

Extremal problems for hypergraph blowups of trees*

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Abstract

In this paper we present a novel approach in extremal set theory which may be viewed as an asymmetric version of Katona’s permutation method. We use it to find more Turán numbers of hypergraphs in the Erdős–Ko–Rado range.

An (a, b) -path P of length $2k - 1$ consists of $2k - 1$ sets of size $r = a + b$ as follows. Take k pairwise disjoint a -element sets $A_0, A_2, \dots, A_{2k-2}$ and other k pairwise disjoint b -element sets $B_1, B_3, \dots, B_{2k-1}$ and order them linearly as $A_0, B_1, A_2, B_3, A_4, \dots$. Define the (hyper)edges of $P_{2k-1}(a, b)$ as the sets of the form $A_i \cup B_{i+1}$ and $B_j \cup A_{j+1}$. The members of P can be represented as r -element intervals of the $ak + bk$ element underlying set.

Our main result is about hypergraphs that are blowups of trees, and implies that for fixed k, a, b , as $n \rightarrow \infty$

$$\text{ex}_r(n, P_{2k-1}(a, b)) = (k - 1) \binom{n}{r - 1} + o(n^{r-1}).$$

This generalizes the Erdős–Gallai theorem for graphs which is the case of $a = b = 1$. We also determine the asymptotics when $a + b$ is even; the remaining cases are still open.

1 Paths

1.1 Definitions concerning r -uniform hypergraphs, Two constructions

An r -uniform hypergraph, or simply r -graph, is a family of r -element subsets of a finite set. We associate an r -graph F with its edge set and call its vertex set $V(F)$. Usually we take $V(F) = [n]$, where $[n]$ is the set of first n integers, $[n] := \{1, 2, 3, \dots, n\}$. We also use the notation $F \subseteq \binom{[n]}{r}$. For a hypergraph H , a vertex subset C of H that intersects all edges of H is called a *vertex cover* of H . Let $\tau(H)$ be the minimum size of a vertex cover of H . Let $\Psi_c(n, r)$ be the r -graph with vertex

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set $[n]$ consisting of all r -edges meeting $[c]$. Then Ψ has the maximum number of r -sets such that $\tau(\Psi) \leq c$. When r and c are fixed and $n \rightarrow \infty$,

$$|\Psi_c(n, r)| = \binom{n}{r} - \binom{n-c}{r} = c \binom{n}{r-1} + o(n^{r-1}). \tag{1}$$

A *crosscut* of a hypergraph H is a set $X \subset V(H)$ such that $|e \cap X| = 1$ for all $e \in H$. Not all hypergraphs have crosscuts. Let $\sigma(H)$ denote the smallest size of a crosscut in a hypergraph H with at least one crosscut. Clearly $\tau(H) \leq \sigma(H)$, since a crosscut is a vertex cover. Here strict inequality may hold, as shown by a double star whose adjacent centers have high degrees. Define $\Psi_c^1(n, r) := \{E \subset [n] : |E| = r, |E \cap [c]| = 1\}$, so it consists of all r -sets intersecting a fixed c -element subset of $V(H)$ at *exactly* one vertex. Then for large enough n , Ψ^1 has the maximum number of r -sets such that $\sigma(\Psi^1) \leq c$. Let us refer to this hypergraph as the *crosscut construction*. When r and c are fixed and $n \rightarrow \infty$,

$$|\Psi_c^1(n, r)| = c \binom{n-c}{r-1} = c \binom{n}{r-1} + o(n^{r-1}). \tag{2}$$

Given an r -graph F , let $\text{ex}_r(n, F)$ denote the maximum number of edges in an r -graph on n vertices that does not contain a copy of F (if the uniformity is obvious from context, we may omit the subscript r). Crosscuts were introduced in [9] to get the following obvious lower bounds

$$\text{ex}(n, F) \geq |\Psi_{\tau(F)-1}(n, r)|, \quad \text{and if crosscut exists then } \text{ex}(n, F) \geq |\Psi_{\sigma(F)-1}^1(n, r)|. \tag{3}$$

Notation. If H is a hypergraph and $e \subset V(H)$, the *neighborhood* of e is $\Gamma_H(e) = \{f \setminus e : e \subseteq f, f \in H\}$ and the *degree* of e is $d_H(e) = |\Gamma_H(e)|$. For an integer p , let the p -*shadow*, $\partial_p H$, be the collection of p -sets that lie in some edge of H . If H is an r -graph, then the $(r-1)$ -shadow of H is simply called the *shadow* and is denoted by ∂H .

Whenever we write $f(n) \sim g(n)$, we always mean $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$ while the other variables of f and g are fixed. This is the case even if the variable n is not indicated.

