\mathcal{G}}$ one immediately obtains the lower bound $\lambda(L_{\mathcal{G}}) \geq |\mathcal{G}|/n^3$. It is well known that for cliques $\mathcal{H} = K_n^3$ this holds with equality and, moreover, that $(1/n, \dots, 1/n)$ is the only point in the simplex Δ_{n-1} , where $L_{\mathcal{H}}$ attains this maximum value.

The main result of this section, Proposition 2.2 below, exhibits a class of almost complete 3-graphs having the same properties. This will allow us later to construct for every given positive integer t a family $\{\mathcal{G}_1, \dots, \mathcal{G}_t\}$ of 3-graphs and a rational number λ_t close to $1/6$ such that $\lambda(\mathcal{G}_i) = |\mathcal{G}_i|/v(\mathcal{G}_i)^3 = \lambda_t$ holds for all $i \in [t]$. The extremal configurations for our hypergraph Turán problem are then going to be balanced blow-ups of $\mathcal{G}_1, \dots, \mathcal{G}_t$. As we can accomplish $v(\mathcal{G}_1) < \dots < v(\mathcal{G}_t)$, this is relevant to Theorem 1.2.

Let us observe that every hypergraph \mathcal{G} satisfying $\lambda(\mathcal{G}) = |\mathcal{G}|/v(\mathcal{G})^3$ needs to be regular in the sense that all vertices have the same degree. In the converse direction, regular hypergraphs can still have much larger Lagrangians than $|\mathcal{G}|/v(\mathcal{G})^3$. For instance, the Lagrangian of the Fano plane is $1/27$ but not $1/49$. To avoid such situations we utilize a design theoretic construction.

For the purposes of this article, by an (n, k) -design we shall mean a k -graph \mathcal{D} on n vertices such that every pair of vertices is covered by a unique edge. With every such design \mathcal{D} we associate the 3-graph

$$H(\mathcal{D}) = \bigcup_{E \in \mathcal{D}} \binom{E}{3}.$$

on $V(\mathcal{D})$. Note that

$$|H(\mathcal{D})| = \binom{k}{3} \frac{\binom{n}{2}}{\binom{k}{2}} = \frac{k-2}{6} n(n-1).$$

It will turn out that for $n \geq 18k$ every 3-graph of the form $\mathcal{G} = K_n^3 \setminus H(\mathcal{D})$, where \mathcal{D} is an (n, k) -design on $[n]$, has the property $\lambda(\mathcal{G}) = |\mathcal{G}|/v(\mathcal{G})^3$. In order to increase our control over the resulting value of $\lambda(\mathcal{G})$ Proposition 2.2 allows the extra flexibility to subtract a very sparse regular 3-graph from \mathcal{G} . Moreover, for reasons related to stability we state slightly more than just the actual value of the Lagrangian.

Proposition 2.2. *Suppose that $n \geq 18k + 37s^3$, \mathcal{D} is an (n, k) -design on $[n]$, and \mathcal{S} is an s -regular 3-graph on $[n]$. If $\mathcal{S} \cap H(\mathcal{D}) = \emptyset$ and $\mathcal{G} = K_n^3 \setminus (H(\mathcal{D}) \cup \mathcal{S})$, then*

$$L_{\mathcal{G}}(x_1, \dots, x_n) + \frac{1}{9} \sum_{i=1}^n \left(x_i - \frac{1}{n}\right)^2 \leq \frac{|\mathcal{G}|}{n^3} = \frac{1}{6} \left(1 - \frac{k+1}{n} + \frac{k-2s}{n^2}\right) \quad (2.1)$$

holds for all $(x_1, \dots, x_n) \in \Delta_{n-1}$ and, consequently,

$$\lambda(\mathcal{G}) = \frac{1}{6} \left(1 - \frac{k+1}{n} + \frac{k-2s}{n^2}\right). \quad (2.2)$$

We start with a simple observation that will come in handily later.

Fact 2.3. *Let \mathcal{G} be a 3-graph with vertex set $[n]$ and let $\alpha \geq 0$ be a real number. If the real numbers $\alpha_1, \dots, \alpha_n \in [-1, \alpha]$ sum up to zero, then*

$$L_{\mathcal{G}}(\alpha_1, \dots, \alpha_n) \leq (\alpha n)^3.$$

Proof of Fact 2.3. Define $P = \{i \in [n] : \alpha_i > 0\}$ to be the set of vertices of \mathcal{G} with positive weight. Let us decompose $L_{\mathcal{G}}(\alpha_1, \dots, \alpha_n) = S_0 + S_1 + S_2 + S_3$ such that for $m \in \{0, 1, 2, 3\}$ the sum S_m consists of all terms $\alpha_i \alpha_j \alpha_k$ contributing to $L_{\mathcal{G}}$ and satisfying $|P \cap \{i, j, k\}| = m$.

As the sums S_0 and S_2 possess no positive terms, we have $S_0, S_2 \leq 0$. Moreover, S_3 has no more than $\binom{|P|}{3} \leq n^3/6$ summands each of which amounts to at most α^3 , wherefore S_3 is at most $(\alpha n)^3/6$. Thus to conclude the argument it is more than enough to show $S_1 \leq (\alpha n)^3/2$.

Writing $W = \sum_{i \in P} \alpha_i$ we have $\sum_{i \in [n] \setminus P} \alpha_i = -W$ and

$$S_1 \leq \sum_{i \in P} \alpha_i \cdot \sum_{jk \in \binom{[n] \setminus P}{2}} \alpha_j \alpha_k \leq W \cdot (W^2/2) = W^3/2,$$

which by $|W| \leq \alpha|P| \leq \alpha n$ completes the proof. \square

Proof of Proposition 2.2. Since the left side of (2.1) is continuous in (x_1, \dots, x_n) and Δ_{n-1} is compact, there exists a point $\xi = (\xi_1, \dots, \xi_n) \in \Delta_{n-1}$ such that

$$\omega = L_{\mathcal{G}}(\xi_1, \dots, \xi_n) + \frac{1}{9} \sum_{i=1}^n \left(\xi_i - \frac{1}{n} \right)^2 - \frac{1}{6} \left(1 - \frac{k+1}{n} + \frac{k-2s}{n^2} \right) \quad (2.3)$$

is maximum. Assume for the sake of contradiction that $\omega > 0$.

Claim 2.4. *There exists an index $i(\star) \in [n]$ such that $\xi_{i(\star)} > \frac{1}{n} + \frac{9s}{n^2}$.*

Proof of Claim 2.4. Define $\alpha_1, \dots, \alpha_n \in [-1, n-1]$ by $\xi_i = (1 + \alpha_i)/n$ for every $i \in [n]$ and observe that

$$\begin{aligned} \omega n^3 &= L_{\mathcal{G}}(1 + \alpha_1, \dots, 1 + \alpha_n) + \frac{n}{9} \sum_{i=1}^n \alpha_i^2 - |\mathcal{G}| \\ &= \sum_{i=1}^n d_{\mathcal{G}}(i) \alpha_i + \sum_{1 \leq i < j \leq n} d_{\mathcal{G}}(i, j) \alpha_i \alpha_j + L_{\mathcal{G}}(\alpha_1, \dots, \alpha_n) + \frac{n}{9} \sum_{i=1}^n \alpha_i^2. \end{aligned}$$

Since all vertices of \mathcal{G} have the same degree and $\sum_{i=1}^n \alpha_i = n(\sum_{i=1}^n \xi_i - 1) = 0$, the first sum on the right side vanishes. Moreover, all pairs of vertices have codegree $n-k$ in $K_n^3 \setminus H(\mathcal{D})$ and thus we obtain

$$\omega n^3 = \left(\frac{n}{9} - \frac{n-k}{2} \right) \sum_{i=1}^n \alpha_i^2 - \sum_{1 \leq i < j \leq n} d_{\mathcal{S}}(i, j) \alpha_i \alpha_j + L_{\mathcal{G}}(\alpha_1, \dots, \alpha_n). \quad (2.4)$$

First case: We have $\xi_1, \dots, \xi_n > 0$.

Collecting the quadratic and cubic terms in (2.4) separately we put

$$Q = \left(\frac{n}{9} - \frac{n-k}{2} \right) \sum_{i=1}^n \alpha_i^2 - \sum_{1 \leq i < j \leq n} d_S(i, j) \alpha_i \alpha_j \quad \text{and} \quad K = L_{\mathcal{G}}(\alpha_1, \dots, \alpha_n),$$

so that

$$\omega n^3 = Q + K.$$

Now for every real number C sufficiently close to 1 the point (ξ'_1, \dots, ξ'_n) defined by $\xi'_i = (1 + C\alpha_i)/n$ belongs to Δ_{n-1} and the maximal choice of ω reveals

$$L_{\mathcal{G}}(\xi'_1, \dots, \xi'_n) + \frac{1}{9} \sum_{i=1}^n \left(\xi'_i - \frac{1}{n} \right)^2 - \frac{1}{6} \left(1 - \frac{k+1}{n} + \frac{k-2s}{n^2} \right) \leq \omega.$$

Multiplying by n^3 and repeating the above calculation we obtain $QC^2 + KC^3 \leq Q + K$ and thus

$$0 \leq (1 - C)[(1 + C)Q + (1 + C + C^2)K] \quad (2.5)$$

whenever $|C - 1|$ is sufficiently small. Letting C tend to 1 from above and below we obtain $2Q + 3K = 0$. Substituting this back into (2.5) we learn

$$0 \leq (1 - C)[(C - 1)Q + (C^2 + C - 2)K] = -(1 - C)^2[Q + (C + 2)K].$$

Thus $Q + 3K \leq (1 - C)K$ holds whenever $|C - 1|$ is sufficiently small, which is only possible if $Q + 3K \leq 0$. Together with $Q + K = \omega n^3 > 0$ this yields $K < 0$ and $\omega n^3 < Q - K = (-1)^2 Q + (-1)^3 Q$. So the maximality of ω tells us that for $C = -1$ we have $(\xi'_1, \dots, \xi'_n) \notin \Delta_{n-1}$. In other words, there is some $i(\star) \in [n]$ such that

$$\xi_{i(\star)} \geq \frac{2}{n} > \frac{1}{n} + \frac{9s}{n^3},$$

as desired.

Second case: There exists some $j(\star) \in [n]$ satisfying $\xi_{j(\star)} = 0$.

Now $\alpha_{j(\star)} = -1$ and, consequently,

$$\sum_{i=1}^n \alpha_i^2 \geq 1. \quad (2.6)$$

Next we observe that the hypothesis that \mathcal{S} be s -regular yields

$$- \sum_{1 \leq i < j \leq n} d_S(i, j) \alpha_i \alpha_j \leq \sum_{1 \leq i < j \leq n} d_S(i, j) \frac{\alpha_i^2 + \alpha_j^2}{2} = s \sum_{i=1}^n \alpha_i^2.$$

Combined with (2.4) and the positivity of ω this shows

$$\left(\frac{n-k}{2} - \frac{n}{9} - s \right) \sum_{i=1}^n \alpha_i^2 < L_{\mathcal{G}}(\alpha_1, \dots, \alpha_n). \quad (2.7)$$

Due to $n \geq 18k + 3^7 s^3$ we have

$$\frac{n-k}{2} - \frac{n}{9} - s > \left(\frac{1}{2} - \frac{1}{36} - \frac{1}{9} - \frac{1}{36} \right) n = \frac{n}{3} > (9s)^3$$

and together with (2.6), (2.7) this establishes

$$(9s)^3 < L_{\mathcal{G}}(\alpha_1, \dots, \alpha_n).$$

In view of Fact 2.3 we deduce that $\alpha_{i(\star)} > 9s/n$ holds for some $i(\star) \in [n]$ and now

$$\xi_{i(\star)} = \frac{1 + \alpha_{i(\star)}}{n} > \frac{1}{n} + \frac{9s}{n^2}$$

follows. Thereby Claim 2.4 is proved. \square

Now for every $i \in [n]$ we set

$$D_i = \frac{\partial L_{\mathcal{G}}(x_1, \dots, x_n)}{\partial x_i} \Big|_{(\xi_1, \dots, \xi_n)} = \sum_{jk \in L_i} \xi_j \xi_k,$$

where L_i denotes the link graph of i in \mathcal{G} . Owing to the maximality of ω in (2.3) the Lagrange multiplier method leads to the existence of a real number M such that

$$D_i + \frac{2}{9} \left(\xi_i - \frac{1}{n} \right) = M$$

holds for every vertex $i \in [n]$ with $\xi_i > 0$. Notice that

$$\begin{aligned} M &= M \sum_{j=1}^n \xi_j = \sum_{j=1}^n \xi_j \left(D_j + \frac{2}{9} \left(\xi_j - \frac{1}{n} \right) \right) = 3L_{\mathcal{G}}(\xi_1, \dots, \xi_n) + \frac{2}{9} \sum_{j=1}^n \left(\xi_j - \frac{1}{n} \right)^2 \\ &\stackrel{(2.3)}{>} \frac{1}{2} \left(1 - \frac{k+1}{n} + \frac{k-2s}{n^2} \right) - \frac{1}{9} \sum_{j=1}^n \left(\xi_j - \frac{1}{n} \right)^2. \end{aligned}$$

Altogether, this proves

$$D_i + \frac{2}{9} \left(\xi_i - \frac{1}{n} \right) + \frac{1}{9} \sum_{j=1}^n \left(\xi_j - \frac{1}{n} \right)^2 > \frac{1}{2} \left(1 - \frac{k+1}{n} + \frac{k-2s}{n^2} \right)$$

for every vertex $i \in [n]$ satisfying $\xi_i > 0$.

By our design theoretic construction, the link in $K_n^3 \setminus H(\mathcal{D})$ of every vertex $i \in [n]$ is a q -partite Turán graph with vertex classes of size $k-1$, where $q = (n-1)/(k-1)$ is an integer. Consequently, there exist real numbers β_1, \dots, β_q such that $\xi_i + (\beta_1 + \dots + \beta_q) = 1$ and

$$D_i \leq \sum_{1 \leq v < w \leq q} \beta_v \beta_w \leq \frac{q-1}{2q} (\beta_1 + \dots + \beta_q)^2 = \frac{n-k}{2(n-1)} (1 - \xi_i)^2.$$

Summarizing, we have

$$\frac{2}{9} \left(\xi_i - \frac{1}{n} \right) + \frac{1}{9} \sum_{j=1}^n \left(\xi_j - \frac{1}{n} \right)^2 > \frac{n-k}{2(n-1)} \left(\left(1 - \frac{1}{n} \right)^2 - (1 - \xi_i)^2 \right) - \frac{s}{n^2} \quad (2.8)$$

for every vertex of positive weight. For the rest of the argument we fix a vertex $i(\star) \in [n]$ such that $\xi_{i(\star)}$ is maximal. Let us add the trivial estimate

$$\frac{1}{9} \sum_{j=1}^n \xi_j (\xi_{i(\star)} - \xi_j) \geq 0$$

to the case $i = i(\star)$ of (2.8). Because of

$$\sum_{j=1}^n \left(\xi_j - \frac{1}{n} \right)^2 + \sum_{j=1}^n \xi_j (\xi_{i(\star)} - \xi_j) = \sum_{j=1}^n \xi_j \left(\xi_{i(\star)} - \frac{1}{n} \right) - \frac{1}{n} \sum_{j=1}^n \left(\xi_j - \frac{1}{n} \right) \quad (2.9)$$

$$= \xi_{i(\star)} - \frac{1}{n} \quad (2.10)$$

this yields

$$\begin{aligned} \frac{1}{3} \left(\xi_{i(\star)} - \frac{1}{n} \right) &> \frac{n-k}{2(n-1)} \left(\xi_{i(\star)} - \frac{1}{n} \right) \left(2 - \frac{1}{n} - \xi_{i(\star)} \right) - \frac{s}{n^2} \\ &\geq \frac{n-k}{2n} \left(\xi_{i(\star)} - \frac{1}{n} \right) - \frac{s}{n^2} \geq \frac{4}{9} \left(\xi_{i(\star)} - \frac{1}{n} \right) - \frac{s}{n^2}, \end{aligned}$$

whence

$$\xi_{i(\star)} < \frac{1}{n} + \frac{9s}{n^2}.$$

Owing to the maximal choice of $\xi_{i(\star)}$ this contradicts Claim 2.4. \square

§3. CONSTRUCTIONS AND TURÁN NUMBERS

Given a positive integer t we define in this section the triple systems $\mathcal{G}_1, \dots, \mathcal{G}_t$ and the forbidden family \mathcal{M}_t appearing in Theorem 1.1. For every $i \in [t]$ there will be three integers n_i, k_i, s_i such that $\mathcal{G}_i = K_{n_i}^3 \setminus (H(\mathcal{D}_i) \cup \mathcal{S}_i)$ holds for some (n_i, k_i) -design \mathcal{D}_i on $[n_i]$ and some s_i -regular triple system \mathcal{S}_i on $[n_i]$ that is disjoint to $H(\mathcal{D}_i)$. As we shall have $n_i \gg k_i, s_i$, Proposition 2.2 will imply

$$\lambda(\mathcal{G}_i) = \frac{1}{6} \left(1 - \frac{k_i + 1}{n_i} + \frac{k_i - 2s_i}{n_i^2} \right).$$

Part of our goal is that balanced blow-ups of $\mathcal{G}_1, \dots, \mathcal{G}_t$ should be extremal \mathcal{M}_t -free triple systems and for this reason we need to ensure $\lambda(\mathcal{G}_i) = \dots = \lambda(\mathcal{G}_t)$. We shall achieve this by letting $k_i = 2s_i$ for $i \in [t]$, and by guaranteeing

$$\frac{k_1 + 1}{n_1} = \dots = \frac{k_t + 1}{n_t}. \quad (3.1)$$

The details of this construction are given in Subsection 3.1 and the exact Turán numbers of our families \mathcal{M}_t are determined in Subsection 3.2.

3.1. The extremal configurations and forbidden family. First, we need the following theorem about the existence of designs due to Wilson [31–33].

Theorem 3.1 (Wilson [31–33]). *For every integer $k \geq 2$ there exists a threshold $n_0(k)$ such that for every integer $n \geq n_0(k)$ satisfying the divisibility conditions $(k-1) \mid (n-1)$ and $(k-1)k \mid (n-1)n$ there exists an (n, k) -design.*

Our next lemma deals with the arithmetic properties the numbers k_1, \dots, k_t and n_1, \dots, n_t entering the construction of $\mathcal{G}_1, \dots, \mathcal{G}_t$ need to satisfy. Apart from (3.1) and the divisibility conditions in Theorem 3.1 we will require that n_1, \dots, n_t are divisible by 3 so that $(k_i/2)$ -regular triple systems on n_i vertices exist. Thus the case $q = 3$ of the following lemma is exactly what we need.

Lemma 3.2. *Given positive integers t and q there exist t even integers $3 < k_1 < \dots < k_t$ such that for every constant $C > 0$ there exist t integers $n_1 < \dots < n_t$ with the following properties.*

- (a) *We have $q \mid n_i$, $(k_i - 1) \mid (n_i - 1)$, and $k_i(k_i - 1) \mid n_i(n_i - 1)$ for all $i \in [t]$.*
- (b) *Moreover,*

$$Q = \frac{n_1}{k_1 + 1} = \dots = \frac{n_t}{k_t + 1}$$

is an integer with $Q \geq C$.

Proof of Lemma 3.2. Starting with an arbitrary positive multiple s_1 of q we recursively define integers $1 \leq s_1 < \dots < s_t$ by setting $s_{i+1} = \prod_{j \leq i} s_j(2s_j - 1) + 1$ for every $i \in [t-1]$. Now whenever $1 \leq i < j \leq t$ we have $s_j \equiv 1 \pmod{s_i(2s_i - 1)}$ and, consequently,

$$s_j(2s_j - 1) \equiv 1 \pmod{s_i(2s_i - 1)}.$$

In particular, the numbers

$$s_1(2s_1 - 1), \dots, s_t(2s_t - 1)$$

are pairwise coprime and by the Chinese remainder theorem there exists an even integer $Q \geq C$ such that $Q/2 \equiv s_i^2 \pmod{s_i(2s_i - 1)}$ holds for all $i \in [t]$. Multiplying these congruences by 2 and setting $k_i = 2s_i$ we obtain

$$Q \equiv k_i^2/2 \pmod{k_i(k_i - 1)}. \quad (3.2)$$

Now it is plain that the numbers $n_i = Q(k_i + 1)$ satisfy (b). Moreover, the case $i = 1$ of (3.2) yields $q \mid k_1 \mid Q$ and, therefore, n_1, \dots, n_t are divisible by q . Finally, multiplying (3.2) by $k_i + 1$ we learn

$$n_i \equiv k_i(k_i + 1)(k_i/2) \equiv 2k_i(k_i/2) \equiv k_i^2 \equiv k_i \pmod{k_i(k_i - 1)},$$

for which reason $k_i \mid n_i$ and $(k_i - 1) \mid (n_i - 1)$. So altogether (a) holds as well. \square

Given two r -graphs \mathcal{H}_1 and \mathcal{H}_2 with the same number of vertices a *packing* of \mathcal{H}_1 and \mathcal{H}_2 is a bijection $\varphi: V(\mathcal{H}_1) \rightarrow V(\mathcal{H}_2)$ such that $\varphi(E) \notin \mathcal{H}_2$ for all $E \in \mathcal{H}_1$. In order to proceed with our construction of the triple systems $\mathcal{G}_1, \dots, \mathcal{G}_t$ we need to argue that, under natural assumptions, if \mathcal{D}_i denotes an (n_i, k_i) -design, then there is an s_i -regular 3-graph $\mathcal{S}_i \subseteq K_n^3 \setminus H(\mathcal{D}_i)$, where $s_i = k_i/2$. Provided that $3 \mid n_i$ and $s_i \leq \binom{n-1}{2}$ the existence of some s_i -regular 3-graph $\mathcal{S}_i \subseteq K_n^3$ is a well known fact that follows, e.g., from Baranyai's factorisation theorem [2]. For making \mathcal{S}_i and $H(\mathcal{D}_i)$ disjoint we use a packing argument based on the following result of Lu and Székely.

Theorem 3.3 (Lu-Székely [23]). *Let \mathcal{H}_1 and \mathcal{H}_2 be two r -graphs on n vertices. If*

$$\Delta(\mathcal{H}_1)|\mathcal{H}_2| + \Delta(\mathcal{H}_2)|\mathcal{H}_1| < \frac{1}{er} \binom{n}{r},$$

then there is a packing of \mathcal{H}_1 and \mathcal{H}_2 .

In fact, we only require the following consequence.

Corollary 3.4. *Suppose $3 \mid n$ and that \mathcal{D} is an (n, k) -design on $[n]$. If $s < \frac{n-2}{6e(k-2)}$, then there exists an s -regular 3-graph \mathcal{S} on $[n]$ such that $\mathcal{S} \cap H(\mathcal{D}) = \emptyset$.*

Proof of Corollary 3.4. By $3 \mid n$ and $s \leq \binom{n-1}{2}$ there is an s -regular 3-graph \mathcal{S}' on n vertices. Since

$$\begin{aligned} \Delta(\mathcal{S}')|H(\mathcal{D})| + \Delta(H(\mathcal{D}))|\mathcal{S}'| &= s \frac{k-2}{6} n(n-1) + \frac{k-2}{2} (n-1) \frac{sn}{3} \\ &= s \frac{k-2}{3} n(n-1) < \frac{n-2}{6e(k-2)} \frac{k-2}{3} n(n-1) \\ &= \frac{1}{3e} \binom{n}{3}, \end{aligned}$$

Theorem 3.3 yields a packing $\varphi: V(\mathcal{S}') \rightarrow [n]$ of \mathcal{S}' and $H(\mathcal{D})$. It is clear that $\mathcal{S} = \varphi(\mathcal{S}')$ satisfies the requirements of Corollary 3.4. \square

Now we are ready to present the definition of $\mathcal{G}_1, \dots, \mathcal{G}_t$.

Construction 3.5. *Given a positive integer t perform the following steps.*

- Apply Lemma 3.2 with $q = 3$, thus getting some even integers $3 < k_1 < \dots < k_t$.

- Take an integer $C \geq \max\{n_0(k_1), \dots, n_0(k_t), 2k_t^3, 3^8\}$, where the thresholds $n_0(k_i)$ are given by Theorem 3.1.
- Now Lemma 3.2 applied to C and k_1, \dots, k_t yields integers $C < n_1 < \dots < n_k$ such that, in particular,

$$Q = \frac{n_1}{k_1 + 1} = \dots = \frac{n_t}{k_t + 1}$$

is an integer with $Q \geq C$.