Aims of this paper. We have two aims. First, to find more Turán numbers (or estimates) of hypergraphs in the Erdős–Ko–Rado range. We are especially interested in cases when the excluded configuration is ‘dense’, it has only a few vertices of degree one. Second, we present an asymmetric version of Katona’s permutation method, when we first solve (estimate) the problem only on a wellchosen substructure. The (a, b) -blowups of trees and paths are good examples for both of our aims.

1.2 Paths in graphs

A fundamental result in extremal graph theory is the Erdős–Gallai Theorem [2], that

$$\text{ex}_2(n, P_\ell) \leq \frac{1}{2}(\ell - 1)n, \tag{4}$$

where P_ℓ is the ℓ -edge path. (Warning! This is a non-standard notation). Equality holds in (4) if and only if ℓ divides n and all connected components of G are ℓ -vertex complete graphs. The Turán function $\text{ex}(n, P_\ell)$ was determined exactly for every ℓ and n by Faudree and Schelp [5] and independently by Kopylov [16]. Let $n \equiv r \pmod{\ell}$, $0 \leq r < \ell$. Then $\text{ex}(n, P_\ell) = \frac{1}{2}(\ell-1)n - \frac{1}{2}r(\ell-r)$. They also described the extremal graphs which are either

- vertex disjoint unions of $\lfloor n/\ell \rfloor$ complete graphs K_ℓ and a K_r , or
- ℓ is odd, $\ell = 2k - 1$, and $r = k$ or $k - 1$. Then other extremal graphs with completely different structure can be obtained by taking a vertex disjoint union of m copies of K_ℓ ($0 \leq m < \lfloor n/\ell \rfloor$) and a copy of $\Psi_{k-1}(n - m\ell, 2)$, i.e., an $(n - m\ell)$ -vertex graph with a $(k - 1)$ -set meeting all edges.

This variety of extremal graphs makes the solution difficult.

We generalize these theorems for some hypergraph paths and trees.

1.3 Paths in hypergraphs

Paths of length 2. Two r -sets with intersection size b can be considered as a hypergraph path $P_2(a, b)$ of length two, where $a + b = r$, and $1 \leq a, b \leq r - 1$. If $H \subset \binom{[n]}{r}$ is $P_2(1, r - 1)$ -free then the obvious inequality $r|H| = |\partial(H)| \leq \binom{n}{r-1}$ yields the upper bound in the following result:

$$\frac{1}{r} \binom{n}{r-1} - O(n^{r-2}) < P(n, r, r-1) = \text{ex}_r(n, P_2(1, r-1)) \leq \frac{1}{r} \binom{n}{r-1}. \quad (5)$$

Here for any given r equality holds if n is sufficiently large ($n > n_0(r)$) and certain divisibility conditions are satisfied (see, Keevash [15]).

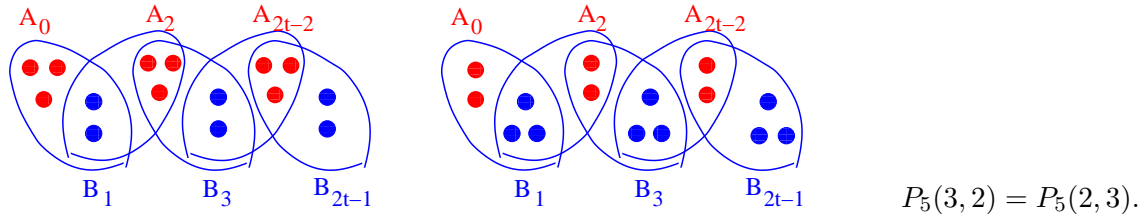
The case $b = 1$ was solved asymptotically by Frankl [6] and the general case was handled in [8].

$$\text{ex}_r(n, P_2(a, b)) = \Theta \left(n^{\max\{a-1, b\}} \right). \quad (6)$$

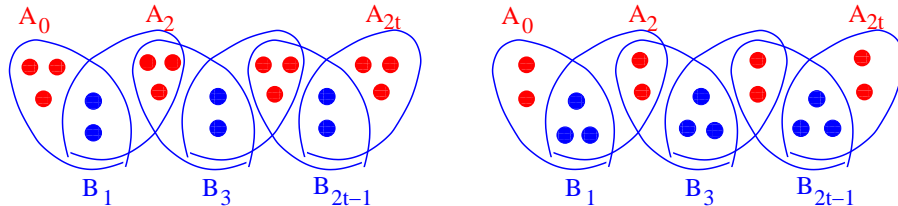
Here the right hand side of (6) is $o(n^{r-1})$ (for $1 \leq a, b \leq r - 1$).

Two disjoint r -sets can be considered as a $P_2(r, 0)$ so (6) also holds for $a = r$ since the maximum size of an intersecting family of r -sets is $\binom{n-1}{r-1}$ for $n \geq 2r$ by the Erdős-Ko-Rado theorem [3].