Now, for every $i \in [t]$

- let \mathcal{D}_i be an (n_i, k_i) -design on $[n_i]$ (as obtained by Theorem 3.1)
- let \mathcal{S}_i be a $(k_i/2)$ -regular 3-graph on $[n_i]$ such that $\mathcal{S}_i \cap H(\mathcal{D}_i) = \emptyset$ (as obtained by Corollary 3.4).
- and, finally, define

$$\mathcal{G}_i = K_{n_i}^3 \setminus (H(\mathcal{D}_i) \cup \mathcal{S}_i).$$

By Proposition 2.2 we have

$$\lambda(\mathcal{G}_i) = \frac{1}{6} \left(1 - \frac{k_i + 1}{n_i} + \frac{k_i - 2k_i/2}{n_i^2} \right) = \frac{1}{6} \left(1 - \frac{1}{Q} \right).$$

for every $i \in [t]$, so some rational λ_t satisfies

$$\lambda_t = \lambda(\mathcal{G}_1) = \dots = \lambda(\mathcal{G}_t) \in [5/32, 1/6]. \quad (3.3)$$

In the remainder of this subsection we introduce the family \mathcal{M}_t . For an r -graph \mathcal{H} and a set $S \subseteq V(\mathcal{H})$ we say that S is *2-covered* in \mathcal{H} if for every pair of vertices in S there is an edge in \mathcal{H} containing it. If this holds for $S = V(\mathcal{H})$ then \mathcal{H} itself is said to be 2-covered.

For all integers $\ell > r \geq 2$ we let \mathcal{K}_ℓ^r denote the family of r -graphs F with at most $\binom{\ell}{2}$ edges that contain a 2-covered set S of ℓ vertices called a *core* of F . The family \mathcal{K}_ℓ^r was first introduced by the second author [24] in order to extend Turán's theorem to hypergraphs. It also plays a key rôle in the construction of the family \mathcal{M} with two extremal configurations in [19]. In the present work, we also need the larger family $\widehat{\mathcal{K}}_\ell^r$ defined to consist of all r -graphs F with at most $\binom{\ell}{r}$ edges that contain a 2-covered set S of ℓ vertices, which is again called a *core* of F .

Let us recall that the *transversal number* of a hypergraph \mathcal{H} is the nonnegative integer

$$\tau(\mathcal{H}) = \min \{|S| : S \subseteq V(\mathcal{H}) \text{ and } S \cap E \neq \emptyset \text{ for all } E \in \mathcal{H}\}.$$

Note that if \mathcal{H} is empty, then we can take $S = \emptyset$, whence $\tau(\mathcal{H}) = 0$ holds in this case. After these preparations, the family \mathcal{M}_t is defined as follows.

Definition 3.6. For every positive integer t the family \mathcal{M}_t consists of all 3-graphs $F \in \bigcup_{\ell \leq n_t} \hat{\mathcal{K}}_\ell^3$ which do not occur as a subgraph in any blow-up of $\mathcal{G}_1, \dots, \mathcal{G}_t$ and which have a core S such that $\tau(F[S]) \geq 2$.

We conclude this subsection with a simple sufficient condition for 3-graphs $F \in \mathcal{K}_{n_t+1}^3$ guaranteeing that they are in \mathcal{M}_t (see Lemma 3.9 below). For this purpose we require the following observation analysing the extent to which $\tau(\mathcal{H}) \geq 2$ is a “local” property of a hypergraph \mathcal{H} .

Fact 3.7. If $r \geq 2$ and \mathcal{H} denotes an r -graph with $\tau(\mathcal{H}) \geq 2$, then there is a subgraph $\mathcal{H}' \subseteq \mathcal{H}$ with at most $r + 1$ edges satisfying $\tau(\mathcal{H}') \geq 2$.

Proof. Pick two distinct edges $E', E'' \in \mathcal{H}$ and write $E' \cap E'' = \{v_1, \dots, v_m\}$, where $0 \leq m \leq r - 1$. For every $i \in [m]$ the assumption that $\{v_i\}$ fails to cover \mathcal{H} yields an edge $E_i \in \mathcal{H}$ such that $v_i \notin E_i$. Now $\mathcal{H}' = \{E', E'', E_1, \dots, E_m\}$ has the desired properties. \square

Notice that the example $\mathcal{H} = K_{r+1}^r$ shows that the bound $|\mathcal{H}'| \leq r + 1$ is optimal.

Lemma 3.8. Suppose that F is a 3-graph and that $S \subseteq V(F)$ is a 2-covered set in F . If $\tau(F[S]) \geq 2$, then F contains a subgraph F' such that $F' \in \mathcal{K}_{|S|}^3$ and $\tau(F'[S]) \geq 2$. Moreover, if $12 \leq s \leq |S|$, then F has a subgraph $F'' \in \mathcal{K}_s^3$ possessing a core S'' such that $\tau(F''[S'']) \geq 2$.

Proof of Lemma 3.8. The case $r = 3$ of Fact 3.7 yields a subgraph \mathcal{G} of $F[S]$ with at most four edges such that $\tau(\mathcal{G}) \geq 2$. Notice that $|\mathcal{G}| \geq 2$ and $|\partial\mathcal{G}| \geq 5$. Since S is 2-covered in F , we can choose for every pair $uw \in \binom{S}{2} \setminus \partial\mathcal{G}$ an edge $e_{uw} \in F$ containing u and w . Now

$$F' = \{e_{uw} : uw \in \binom{S}{2} \setminus \partial\mathcal{G}\} \cup \mathcal{G}$$

has the properties that S is 2-covered in F' and $\tau(F'[S]) \geq 2$. Together with

$$|F'| \leq \binom{\ell}{2} - |\partial\mathcal{G}| + |\mathcal{G}| \leq \binom{\ell}{2} - 5 + 4 < \binom{\ell}{2}$$

this proves $F' \in \mathcal{K}_{|S|}^3$. Moreover, if any $s \in [12, |S|]$ is given, we can take a set S'' of size s with $V(\mathcal{G}) \subseteq S'' \subseteq S$ and apply the first part of the lemma to S'' rather than S . \square

Lemma 3.9. If S denotes a core of $F \in \mathcal{K}_{n_t+1}^3$ and $\tau(F[S]) \geq 2$, then $F \in \mathcal{M}_t$.

Proof. By the previous lemma and $n_t \geq 12$ there exists a set $S'' \subseteq S$ such that $|S''| = n_t$ and $\tau(F[S'']) \geq 2$. Since $|F| \leq \binom{n_t+1}{2} \leq \binom{n_t}{3}$, we can regard F as a member of $\hat{\mathcal{K}}_{n_t}^3$ with core S'' and it remains to prove that F cannot be \mathcal{G}_i -colorable for any $i \in [t]$. This is due to the fact that the shadows of blow-ups of \mathcal{G}_i are complete n_i -partite graphs, while S induces a K_{n_t+1} in ∂F . \square

3.2. Turán numbers of \mathcal{M}_t . Having now introduced the main protagonists $\mathcal{G}_1, \dots, \mathcal{G}_t$ and \mathcal{M}_t we shall determine the extremal numbers $\text{ex}(n, \mathcal{M}_t)$ in this subsection. More precisely, setting

$$\mathfrak{M}(n) = \max \{|\mathcal{G}| : \mathcal{G} \text{ is } \mathcal{G}_i\text{-colorable for some } i \in [t] \text{ and } v(\mathcal{G}) = n\}$$

for every positive integer n we shall prove the following result.

Theorem 3.10. *The equality $\text{ex}(n, \mathcal{M}_t) = \mathfrak{M}(n)$ holds for all positive integers n .*

Notice that in view of Lemma 2.1 and (3.3) this implies $\text{ex}(n, \mathcal{M}_t) \leq \lambda_t n^3$ for every positive integer n . Moreover, whenever n is a multiple of n_i for some $i \in [t]$, the balanced blow-up of \mathcal{G}_i with factor n/n_i exemplifies that this holds with equality. For these reasons, Theorem 3.10 is stronger than Theorem 1.1 (a). Let us start with the lower bound on $\text{ex}(n, \mathcal{M}_t)$.

Fact 3.11. *We have $\text{ex}(n, \mathcal{M}_t) \geq \mathfrak{M}(n)$ for every positive integer n .*

Proof of 3.11. This is an immediate consequence of the fact that by Definition 3.6 for every $i \in [t]$ all blow-ups of \mathcal{G}_i are \mathcal{M}_t -free. \square

Our proof for the upper bound uses the Zykov symmetrization method [34]. The applicability of this technique in the current situation hinges on the fact that if a hypergraph \mathcal{H} is \mathcal{M}_t -free, then there is no homomorphism from a member of \mathcal{M}_t to \mathcal{H} (see Proposition 3.12 below). Let us recall that given two r -graphs F and \mathcal{H} a map $\varphi: V(F) \rightarrow V(\mathcal{H})$ is said to be a *homomorphism* if φ preserves edges, i.e., if $\varphi(E) \in \mathcal{H}$ holds for all $E \in F$. Further, \mathcal{H} is *F -hom-free* if there is no homomorphism from F to \mathcal{H} or, in other words, if F fails to be \mathcal{H} -colourable. For a family \mathcal{F} of r -graphs, we say that \mathcal{H} is *\mathcal{F} -hom-free* if it is F -hom-free for every $F \in \mathcal{F}$.

Proposition 3.12. *A 3-graph \mathcal{H} is \mathcal{M}_t -hom-free if and only if it is \mathcal{M}_t -free.*

Proof of Proposition 3.12. Notice that the forward implication is clear. Now suppose conversely that \mathcal{H} fails to be \mathcal{M}_t -hom-free, i.e., that there is a homomorphism $\varphi: V(F) \rightarrow V(\mathcal{H})$ for some $F \in \mathcal{M}_t$. Clearly the restriction of φ to a core S of F is injective. So $\varphi(F) \in \hat{\mathcal{K}}_{|S|}^3 \cap \mathcal{M}_t$ and in view of $\varphi(F) \subseteq \mathcal{H}$ it follows that \mathcal{H} fails to be \mathcal{M}_t -free. \square

As an immediate consequence of Definition 3.6, semibipartite triple systems are \mathcal{M}_t -free. We analyze the semibipartite case as follows.

Lemma 3.13. *If \mathcal{H} denotes a semibipartite triple system on n vertices, then*

$$|\mathcal{H}| \leq \min\{2n^3/27, \mathfrak{M}(n)\}.$$

Proof. Fix a partition $V(\mathcal{H}) = A \cup B$ such that $|E \cap A| = 1$ holds for every $E \in \mathcal{H}$. Now the AM-GM inequality yields

$$|\mathcal{H}| \leq |A| \binom{|B|}{2} \leq \frac{2|A| \cdot |B| \cdot |B|}{4} \leq \frac{1}{4} \left(\frac{2|A| + |B| + |B|}{3} \right)^3 = \frac{2n^3}{27}$$

and it remains to show $|\mathcal{H}| \leq \mathfrak{M}(n)$. If n is large this is an immediate consequence of $\mathfrak{M}(n) = (\lambda_t - o(1))n^3$ and $\lambda_t \geq 5/32 > 2/27$, but for a complete proof addressing all values of n we need to argue more carefully.

To this end we consider a random map $\varphi: [n] \rightarrow [n_1]$ together with the random blow-up $\hat{\mathcal{G}}$ of \mathcal{G}_1 determined by φ . Explicitly $\hat{\mathcal{G}}$ has vertex set $[n]$ and a triple ijk forms an edge of $\hat{\mathcal{G}}$ if and only if $\varphi(i)\varphi(j)\varphi(k) \in \mathcal{G}_1$. Now every potential edge of $\hat{\mathcal{G}}$ is present with probability $\frac{6|\mathcal{G}_1|}{n_1^3} = 6\lambda_t$ and thus the expectation of $|\hat{\mathcal{G}}|$ is $6\lambda_t \binom{n}{3}$. So by averaging we obtain

$$\mathfrak{M}(n) \geq 6\lambda_t \binom{n}{3} \geq \frac{15}{16} \binom{n}{3}, \quad (3.4)$$

which for $n \geq 5$ implies the desired estimate $\mathfrak{M}(n) \geq 2n^3/27$. Moreover, (3.4) yields $\mathfrak{M}(4) \geq 3$, which still suffices for the case $n = 4$ of our lemma. Finally, the case $n \leq 3$ is trivial. \square

The central notion in arguments based on Zykov symmetrization is the following: Given an r -graph \mathcal{H} , two non-adjacent vertices $u, v \in V(\mathcal{H})$ are said to be *equivalent* if $L_{\mathcal{H}}(u) = L_{\mathcal{H}}(v)$. Evidently, equivalence is an equivalence relation. Since any two equivalent vertices have the same degree and the same link, we can write $d_{\mathcal{H}}(C)$ and $L_{\mathcal{H}}(C)$ for the common degree and the common link of all vertices in an equivalence class C , respectively.

Lemma 3.14. *Let \mathcal{H} be an \mathcal{M}_t -free 3-graph with equivalence classes C_1, \dots, C_m . If for all distinct $k, \ell \in [m]$ the shadow $\partial\mathcal{H}$ induces a complete bipartite graph between C_k and C_ℓ , then \mathcal{H} is either semibipartite or \mathcal{G}_i -colourable for some $i \in [t]$.*

Proof of Lemma 3.14. Let $T \subseteq V(\mathcal{H})$ be a set containing exactly one vertex from each equivalence class of \mathcal{H} , and let \mathcal{T} be the subgraph of \mathcal{H} induced by T . By assumption, \mathcal{T} is 2-covered, $|T| = m$, and \mathcal{H} is a blow-up of \mathcal{T} . If $\tau(\mathcal{T}) < 2$, then \mathcal{T} is a star and \mathcal{H} is semibipartite. So we may assume $\tau(\mathcal{T}) \geq 2$ from now on.

Since \mathcal{T} is 2-covered and $|\mathcal{T}| \leq \binom{m}{3}$ we have $\mathcal{T} \in \hat{\mathcal{K}}_m^3$. So if $m \leq n_t$, then in view of Definition 3.6 and $\mathcal{T} \notin \mathcal{M}_t$ there exists an index $i \in [t]$ such that \mathcal{T} is \mathcal{G}_i -colorable. As \mathcal{H} is a blow-up of \mathcal{T} , it follows that \mathcal{H} is \mathcal{G}_i -colorable as well.

Now assume for the sake of contradiction that $m > n_t$. Since $n_t \geq 12$, Lemma 3.8 leads to a subgraph $\mathcal{T}'' \in \mathcal{K}_{n_t+1}^3$ of \mathcal{T} having a core S'' such that $\tau(\mathcal{T}''[S'']) \geq 2$. By Lemma 3.9 this contradicts \mathcal{H} being \mathcal{M}_t -free. \square

Now we are ready to establish the main result of this subsection.

Proof of Theorem 3.10. Fix some positive integer n . By Fact 3.11 it suffices to establish the upper bound $\text{ex}(n, \mathcal{M}_t) \leq \mathfrak{M}(n)$. Arguing indirectly we choose an \mathcal{M}_t -free triple system \mathcal{H} on n vertices with more than $\mathfrak{M}(n)$ edges such that the number m of equivalence classes of \mathcal{H} is minimal. Let C_1, \dots, C_m be the equivalence classes of \mathcal{H} .

By Lemma 3.13 we know that \mathcal{H} is not semibipartite and the definition of $\mathfrak{M}(n)$ implies that \mathcal{H} fails to be \mathcal{G}_i -colorable for every $i \in [t]$. For these reasons, Lemma 3.14 tells us that ∂H is not the complete m -partite graph with vertex classes C_1, \dots, C_m . Without loss of generality we may assume that at least one possible edge between C_1 and C_2 is missing in ∂H . Due to the definition of equivalence there are actually no edges between C_1 and C_2 in ∂H . By symmetry we may suppose further that $d_{\mathcal{H}}(C_1) \leq d_{\mathcal{H}}(C_2)$.

Now let \mathcal{H}' be the unique 3-graph satisfying $V(\mathcal{H}') = V(\mathcal{H})$, $\mathcal{H}' - C_1 = \mathcal{H} - C_1$, and $L_{\mathcal{H}'}(v) = L_{\mathcal{H}}(w)$ for all $v \in C_1$ and $w \in C_2$. Observe that $\{C_1 \cup C_2, C_3, \dots, C_m\}$ refines the partition of $V(\mathcal{H}')$ into the equivalence classes of \mathcal{H}' and

$$|\mathcal{H}'| = |\mathcal{H}| + |C_1|(d_{\mathcal{H}}(C_2) - d_{\mathcal{H}}(C_1)) \geq |\mathcal{H}| > \mathfrak{M}(n).$$

So our minimal choice of m implies that \mathcal{H}' cannot be \mathcal{M}_t -free. As there exists a homomorphism from \mathcal{H}' to \mathcal{H} , it follows that \mathcal{H} fails to be \mathcal{M}_t -hom-free. But owing to Proposition 3.12 this contradicts \mathcal{H} being \mathcal{M}_t -free. \square

§4. STABILITY

In this section we prove most of Theorem 1.1 (b) – only the proof of $\xi(\mathcal{M}_t) = t$ is postponed to Section 5. Our goal is to show that after deleting a small number of low-degree vertices an “almost extremal” \mathcal{M}_t -free 3-graph becomes \mathcal{G}_i -colorable for some $i \in [t]$. More precisely, we aim for the following result.

Theorem 4.1. *If $\varepsilon > 0$ is sufficiently small, n is sufficiently large, and \mathcal{H} is an \mathcal{M}_t -free 3-graph on n vertices with $|\mathcal{H}| \geq (\lambda_t - \varepsilon)n^3$, then the set*

$$Z = \{u \in V(\mathcal{H}) : d_{\mathcal{H}}(u) \leq (3\lambda_t - 2\varepsilon^{1/2})n^2\}$$

has size at most $\varepsilon^{1/2}n$ and the 3-graph $\mathcal{H} - Z$ is \mathcal{G}_i -colorable for some $i \in [t]$.

As the proof of this result will occupy the entire section, we would like to start with a quick overview. The argument is somewhat similar in spirit to [3, 19, 26] and ultimately it is based on the Zykov symmetrization method [34]. There are certain kinds of complications that often arise when one uses this strategy in order to establish stability results and we overcome several of these common difficulties by introducing the so-called Ψ -trick in

Subsection 4.1. By means of this trick, the problem to prove Theorem 4.1 gets reduced to an apparently much simpler task: If a triple system \mathcal{H} with n vertices and minimum degree $(3\lambda_t - o(1))n^2$ can be made \mathcal{G}_i -colorable by deleting a single vertex, then, actually, \mathcal{H} itself is \mathcal{G}_i -colorable (see Lemma 4.3). The Ψ -trick can also be used to reprove some known stability results with improved control over the dependence of the constants (see [20]).

The proof of Lemma 4.3 is still quite long. We will collect some auxiliary results in Subsection 4.2 and defer the main part of the argument to Subsection 4.3

4.1. General preliminaries. This subsection reduces the task of proving Theorem 4.1 to the apparently much simpler task of verifying Lemma 4.3 below. There are only few “special properties” of \mathcal{M}_t we are going to utilize in the course of this reduction and we refer to [20] for a more systematic treatment.

Throughout this subsection we use the following notation: For every 3-graph \mathcal{H} on n vertices and every $\varepsilon > 0$ we set

$$Z_\varepsilon(\mathcal{H}) = \{u \in V(\mathcal{H}) : d_{\mathcal{H}}(u) \leq (3\lambda_t - 2\varepsilon^{1/2})n^2\}.$$

Lemma 4.2. *If $\varepsilon \in (0, 1)$, $n \geq \varepsilon^{-1/2}$ and \mathcal{H} is an \mathcal{M}_t -free 3-graph on n vertices with at least $(\lambda - \varepsilon)n^3$ edges, then*

- (a) *the set $Z_\varepsilon(\mathcal{H})$ has at most the size $\varepsilon^{1/2}n$*
- (b) *and the subgraph $\mathcal{H}' = \mathcal{H} - Z_\varepsilon(\mathcal{H})$ of \mathcal{H} satisfies $\delta(\mathcal{H}') \geq (3\lambda_t - 3\varepsilon^{1/2})n^2$ as well as $|\mathcal{H}'| \geq (\lambda_t - 2\varepsilon^{1/2})n^3$.*

Proof of Lemma 4.2. Set $Z = Z_\varepsilon(\mathcal{H})$. Assuming that part (a) fails we can take a set $X \subseteq Z$ of size $\frac{2}{3}\varepsilon^{1/2}n \leq |X| \leq 2\varepsilon^{1/2}n$. The definition of Z leads to

$$\begin{aligned} |\mathcal{H} - X| &\geq (\lambda_t - \varepsilon)n^3 - |X|(3\lambda_t - 2\varepsilon^{1/2})n^2 \\ &\geq (\lambda_t - \varepsilon)n^3 - |X|(3\lambda_t - 2\varepsilon^{1/2})n^2 - \frac{3}{4}n(|X| - \frac{2}{3}\varepsilon^{1/2}n)(2\varepsilon^{1/2}n - |X|) \\ &= \lambda_t(n - |X|)^3 + 3(1/4 - \lambda_t)n|X|^2 + \lambda_t|X|^3 > \lambda_t(n - |X|)^3, \end{aligned}$$

where we used $\lambda_t < 1/6 < 1/4$ in the last step. However, by Theorem 1.1 (a) this contradicts the fact that $\mathcal{H} - X$ is \mathcal{M}_t -free.

Now we prove part (b). For every $u \in V(\mathcal{H}')$ the definition of Z and (a) yield

$$d_{\mathcal{H}'}(u) \geq d_{\mathcal{H}}(u) - |Z|n \geq (3\lambda_t - 2\varepsilon^{1/2})n^2 - \varepsilon^{1/2}n^2 = (3\lambda_t - 3\varepsilon^{1/2})n^2.$$

Similarly, we have

$$|\mathcal{H}'| \geq |\mathcal{H}| - |Z|n^2 \geq (\lambda_t - \varepsilon)n^3 - \varepsilon^{1/2}n^3 > (\lambda_t - 2\varepsilon^{1/2})n^3. \quad \square$$

The following lemma will be shown to imply Theorem 4.1.

Lemma 4.3. *There exist constants $\zeta \in (0, 1)$ and $N_0 \in \mathbb{N}$ such that the following holds for all $n \geq N_0$. Let \mathcal{H} be an \mathcal{M}_t -free 3-graph on n vertices with at least $(\lambda_t - \zeta)n^3$ edges and $\delta(\mathcal{H}) > (3\lambda_t - \zeta)n^2$. If there exists a vertex $v \in V(\mathcal{H})$ such that $\mathcal{H} - v$ is \mathcal{G}_i -colorable for some $i \in [t]$, then \mathcal{H} itself is \mathcal{G}_i -colorable as well.*

We postpone the proof of this result to Subsection 4.3. The deduction of Theorem 4.1 from Lemma 4.3 factorises through the following statement.

Lemma 4.4. *There exists $\varepsilon \in (0, 1/16)$ such that the following holds for every sufficiently large integer n . Let \mathcal{H} denote an \mathcal{M}_t -free 3-graph with n vertices and at least $(\lambda_t - \varepsilon)n^3$ edges. If $Q \subseteq V(\mathcal{H})$ has size $|Q| \leq 2\varepsilon^{1/2}n$ and $\mathcal{H} - Q$ is \mathcal{G}_i -colourable for some $i \in [t]$, then $\mathcal{H} - Z_\varepsilon(\mathcal{H})$ is \mathcal{G}_i -colourable as well.*

Proof of Lemma 4.4 using Lemma 4.3. We show that $\varepsilon = \zeta^2/25$ has the desired property, where ζ denotes the constant provided by Lemma 4.3. Given a sufficiently large 3-graph \mathcal{H} and a set Q as described in the statement of Lemma 4.4 we set $Q' = Q \setminus Z_\varepsilon(\mathcal{H})$ and $V' = V(\mathcal{H}) \setminus (Z_\varepsilon(\mathcal{H}) \cup Q)$.

By our assumption, there is an index $i(\star) \in [t]$ such that $\mathcal{H}[V']$ is $\mathcal{G}_{i(\star)}$ -colorable. Choose a set $S \subseteq Q'$ of maximum size such that $\mathcal{H}[V' \cup S]$ is still $\mathcal{G}_{i(\star)}$ -colorable. If $S = Q'$ we are done, so suppose for the sake of contradiction that there exists a vertex $v \in Q' \setminus S$.

Due to the maximality of S the triple system $\mathcal{H}' = \mathcal{H}[V' \cup S \cup \{v\}]$ is not $\mathcal{G}_{i(\star)}$ -colorable. On the other hand, Lemma 4.2 (a) and $|Q| \leq 2\varepsilon^{1/2}n$ entail

$$\begin{aligned} \delta(\mathcal{H}') &> (3\lambda_t - 2\varepsilon^{1/2})n^2 - |Z(\mathcal{H}) \cup Q|n > (3\lambda_t - 5\varepsilon^{1/2})n^2 \\ \text{and} \quad |\mathcal{H}'| &> (\lambda_t - \varepsilon)n^3 - |Z(\mathcal{H}) \cup Q|n^2 > (\lambda_t - 4\varepsilon^{1/2})n^3. \end{aligned}$$

So by Lemma 4.3 and $\zeta = 5\varepsilon^{1/2}$ the $\mathcal{G}_{i(\star)}$ -colorability of $\mathcal{H}' - v = \mathcal{H}[V' \cup S]$ implies that \mathcal{H}' itself is $\mathcal{G}_{i(\star)}$ -colorable as well. This contradiction completes the proof of Lemma 4.4 \square

It remains to deduce Theorem 4.1. The argument involves the following invariant of 3-graphs: Given a 3-graph \mathcal{H} with equivalence classes C_1, \dots, C_m we set $\Psi(\mathcal{H}) = \sum_{i=1}^m |C_i|^2$.

Proof of Theorem 4.1 using Lemma 4.4. Let ε be the constant delivered by Lemma 4.4 and fix a sufficiently large natural number n . Assuming that the conclusion of Theorem 4.1 fails for our values of ε and n we pick a counterexample \mathcal{H} such that the pair $(|\mathcal{H}|, \Psi(\mathcal{H}))$ is lexicographically maximal. Let C_1, \dots, C_m be the equivalence classes of \mathcal{H} .

Recall that Lemma 4.2 (a) tells us $|Z_\varepsilon(\mathcal{H})| \leq \varepsilon^{1/2}n$. Since \mathcal{H} is a counterexample, it cannot be \mathcal{G}_i -colorable for any $i \in [t]$. Moreover, (3.3) yields

$$|\mathcal{H}| > (\lambda_t - \varepsilon)n^3 \geq (5/32 - 1/16)n^3 = 3n^3/32 > 2n^3/27$$

and thus \mathcal{H} cannot be semibipartite. So by Lemma 3.14 there exist two equivalence classes, say C_1 and C_2 , such that $\partial\mathcal{H}$ possesses no edges from C_1 to C_2 . We may assume that $(d_{\mathcal{H}}(C_1), |C_1|) \leq_{\text{lex}} (d_{\mathcal{H}}(C_2), |C_2|)$, where \leq_{lex} indicates the lexicographic ordering on \mathbb{N}^2 .

Pick arbitrary vertices $v_1 \in C_1$ and $v_2 \in C_2$ and symmetrize only them. That is, we let \mathcal{H}' be the 3-graph with $V(\mathcal{H}') = V(\mathcal{H})$, $\mathcal{H}' - v_1 = \mathcal{H} - v_1$ and $L_{\mathcal{H}'}(v_1) = L_{\mathcal{H}}(v_2)$. Clearly, if $d_{\mathcal{H}}(v_1) < d_{\mathcal{H}}(v_2)$, then $|\mathcal{H}'| > |\mathcal{H}|$. Moreover, if $d_{\mathcal{H}}(v_1) = d_{\mathcal{H}}(v_2)$, then $|\mathcal{H}'| = |\mathcal{H}|$, $|C_1| \leq |C_2|$, and

$$\Psi(\mathcal{H}') - \Psi(\mathcal{H}) \geq (|C_1| - 1)^2 + (|C_2| + 1)^2 - |C_1|^2 - |C_2|^2 = 2(|C_2| - |C_1| + 1) \geq 2.$$

In both cases $(|\mathcal{H}'|, \Psi(\mathcal{H}'))$ is lexicographically larger than $(|\mathcal{H}|, \Psi(\mathcal{H}))$ and our choice of \mathcal{H} implies that $\mathcal{H}' - Z_{\varepsilon}(\mathcal{H}')$ is \mathcal{G}_i -colourable for some $i \in [t]$. By Lemma 4.2 (a) the set $Q = Z_{\varepsilon}(\mathcal{H}') \cup \{v_1\}$ has size $|Q| \leq \varepsilon^{1/2}n + 1 < 2\varepsilon^{1/2}n$. Since the hypergraph $\mathcal{H} - Q = \mathcal{H}' - Q$ is \mathcal{G}_i -colourable, Lemma 4.4 implies that $\mathcal{H} - Z(\mathcal{H})$ is \mathcal{G}_i -colourable too. This contradiction to the choice of \mathcal{H} establishes Theorem 4.1. \square

4.2. Transversals. Roughly speaking, the hypergraph $\mathcal{H} - v$ appearing in Lemma 4.3 arises from an almost balanced blow-up of \mathcal{G}_i by deleting a small number of edges. When we randomly select one vertex from each partition class of $\mathcal{H} - v$ it is thus very likely that the resulting transversal induces a copy of \mathcal{G}_i . In the proof of Lemma 4.3 there are several places where we argue similarly in situations where some vertices from the transversals have been selected in advance. The precise statement we shall use in these cases is Lemma 4.5 below.

Consider a 3-graph with $V(\mathcal{G}) = [m]$ and pairwise disjoint sets V_1, \dots, V_m . The blow-up $\mathcal{G}[V_1, \dots, V_m]$ of \mathcal{G} is obtained from \mathcal{G} by replacing each vertex $j \in [m]$ with the set V_j and each edge $\{j_1, j_2, j_3\} \in \mathcal{G}$ with the complete 3-partite 3-graph with vertex classes V_{j_1} , V_{j_2} , and V_{j_3} . For a 3-graph \mathcal{H} we say that a partition $V(\mathcal{H}) = \bigcup_{j \in [m]} V_j$ is a \mathcal{G} -coloring of \mathcal{H} if $\mathcal{H} \subseteq \mathcal{G}[V_1, \dots, V_m]$.

Lemma 4.5. *Fix a real $\eta \in (0, 1)$ and integers $m, n \geq 1$. Let \mathcal{G} be a 3-graph with vertex set $[m]$ and let \mathcal{H} be a further 3-graph with $v(\mathcal{H}) = n$. Consider a vertex partition $V(\mathcal{H}) = \bigcup_{i \in [m]} V_i$ and the associated blow-up $\hat{\mathcal{G}} = \mathcal{G}[V_1, \dots, V_m]$ of \mathcal{G} . If two sets $T \subseteq [m]$ and $S \subseteq \bigcup_{j \notin T} V_j$ have the properties*

- (a) $|V_j| \geq (|S| + 1)|T|\eta^{1/3}n$ for all $j \in T$,
- (b) $|\mathcal{H}[V_{j_1}, V_{j_2}, V_{j_3}]| \geq |\hat{\mathcal{G}}[V_{j_1}, V_{j_2}, V_{j_3}]| - \eta n^3$ for all $\{j_1, j_2, j_3\} \in \binom{T}{3}$,
- (c) and $|L_{\mathcal{H}}(v)[V_{j_1}, V_{j_2}]| \geq |L_{\hat{\mathcal{G}}}(v)[V_{j_1}, V_{j_2}]| - \eta n^2$ for all $v \in S$ and $\{j_1, j_2\} \in \binom{T}{2}$,

then there exists a selection of vertices $u_j \in V_j$ for all $j \in [T]$ such that $U = \{u_j : j \in T\}$ satisfies $\widehat{\mathcal{G}}[U] \subseteq \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] \subseteq L_{\mathcal{H}}(v)[U]$ for all $v \in S$. In particular, if $\mathcal{H} \subseteq \widehat{\mathcal{G}}$, then $\widehat{\mathcal{G}}[U] = \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] = L_{\mathcal{H}}(v)[U]$ for all $v \in S$.

Proof of Lemma 4.5. Choose for $j \in T$ the vertices $u_j \in V_j$ independently and uniformly at random and let $U = \{u_j : j \in T\}$ be the random transversal consisting of these vertices. By (a) and (b) we have

$$\mathbb{P}(\{u_{j_1}, u_{j_2}, u_{j_3}\} \notin \mathcal{H}) = 1 - \frac{|\mathcal{H}[V_{j_1}, V_{j_2}, V_{j_3}]|}{|V_{j_1}||V_{j_2}||V_{j_3}|} \leq \frac{\eta n^3}{|V_{j_1}||V_{j_2}||V_{j_3}|} \leq \frac{1}{(|S| + 1)^3 |T|^3}$$

for all edges $\{j_1, j_2, j_3\} \in \mathcal{G}$. Similarly (a) and (c) lead to

$$\begin{aligned} \mathbb{P}(\{u_{j_1}, u_{j_2}\} \notin L_{\mathcal{H}}(v) \mid \{u_{j_1}, u_{j_2}\} \in L_{\widehat{\mathcal{G}}}(v)) &= 1 - \frac{|L_{\mathcal{H}}(v)[V_{j_1}, V_{j_2}]|}{|V_{j_1}||V_{j_2}|} \leq \frac{\eta n^2}{|V_{j_1}||V_{j_2}|} \\ &\leq \frac{\eta^{1/3}}{(|S| + 1)^2 |T|^2} \end{aligned}$$

for all $v \in S$ and all distinct $j_1, j_2 \in [m]$. Therefore, the union bound reveals

$$\begin{aligned} \mathbb{P}(\widehat{\mathcal{G}}[U] \not\subseteq \mathcal{H}[U]) &\leq \binom{|T|}{3} \frac{1}{(|S| + 1)^3 |T|^3} < \frac{1}{6} \\ \text{and} \quad \mathbb{P}(L_{\widehat{\mathcal{G}}}(v) \not\subseteq L_{\mathcal{H}}(v)) &\leq \binom{|T|}{2} \frac{\eta^{1/3}}{(|S| + 1)^2 |T|^2} < \frac{1}{2(|S| + 1)} \quad \text{for every } v \in S. \end{aligned}$$

Altogether, the probability that U fails to have the desired properties is at most

$$\frac{1}{6} + \frac{|S|}{2(|S| + 1)} < \frac{2}{3}.$$

So the probability that U has these properties is positive. \square

In practice the sets U obtained by means of Lemma 4.5 will be 2-covered and thus they will be cores of some subgraphs $F \in \widehat{\mathcal{K}}_{|U|}^3$ of \mathcal{H} . In such situations F will be \mathcal{M}_t -free and in order to exploit this fact we need to know that for $i \neq j$ the triple system \mathcal{G}_i is in some sense far from being \mathcal{G}_j -colorable (see Lemma 4.7 below). The verification of this statement requires that we take a closer look into Construction 3.5 and the observation that follows summarizes everything we need in the sequel.

Observation 4.6. *The triple systems $\mathcal{G}_1, \dots, \mathcal{G}_t$ have the following properties.*

(a) *For $i \in [t]$ and $v \in \mathcal{G}_i$ the clique number $\omega(L_{\mathcal{G}_i}(v))$ of the link graph $L_{\mathcal{G}_i}(v)$ satisfies*

$$\frac{n_i - 1}{k_i - 1} - \frac{k_i}{2} \leq \omega(L_{\mathcal{G}_i}(v)) \leq \frac{n_i - 1}{k_i - 1}.$$

(b) We have

$$\frac{n_i - 1}{k_i - 1} - \frac{n_{i+1} - 1}{k_{i+1} - 1} > \frac{Q}{k_i^2}$$

for every $i \in [t - 1]$, where

$$Q = \frac{n_1}{k_1 + 1} = \cdots = \frac{n_t}{k_t + 1} \geq 2k_t^3 \geq 16.$$

(c) For $i \in [t]$ the 3-graph \mathcal{G}_i is regular with degree $3\lambda_t n_i^2$ and

$$7n_i/8 \leq n_i - 3k_i/2 \leq \delta_2(\mathcal{G}_i) \leq \Delta_2(\mathcal{G}_i) \leq n_i - k_i.$$

Proof. Part (a) follows from the fact that due to $\mathcal{G}_i = (K_{n_i}^3 \setminus H(\mathcal{D}_i)) \setminus \mathcal{S}_i$ the link $L_{\mathcal{G}_i}(v)$ arises from an $((n_i - 1)/(k_i - 1))$ -partite Turán graph by the deletion of $k_i/2$ edges. The proof of part (c) is similar. For part (b) it suffices to calculate

$$\begin{aligned} \frac{n_i - 1}{k_i - 1} - \frac{n_{i+1} - 1}{k_{i+1} - 1} &= \frac{Q(k_i + 1) - 1}{k_i - 1} - \frac{Q(k_{i+1} + 1) - 1}{k_{i+1} - 1} \\ &= (2Q - 1) \left(\frac{1}{k_i - 1} - \frac{1}{k_{i+1} - 1} \right) \\ &\geq Q \left(\frac{1}{k_i - 1} - \frac{1}{k_i} \right) > \frac{Q}{k_i^2}. \end{aligned} \quad \square$$

As indicated earlier, this has the following consequence.

Lemma 4.7. *If $i \in [t]$ and the triple system \mathcal{G}'_i arises from \mathcal{G}_i by the deletion of at most $Q/(2k_i^2)$ vertices, then \mathcal{G}'_i fails to be \mathcal{G}_j -colorable for every $j \in [t] \setminus \{i\}$.*

Proof of Lemma 4.7. Suppose first that $j \in [i - 1]$. Due to

$$\delta_2(\mathcal{G}'_i) \geq \delta_2(\mathcal{G}_i) - \frac{Q}{2k_i^2} \geq 1$$

we know that \mathcal{G}'_i is 2-covered. Together with

$$v(\mathcal{G}'_i) = n_i - Q/(2k_i^2) > Q(k_i + 1) - Q \geq Q(k_j + 1) = n_j$$

it follows that \mathcal{G}'_i is indeed not \mathcal{G}_j -colorable.

If $j \in (i, t]$ we take an arbitrary vertex $v \in V(\mathcal{G}'_i)$. The parts (a) and (b) of Observation 4.6 yield

$$\omega(L_{\mathcal{G}'_i}(v)) \geq \omega(L_{\mathcal{G}_i}(v)) - \frac{Q}{2k_i^2} \geq \frac{n_i - 1}{k_i - 1} - \frac{k_i}{2} - \frac{Q}{2k_i^2} > \frac{n_j - 1}{k_j - 1}.$$

On the other hand, by Observation 4.6 (a) again, any \mathcal{G}_j -coloring of \mathcal{G}'_i would show that

$$\omega(L_{\mathcal{G}'_i}(v)) \leq \omega(L_{\mathcal{G}_j}(v)) \leq \frac{n_j - 1}{k_j - 1}. \quad \square$$

On most occasions the following corollary of Lemma 4.7 will suffice.

Corollary 4.8. *If $i \in [t]$, the 3-graph \mathcal{H} is \mathcal{M}_t -free and $U \subseteq V(\mathcal{H})$ denotes a 2-covered set of size $n_i + 1$, then $\mathcal{H}[U]$ is \mathcal{G}_i -free.*

Proof. Assume for the sake of contradiction that $\mathcal{H}[U]$ has a subgraph isomorphic to \mathcal{G}_i . If $i < t$ we can take a subgraph $F \in \hat{\mathcal{K}}_{n_i+1}^3$ of \mathcal{H} with $F[U] = \mathcal{H}[U]$ having U as a core. As $\mathcal{H}[U]$ contains a copy of \mathcal{G}_i , we have $\tau(F[U]) \geq 2$. Now $F \notin \mathcal{M}_t$ implies that F is \mathcal{G}_j -colorable for some $j \in [t]$. In particular, \mathcal{G}_i is \mathcal{G}_j -colorable and by Lemma 4.7 this leads to $i = j$. In other words, F is \mathcal{G}_i -colorable, contrary to the fact that ∂F contains a copy of K_{n_i+1} .

It remains to discuss the case $i = t$. Now Lemma 3.8 yields a subgraph F' of F which belongs to $\mathcal{K}_{n_t+1}^3$, and whose induced subgraph on its core has covering number at least 2. By Lemma 3.9 this contradicts \mathcal{H} being \mathcal{M}_t -free. \square

4.3. Proof of the main lemma. This entire subsection is devoted to the proof of Lemma 4.3. Select constants ζ and N_0 fitting into the hierarchy

$$N_0^{-1} \ll \zeta \ll n_t^{-1}.$$

Consider an \mathcal{M}_t -free 3-graph \mathcal{H} on $n \geq N_0$ vertices satisfying $|\mathcal{H}| \geq (\lambda_t - \zeta)n^3$ and $\delta(\mathcal{H}) \geq (3\lambda - \zeta)n^2$ such that for some $v \in V(\mathcal{H})$ and $i \in [t]$ the 3-graph $\mathcal{H}_v = \mathcal{H} \setminus \{v\}$ is \mathcal{G}_i -colorable. Set $V = V(\mathcal{H})$ and fix a partition $\bigcup_{i \in [n_i]} V_i = V \setminus \{v\}$ exemplifying the \mathcal{G}_i -colorability of \mathcal{H}_v . We divide the argument that follows into three main parts each of which consists of several claims.

Part I. Analysis of \mathcal{H}_v . The three claims that follow only deal with \mathcal{H}_v but say nothing about v and its link.

Claim 4.9. *We have $|V_j| = n/n_i \pm 5\zeta^{1/2}n$ for every $j \in [n_i]$.*

Proof of Claim 4.9. Set $x_j = |V_j|/(n-1)$ for every $j \in [n_i]$. By Proposition 2.2 (and the proof of Lemma 2.1) we obtain

$$|\mathcal{H}_v| = L_{\mathcal{G}_i}(x_1, \dots, x_{n_i})(n-1)^3 \leq \left(\lambda_t - \frac{1}{9} \sum_{j \in [n_i]} \left(x_j - \frac{1}{n_i} \right)^2 \right) n^3.$$

Combined with

$$|\mathcal{H}_v| \geq (\lambda_t - \zeta)n^3 - d_{\mathcal{H}}(v) > (\lambda_t - 2\zeta)n^3$$

this leads to $\frac{1}{9} \sum_{j \in [n_i]} (x_j - 1/n_i)^2 \leq 2\zeta$, whence $x_j = 1/n_i \pm (18\zeta)^{1/2}$ and

$$||V_j| - n/n_i| \leq (n-1)|x_j - 1/n_i| + 1/n_i \leq (18\zeta)^{1/2}n + 1/n_i \leq 5\zeta^{1/2}n. \quad \square$$

Recall that the sets V_1, \dots, V_{n_i} have been chosen in such a way that \mathcal{H}_v is a subgraph of the blow-up $\widehat{\mathcal{G}}_i = \mathcal{G}_i[V_1, \dots, V_{n_i}]$ of \mathcal{G}_i . Our next objective is to compare the links of an arbitrary vertex $u \in V \setminus \{v\}$ in \mathcal{H}_v and in $\widehat{\mathcal{G}}_i$. As a consequence of $\mathcal{H}_v \subseteq \widehat{\mathcal{G}}_i$ we know $L_{\mathcal{H}_v}(u) \subseteq L_{\widehat{\mathcal{G}}_i}(u)$ and $|L_{\mathcal{H}_v}(u)| \leq |L_{\widehat{\mathcal{G}}_i}(u)|$. Members of $L_{\widehat{\mathcal{G}}_i}(u) \setminus L_{\mathcal{H}_v}(u)$ are referred to as the *missing pairs of u* . By Lemma 2.1 the global number of missing edges can be bounded from above by

$$|\widehat{\mathcal{G}}_i \setminus \mathcal{H}_v| \leq \lambda_t(n-1)^3 - (\lambda_t - \zeta)n^3 + d_{\mathcal{H}}(v) \leq 2\zeta n^3. \quad (4.1)$$

Locally we obtain the following.

Claim 4.10. *Every $u \in V \setminus \{v\}$ satisfies $|L_{\widehat{\mathcal{G}}_i}(u)| < (3\lambda_t + 6n_i\zeta^{1/2})n^2$. Moreover the number of missing pairs of u is bounded by $|L_{\widehat{\mathcal{G}}_i}(u) \setminus L_{\mathcal{H}_v}(u)| < 7\zeta^{1/2}n_in^2$.*

Proof of Claim 4.10. Since \mathcal{G}_i is $(3\lambda_t n_i^2)$ -regular, Claim 4.9 yields

$$|L_{\widehat{\mathcal{G}}_i}(u)| \leq 3\lambda_t n_i^2 \left(\frac{n}{n_i} + 5\zeta^{1/2}n \right)^2 = 3\lambda_t n^2 (1 + 5\zeta^{1/2}n_i)^2 < (3\lambda_t + 6\zeta^{1/2}n_i)n^2,$$

where we used $\lambda_t < 1/6$ and our hierarchy $\zeta \ll n_i^{-1}$. Owing to the minimum degree condition $\delta(\mathcal{H}) \geq (3\lambda_t - \zeta)n^2$ this entails the upper bound

$$|L_{\widehat{\mathcal{G}}_i}(u) \setminus L_{\mathcal{H}_v}(u)| \leq (3\lambda_t n^2 + 6\zeta^{1/2}n_in^2) - (3\lambda_t n^2 - \zeta n^2 - n) < 7\zeta^{1/2}n_in^2$$

on the number of missing pairs of u . □

It can now be shown that in \mathcal{H}_v all neighborhoods have roughly the expected size $\frac{n_i-1}{n_i}n$, but for our concerns it suffices to establish a lower bound.

Claim 4.11. *We have $|N_{\mathcal{H}_v}(u)| \geq \frac{n_i-1}{n_i}n - 17\zeta^{1/2}n_in$ for every $u \in V \setminus \{v\}$.*

Proof of Claim 4.11. Let $j \in [n_i]$ be the index satisfying $u \in V_j$. Since every vertex in $V \setminus (V_j \cup N_{\mathcal{H}_v}(u) \cup \{v\})$ belongs to at least $\delta_2(\mathcal{G}) \cdot \min\{|V_\ell| : \ell \in [n_i]\}$ missing pairs of u , and every missing pair is counted at most twice in this manner, Claim 4.10 yields

$$|V \setminus (V_j \cup N_{\mathcal{H}_v}(u) \cup \{v\})| \cdot \delta_2(\mathcal{G}) \cdot \min\{|V_\ell| : \ell \in [n_i]\} < 14\zeta^{1/2}n_in^2.$$

So by Observation 4.6 (c) and Claim 4.9 the assumption $|N_{\mathcal{H}_v}(u)| < \frac{n_i-1}{n_i}n - 17\zeta^{1/2}n_in$ would yield the contradiction

$$(17\zeta^{1/2}n_in - 5\zeta^{1/2}n - 1) \cdot \frac{7n_i}{8} \cdot \left(\frac{n}{n_i} - 5\zeta^{1/2}n \right) < 14\zeta^{1/2}n_in^2.$$

Thereby Claim 4.11 is proved. □

Part II. Choice of a vertex class for v . Our strategy for showing that \mathcal{H} is \mathcal{G}_i -colorable is to adjoin v to one the partition classes V_1, \dots, V_{n_i} . In fact, there is only one of these classes v fits into. Before finding this class we show a statement that has to hold if our plan is sound.

Claim 4.12. *We have $L_{\mathcal{H}}(v) \cap \binom{V_j}{2} = \emptyset$ for every $j \in [n_i]$.*

Proof of Claim 4.12. Without loss of generality we may assume that $j = 1$. Let $u_0, u_1 \in V_1$ be two distinct vertices. By Lemma 4.5 applied to $S = \{u_0, u_1\}$ and $T = [2, n_i]$ there exist vertices $u_j \in V_j$ for $j \in [2, n_i]$ such that the subgraphs of \mathcal{H} induced by $\{u_0, u_2, \dots, u_{n_i}\}$ and $\{u_1, u_2, \dots, u_{n_i}\}$ are isomorphic to \mathcal{G}_i . Now Corollary 4.8 informs us that the set $U = \{u_0, u_1, \dots, u_{n_i}\}$ cannot be 2-covered, for which reason $u_0 u_1 \notin \partial \mathcal{H}$. So, in particular, we have $u_0 u_1 \notin L_{\mathcal{H}}(v)$. \square

Claim 4.13. *There exists $j \in [n_i]$ such that $|N_{\mathcal{H}}(v) \cap V_j| < \zeta^{1/7} n$.*

Proof of Claim 4.13. Suppose for the sake of contradiction that the sets $W_j = N_{\mathcal{H}}(v) \cap V_j$ satisfy $|W_j| \geq \zeta^{1/7} n$ for every $j \in [n_i]$. Applying Lemma 4.5 to W_j here in place of V_j there and to $S = \emptyset$, $T = [n_i]$ we obtain vertices $u_j \in V_j$ for all $j \in [n_i]$ such that the set $U = \{u_1, \dots, u_{n_i}\}$ induces a copy of \mathcal{G}_i in \mathcal{H} . But now the 2-covered set $U \cup \{v\}$ contradicts Corollary 4.8. \square

It will turn out later that the index j delivered by Claim 4.13 is unique. Without loss of generality we may assume that

$$|N_{\mathcal{H}}(v) \cap V_1| < \zeta^{1/7} n. \quad (4.2)$$

Part III. The link of v . It remains to show that $L_{\mathcal{H}}(v) \subseteq L_{\hat{\mathcal{G}}_i}(V_1)$. To this end we define

$$N_v(u) = \{j \in [n_i] : |N_{\mathcal{H}}(u, v) \cap V_j| \geq \zeta^{1/7} n\}$$

for every $u \in N_{\mathcal{H}}(v)$. The upper bound on $\Delta_2(\mathcal{G}_i)$ in Observation 4.6 (c) transfers to these sets as follows.

Claim 4.14. *We have $|N_v(u)| \leq n_i - k_i$ for every $u \in N_{\mathcal{H}}(v)$.*

Proof of Claim 4.14. Assume for the sake of contradiction that there is a set $N_{\star} \subseteq N_v(u)$ such that $|N_{\star}| = n_i - k_i + 1 < n_i - 2$. As in the proof of Claim 4.13 there exist vertices $u_j \in N_{\mathcal{H}}(u, v) \cap V_j$ for $j \in N_{\star}$ such that $\mathcal{G}_i[N_{\star}]$ is isomorphic to $\mathcal{H}[U]$, where $U = \{u_j : j \in N_{\star}\}$.

Now we consider the 3-graph $F = \mathcal{H}[U \cup \{u, v\}]$. Clearly $U \cup \{u, v\}$ is 2-covered in F and $\tau(F) \geq \tau(\mathcal{G}_i[N_{\star}]) \geq 2$. So $F \notin \mathcal{M}_t$ tells us that F is \mathcal{G}_s -colorable for some $s \in [t]$.

On the other hand by Lemma 4.7 and $|U| \geq n_i - k_i + 2 > n_i - Q/(2k_i^2)$ the subgraph $F[U]$ of F cannot be \mathcal{G}_s -colorable for any $s \in [t] \setminus \{i\}$.

Summarizing this discussion, F is \mathcal{G}_i -colorable. As F is also 2-covered, F is actually isomorphic to a subgraph of \mathcal{G}_i and, consequently, $n_i - k_i < |N_\star| = d_F(u, v) \leq \Delta_2(\mathcal{G}_i)$, contrary to Observation 4.6 (c). \square

Claim 4.15. *We have $|N_{\mathcal{H}}(v) \cap V_j| \geq \zeta^{1/7}n$ for every $j \in [2, n_i]$.*

Proof of Claim 4.15. The minimum degree condition imposed on \mathcal{H} and $6\lambda_t = 1 - \frac{k_i+1}{n_i}$ yield

$$\left(1 - \frac{k_i + 1}{n_i} - 2\zeta\right)n^2 = 2(3\lambda_t - \zeta)n^2 \leq 2d_{\mathcal{H}}(v) \leq \Delta(L_{\mathcal{H}}(v))|N_{\mathcal{H}}(v)|.$$

Claim 4.14 allows us to bound the first factor on the right side from above by

$$\Delta(L_{\mathcal{H}}(v)) \leq (n_i - k_i) \left(\frac{n}{n_i} + 5\zeta^{1/2}n \right) + k_i\zeta^{1/7}n < \frac{n_i - k_i}{n_i}n + 2k_i\zeta^{1/7}n.$$

Altogether we obtain

$$\frac{n_i - (k_i + 1) - 2n_i\zeta}{(n_i - k_i) + 2k_in_i\zeta^{1/7}} \leq \frac{|N_{\mathcal{H}}(v)|}{n},$$

which due to

$$\frac{n_i - (k_i + 1)}{n_i - k_i} = 1 - \frac{1}{n_i - k_i} > 1 - \frac{5/4}{n_i}$$

and $\zeta \ll n_i^{-1}$ implies

$$\left(1 - \frac{3/2}{n_i}\right)n \leq |N_{\mathcal{H}}(v)|.$$

On the other hand, setting $I = \{j \in [2, n_i] : |N_{\mathcal{H}}(v) \cap V_j| \geq \zeta^{1/7}n\}$ Claim 4.9 and (4.2) lead to

$$|N_{\mathcal{H}}(v)| \leq |I| \left(\frac{1}{n_i} + 5\zeta^{1/2} \right) n + \zeta^{1/7}n_in.$$

Combining both estimates we arrive at $|I| > n_i - 7/4$, whence $I = [2, n_i]$. \square

Claim 4.16. *We have $N_{\mathcal{H}}(v) \cap V_1 = \emptyset$.*

Proof of Claim 4.16. Suppose that there exists $u_1 \in N_{\mathcal{H}}(v) \cap V_1$. Owing Claim 4.15 we can apply Lemma 4.5 with $S = \{u_1\}$ and $T = [2, n_i]$ in order to obtain vertices $u_j \in N_{\mathcal{H}}(v) \cap V_j$ for $j \in [2, n_i]$ such that \mathcal{H} induces a copy of \mathcal{G}_i on $U = \{u_1, \dots, u_{n_i}\}$. Since $U \cup \{v\}$ is 2-covered, this contradicts Corollary 4.8. \square

Let us recall that $L_{\hat{\mathcal{G}}_i}(V_1)$ denotes the common $\hat{\mathcal{G}}_i$ -link of all vertices in V_1 .

Claim 4.17. *We have $L_{\mathcal{H}}(v) \subseteq L_{\hat{\mathcal{G}}_i}(V_1)$.*

Proof of Claim 4.17. Due to the Claims 4.12 and 4.16 we know that $L_{\mathcal{H}}(v)$ is an $(n_i - 1)$ -partite graph with vertex classes V_2, \dots, V_{n_i} . So if Claim 4.17 fails we may assume without loss of generality $123 \notin \mathcal{G}_i$ and that there exists a pair $u_2 u_3 \in L_{\mathcal{H}}(v)$ with $u_2 \in V_2$, $u_3 \in V_3$.

Since $|V_1| > n/(2n_i)$ and $|N_{\mathcal{H}}(v) \cap V_j| \geq \zeta^{1/7} n$ for $j \in [4, n_i]$, Lemma 4.5 applied to $S = \{u_2, u_3\}$ and $T = \{1, 4, \dots, n_i\}$ delivers vertices $u_1 \in V_1$ and $u_j \in N_{\mathcal{H}}(v) \cap V_j$ for $j \in [4, n_i]$ such that the set $U' = \{u_1, u_4, \dots, u_{n_i}\}$ satisfies

$$\mathcal{H}[U'] = \widehat{\mathcal{G}}_i[U'] \quad \text{and} \quad L_{\mathcal{H}}(u_\ell)[U'] = L_{\widehat{\mathcal{G}}_i}(u_\ell)[U'] \quad \text{for } \ell = 2, 3. \quad (4.3)$$

Consider the set $U = \{u_1, \dots, u_{n_i}\}$. Because of (4.3) and $123 \notin \mathcal{G}_i$ the map $i \mapsto u_i$ is an embedding of $L_{\mathcal{G}_i}(1)$ into \mathcal{H} and for this reason we have

$$d_{\mathcal{H}[U]}(u_1) \geq d_{\mathcal{G}_i}(1). \quad (4.4)$$

Next we choose for every $j \in [4, n_i]$ an edge $e_j \in \mathcal{H}$ such that $u_j, v \in e_j$ and observe that U is 2-covered in the 3-graph

$$F = \{vu_2u_3\} \cup \{e_j : 4 \leq j \leq n_i\} \cup \mathcal{H}[U].$$

Moreover, $|F| \leq |\mathcal{G}_i| + n_i - 2 < \binom{n_i}{3}$ implies $F \in \widehat{\mathcal{K}}_{n_i}^3$. Since $F[U'] = \mathcal{H}[U']$ is isomorphic to $\mathcal{G}_i - \{2, 3\}$, Lemma 4.7 tells us that F cannot be \mathcal{G}_j -colorable for any $j \in [t] \setminus \{i\}$. But on the other hand we have $\tau(F[U]) \geq 2$ and $F \notin \mathcal{M}_t$, so altogether F is \mathcal{G}_i -colorable.

Fix a homomorphism $\varphi: V(F) \rightarrow V(\mathcal{G}_i)$ from F to \mathcal{G}_i . Since U and $U_v = U \cup \{v\} \setminus \{u_1\}$ are 2-covered subsets of F whose size is $n_i = v(\mathcal{G}_i)$, the map φ has to be bijective on U and U_v , which is only possible if $\varphi(v) = \varphi(u_1)$. Now φ embeds the link $L_{F[U]}(u_1)$ into the link $L_{\mathcal{G}_i}(\varphi(u_1))$. Moreover, $vu_2u_3 \in F$ implies that $\varphi(u_2)\varphi(u_3)$ belongs to the link $L_{\mathcal{G}_i}(\varphi(u_1))$ as well and by $123 \notin \mathcal{G}_i$ this edge is not in the image $\varphi(L_{F[U]}(u_1))$. Altogether this proves $d_{F[U]}(u_1) + 1 \leq d_{\mathcal{G}_i}(\varphi(u_1))$, which in view of $F[U] = \mathcal{H}[U]$ and (4.4) contradicts the regularity of \mathcal{G}_i . \square

By Claim 4.17 the partition $\bigcup_{j \in [n_i]} \widehat{V}_j$, where

$$\widehat{V}_j = \begin{cases} V_1 \cup \{v\} & \text{if } j = 1 \\ V_j & \text{if } 2 \leq j \leq n_i \end{cases}$$

is a \mathcal{G}_i -coloring of \mathcal{H} . This completes the proof of Lemma 4.3.

§5. FEASIBLE REGION OF \mathcal{M}_t AND $\xi(\mathcal{M}_t)$

We prove Theorem 1.2 and that $\xi(\mathcal{M}_t) = t$ in this section. First, let us show a simple lemma.

Lemma 5.1. *Suppose that \mathcal{H} is an n -vertex \mathcal{G}_i -colorable 3-graph for some $i \in [t]$. If $|\mathcal{H}| \geq (\lambda_t - \varepsilon)n^3$, then $|\partial\mathcal{H}| \geq \left(\frac{n_i-1}{2n_i} - 3\varepsilon^{1/2}n_i\right)n^2$.*

Proof of Lemma 5.1. Let $V(\mathcal{H}) = \bigcup_{j \in [n_i]} V_j$ be a \mathcal{G}_i -coloring of \mathcal{H} . Now by Proposition 2.2, $|V_j| = (1/n_i \pm 3\varepsilon^{1/2})n$ for all $j \in [n_i]$. Call a pair $\{u, v\}$ with $u \in V_j, v \in V_k$ and $j \neq k$ missing if $uv \notin \partial\mathcal{H}$, and let M denote the set of all missing pairs. Since $\delta_2(\mathcal{G}_i) \geq 7n_i/8$, we obtain

$$|M| \cdot \frac{7n_i}{8} \cdot \left(\frac{1}{n_i} - 3\varepsilon^{1/2}\right)n \leq 3\varepsilon n^3,$$

which yields $|M| < 4\varepsilon n^2$. Therefore,

$$|\partial\mathcal{H}| > \binom{n_i}{2} \times \left(\frac{1}{n_i} - 3\varepsilon^{1/2}\right)^2 n^2 - |M| > \frac{n_i-1}{2n_i} n^2 - 3\varepsilon^{1/2} n_i n^2. \quad \square$$

We remark that the stronger conclusion $|\partial\mathcal{H}| \geq \left(\frac{n_i-1}{2n_i} - 5\varepsilon n_i\right)n^2$ could be shown by arguing more carefully, but this is immaterial to what follows.

Proof of Theorem 1.2. Recall from Section 3 that semibipartite 3-graphs are \mathcal{M}_t -free. This yields $\text{proj}\Omega(\mathcal{M}_t) = [0, 1]$, as for every $x \in [0, 1]$ there exists a good sequence of semibipartite 3-graphs such that the edge densities of their shadows converges to x .

Theorem 1.1 (a) implies that $g(\mathcal{M}_t, x) \leq 6\lambda_t$ for all $x \in [0, 1]$. Furthermore for every $i \in [t]$ the sequence of balanced blow-ups of \mathcal{G}_i shows the equality $g(\mathcal{M}_t, 1 - 1/n_i) = 6\lambda_t$. So, in order to finish the proof it suffices to show that if some $x \in [0, 1]$ satisfies $g(\mathcal{M}_t, x) = 6\lambda_t$, then there is an index $i \in [t]$ such that $x = 1 - 1/n_i$.

Fix such an $x \in [0, 1]$ and let $(\mathcal{H}_n)_{n=1}^\infty$ be a good sequence of \mathcal{M}_t -free 3-graphs realizing $(x, 6\lambda_t)$. Consider an arbitrary $\delta > 0$ and let $\varepsilon > 0, N_0$ be the constants guaranteed by Theorem 1.1 (b). Without loss of generality we may assume $\varepsilon \leq \delta$. By our choice of $(\mathcal{H}_n)_{n=1}^\infty$ there exists $n_0 \in \mathbb{N}$ such that

$$d(\mathcal{H}_n) = 6\lambda_t \pm \varepsilon \quad \text{and} \quad d(\partial\mathcal{H}_n) = x \pm \varepsilon$$

hold for all $n \geq n_0$. By Theorem 1.1 (b), for every $n \geq \max\{n_0, N_0\}$ the 3-graph \mathcal{H}_n is \mathcal{G}_i -colorable for some $i = i(n) \in [t]$ after removing at most $\delta v(\mathcal{H}_n)$ vertices. Therefore,

$$|\partial\mathcal{H}_n| \leq \left(\frac{n_i-1}{2n_i} + \delta\right) v(\mathcal{H}_n)^2,$$

and, on the other hand, by Lemma 5.1,

$$|\partial\mathcal{H}_n| > \left(\frac{n_i-1}{2n_i} - 3\varepsilon^{1/2}n_i\right) (1-\delta)^2 v(\mathcal{H}_n)^2 > \frac{n_i-1}{2n_i} v(\mathcal{H}_n)^2 - (3\varepsilon^{1/2}n_i + 2\delta) v(\mathcal{H}_n)^2.$$

Summarizing and taking $\varepsilon \leq \delta$ into account we arrive at

$$\frac{n_i-1}{n_i} - (6\delta^{1/2}n_t + 4\delta) < d(\partial\mathcal{H}_n) \leq \frac{n_i-1}{n_i} + 2\delta, \quad (5.1)$$

where, let us recall, $i = i(n)$ might depend on n . So what (5.1) means is that if we set

$$I_i(\delta) = \left[\frac{n_i - 1}{n_i} - 6\delta^{1/2}n_i - 4\delta, \frac{n_i - 1}{n_i} + 2\delta \right]$$

for every $i \in [t]$, then

$$d(\partial\mathcal{H}_n) \in I_1(\delta) \cup \cdots \cup I_t(\delta)$$

holds for every $n \geq n_0$. As the set on the right side is closed we obtain

$$x \in I_1(\delta) \cup \cdots \cup I_t(\delta)$$

in the limit $n \rightarrow \infty$. Since $\delta > 0$ was arbitrary,

$$x \in \bigcap_{\delta > 0} (I_1(\delta) \cup \cdots \cup I_t(\delta)) = \{1 - 1/n_i : i \in [t]\}$$

follows. □

Recall that we already proved that \mathcal{M}_t is t -stable, which, by definition, shows that $\xi(\mathcal{M}_t) \leq t$. Therefore, in order to prove $\xi(\mathcal{M}_t) = t$ it suffices to show that $\xi(\mathcal{M}_t) \geq t$, and this is an easy consequence of the following proposition and Theorem 1.2.

Proposition 5.2. *Let \mathcal{F} be a family of r -graphs and let M be the set of global maxima of $g(\mathcal{F})$. If M is finite, then $|M| \leq \xi(\mathcal{F})$.*

The proof of this result involves the *edit distance* of hypergraphs: Given two r -graphs H and H' with the same number of vertices we set

$$d_1(H, H') = \min\{|H \triangle H''| : V(H'') = V(H) \text{ and } H'' \cong H'\}.$$

It is well known and easy to confirm that this distance satisfies the triangle inequality.

Proof of Proposition 5.2. If \mathcal{F} is degenerate, then $g(\mathcal{F})$ is the constant function whose value is always 0 and M is infinite. So we may assume that the Turán density $y = \pi(\mathcal{F})$ is positive. Let us write $M = \{(x_i, y) : i \in [m]\}$ such that $x_1 < \cdots < x_m$ and $m = |M|$. For every $i \in [m]$ we select a good sequence $(\mathcal{H}_i(n))_{n=1}^\infty$ of \mathcal{F} -free r -graphs realizing (x_i, y) . Without loss of generality we have $v(\mathcal{H}_i(n)) = n$ for every positive integer n . Now suppose for the sake of contradiction that $t = \xi(\mathcal{F})$ is smaller than m .

Claim 5.3. *For every $\delta > 0$ there are distinct $i, j \in [m]$ and $n > 1/\delta$ such that*

$$d_1(\mathcal{H}_i(n), \mathcal{H}_j(n)) \leq \delta n^r \quad \text{and} \quad \min\{|\mathcal{H}_i(n)|, |\mathcal{H}_j(n)|\} \geq (y - \delta) \binom{n}{r}.$$

Proof of Claim 5.3. By the definition of $\xi(\mathcal{F}) = t$ there are $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that for every $n \geq n_0$ there exists a family $\{\mathcal{G}_1(n), \dots, \mathcal{G}_t(n)\}$ of r -graphs on n vertices such that for every \mathcal{F} -free r -graph \mathcal{H} with $v(\mathcal{H}) = n$ and $|\mathcal{H}| \geq (y - \varepsilon) \binom{n}{r}$ there is some $s \in [t]$ such that $d_1(\mathcal{H}, \mathcal{G}_s(n)) \leq (\delta/2)n^r$. As usual, we may suppose that $\varepsilon \leq \delta$.

Now choose $n \geq n_0, \delta^{-1}$ such that for every $i \in [m]$ we have $d(\mathcal{H}_i(n)) \geq y - \varepsilon$. Stability allows us to select for every $i \in [m]$ an index $s(i) \in [t]$ such that $d_1(\mathcal{H}_i(n), \mathcal{G}_{s(i)}(n)) \leq \delta n^r$. By $t < m$ the map $i \mapsto s(i)$ cannot be injective, i.e., there are distinct $i, j \in [m]$ and $s \in [t]$ such that $s(i) = s(j) = s$. Now the triangle inequality yields

$$d_1(\mathcal{H}_i(n), \mathcal{H}_j(n)) \leq d_1(\mathcal{H}_i(n), \mathcal{G}_s(n)) + d_1(\mathcal{G}_s(n), \mathcal{H}_j(n)) \leq \delta n^r,$$

as desired. \square

Notice that, as stated, Claim 5.3 allows i and j to depend on δ . However, a quick thought reveals that there actually have to be two indices $i < j$ that work for every $\delta > 0$. Now we intend to contradict the finiteness of M by proving $[x_i, x_j] \times \{y\} \subseteq M$.

To this end, let $x \in [x_i, x_j]$ and a large integer N be given. It suffices to construct an \mathcal{F} -free r -graph \mathcal{H} satisfying $v(\mathcal{H}) > N$, $d(\partial\mathcal{H}) = x \pm 1/N$ and $d(\mathcal{H}) = y \pm 1/N$. By Claim 5.3 applied to $\delta \ll N^{-1}$ there is some $n > N$ such that $d_1(\mathcal{H}_i(n), \mathcal{H}_j(n)) \leq \delta n^r$ and $\min\{|\mathcal{H}_i(n)|, |\mathcal{H}_j(n)|\} \geq (y - \delta) \binom{n}{r}$. Assume without loss of generality that

$$|\mathcal{H}_i(n) \Delta \mathcal{H}_j(n)| \leq \delta n^r.$$

Now consider the following process transforming $\mathcal{H}_i(n)$ into $\mathcal{H}_j(n)$: Start with $\mathcal{H}_i(n)$ and remove edges one by one until $\mathcal{H}_i(n) \cap \mathcal{H}_j(n)$ is reached. Then, keep adding edges one by one until you arrive at $\mathcal{H}_j(n)$. Every r -graph occurring along the way is \mathcal{F} -free. Moreover, since deleting or adding an edge can affect the size of the shadow by at most r , in every step of the process the shadow density changes by at most $r/\binom{n}{r-1}$. Thus at some moment we pass an r -graph \mathcal{H} such that $|d(\partial\mathcal{H}) - x| \leq r/\binom{n}{r-1} \leq \delta$. Finally, $d(\mathcal{H}) \geq d(\mathcal{H}_i(n) \cap \mathcal{H}_j(n)) \geq d(\mathcal{H}_i(n)) - |\mathcal{H}_i(n) \Delta \mathcal{H}_j(n)|/\binom{n}{r} \geq y - O(\delta)$ completes the proof that \mathcal{H} has all desired properties. \square

§6. CONCLUDING REMARKS

For every positive integer t we constructed a family of 3-graphs $\{\mathcal{G}_1, \dots, \mathcal{G}_t\}$ that have the same Lagrangian λ_t , and we showed that there is a family \mathcal{M}_t of 3-graphs whose extremal configurations are balanced blow-ups of $\mathcal{G}_1, \dots, \mathcal{G}_t$, and whose stability number is $\xi(\mathcal{M}_t) = t$. Notice that our choice of λ_t is very close to $1/6$, which is the supremum of the Lagrangians of all 3-graphs. It would be interesting to find for every integer $t \geq 2$ the minimum value (if it exists) of $\lambda = \lambda(t)$ so that there exists a t -stable family \mathcal{F}_t with

$\pi(\mathcal{F}_t) = 6\lambda$. A result of Erdős [4] implies that there are no Turán densities in the interval $(0, 2/9)$. This motivates the following question.

Problem 6.1. *Does there exist a family \mathcal{F} of triple systems with $\pi(\mathcal{F}) = 2/9$ but $\xi(\mathcal{F}) \neq 1$?*

For a family \mathcal{F} of r -graphs let $M(\mathcal{F}) = \{x \in \text{proj}\Omega(\mathcal{F}) : g(\mathcal{F})(x) = \pi(\mathcal{F})\}$ be the set of abscissae of the global maxima of its feasible region function. As we have shown here, $|M(\mathcal{F})|$ can be every finite cardinal except zero. It would be interesting to know whether $M(\mathcal{F})$ can be infinite and, in case the answer is affirmative, there immediately arise further questions.

Problem 6.2. *For $r \geq 3$ does there exist a non-degenerate family \mathcal{F} of r -graphs so that $g(\mathcal{F})$ has infinitely many global maxima? If so, can the set $M(\mathcal{F})$ be uncountable? Can it even contain a non-trivial interval?*

Notice that if the last question on intervals has a negative answer, then in Proposition 5.2 the assumption that M should be finite can be omitted. In fact, it is somewhat bizarre that we do not know the following.

Problem 6.3. *Let \mathcal{F} be a non-degenerate family of r -graphs such that $M(\mathcal{F})$ is infinite. Can it nevertheless happen that \mathcal{F} has finite stability number?*

In a forthcoming work [21] we will show an extension of our results about triples systems to r -graphs for all $r \geq 4$ and exhibit a family \mathcal{M}_t^r that is t -stable such that the function $g(\mathcal{M}_t^r)$ has exactly t -global maxima.

Added in proof. After the submission of this article, all questions raised in Section 6 have been answered affirmatively. In [11] Hou, Li, Zhang, and the first two authors construct a family of 3-graphs whose Turán density is $2/9$ and whose stability number is infinite, thus solving Problem 6.1 in a very strong form. Moreover, they show that $M(\mathcal{F})$ can indeed contain an interval, which is the most bizarre possibility mentioned in Problem 6.2. Moreover, Pikhurko and the first author show in [22] that $M(\mathcal{F})$ can even look like a Cantor-type set. In these examples, the stability number is always infinite.

Balogh, Clemen, and Luo [1], on the other hand, show that there also exists a 1-stable family \mathcal{F} such that $M(\mathcal{F})$ contains a nontrivial interval, thus giving a different solution to Problem 6.2. Moreover, they obtained a positive solution to Problem 6.3.

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