Definition. Suppose that a, b, ℓ are positive integers, $r = a + b$. The (a, b) -path $P_\ell(a, b)$ of length ℓ is an r -uniform hypergraph obtained from a (graph) path P_ℓ by blowing up its vertices to a -sets and b -sets. More precisely, an (a, b) -path $P_\ell(a, b)$ of length $2k - 1$ consists of $2k - 1$ sets of size $r = a + b$ as follows. Take $2k$ pairwise disjoint sets $A_0, A_2, \dots, A_{2k-1}$ with $|A_i| = a$ and $B_1, B_3, \dots, B_{2k-1}$ with $|B_j| = b$ and define the (hyper)edges of $P_{2k-1}(a, b)$ as the sets of the form $A_i \cup B_{i+1}$ and $B_j \cup A_{j+1}$. If the $ak + bk$ elements are ordered linearly, then the members of P can be represented as intervals of length r . By adding one more set A_{2k} to the underlying set together with the hyperedge $B_{2k-1} \cup A_{2k}$ we obtain the (a, b) -path of even length, $P_{2k}(a, b)$.



While $P_{2k-1}(a, b) = P_{2k-1}(b, a)$ we have that $P_{2k}(a, b) \neq P_{2k}(b, a)$ for $a \neq b$.



On (a, b) -paths of length 3.

In the case $\ell = 3$ an (a, b) -path has three r -sets, two of them are disjoint and they cover the third in a prescribed way. For given $1 \leq a, b < r$, $r = a + b$ and for $n > n_2(r)$, Füredi and Özkahya [13] showed that

$$\text{ex}_r(n, P_3(a, b)) = \binom{n-1}{r-1}.$$

Longer paths.

Our first goal is to prove a nontrivial extension of the Erdős–Gallai Theorem (4) for r -graphs.

There are several ways to define a hypergraph path P . One of the most difficult cases appears to be the case when P is a *tight path* of length ℓ , namely the r -graph $Tight P_\ell^r$ with edges $\{1, 2, \dots, r\}, \{2, 3, \dots, r + 1\}, \dots, \{\ell, \ell + 1, \dots, \ell + r - 1\}$. The best known results [11] for this special case are

$$\frac{\ell-1}{r} \binom{n}{r-1} \leq \text{ex}_r(n, Tight P_\ell^r) \leq \begin{cases} \frac{\ell-1}{2} \binom{n}{r-1} & \text{if } r \text{ is even,} \\ \frac{1}{2}(\ell + \lfloor \frac{\ell-1}{r} \rfloor) \binom{n}{r-1} & \text{if } r \text{ is odd,} \end{cases}$$

where the lower bound holds as long as certain designs exist.

Another possibility is the r -uniform *loose path* (also called *linear path*) $Lin P_\ell^r$, which is obtained from P_ℓ^2 by enlarging each edge with a new set of $(r - 2)$ vertices such that these new $(r - 2)$ -sets are pairwise disjoint (so $|V(P_\ell^r)| = \ell(r - 1) + 1$). Recently, the authors [12, 17] determined $\text{ex}_r(n, Lin P_\ell^r)$ exactly for large n , extending a work of Frankl [6] who solved the case $\ell = 2$ by answering a question of Erdős and Sós [22] (see [19] for a solution for all n when $r = 4$).

Here we consider the (a, b) -blowup of P_ℓ . Since the case $\ell = 2$ behaves somewhat differently, see (5) and (6), we only discuss the case $\ell \geq 3$.

Suppose that $a + b = r$, $a, b \geq 1$, $r \geq 3$ and suppose that $\ell \in \{2k - 1, 2k\}$, $\ell \geq 4$. Furthermore, suppose that these values are fixed and $n \rightarrow \infty$ or $n > n_3(r, k)$. Recall that $\Psi_{t-1}(n, r) := \{E \subset$

$[n] : |E| = r, E \cap [k-1] \neq \emptyset$. We have the lower bound

$$\begin{aligned} \text{ex}_r(n, P_{2k}(a, b)) &\geq \text{ex}_r(n, P_{2k-1}(a, b)) \\ &\geq |\Psi_{k-1}(n, r)| = \binom{n}{r} - \binom{n-k+1}{r} = (k-1) \binom{n}{r-1} + o(n^{r-1}). \end{aligned}$$

Our main result (Theorem 7) implies that here equality holds for at least 75% of the cases.

Theorem 1. *Suppose that $a + b = r$, $a, b \geq 1$, $k \geq 2$. Concerning the Turan number of $P_\ell(a, b)$, the (a, b) blowup of a path of length ℓ , we have*

$$\begin{aligned} \text{ex}_r(n, P_{2k-1}(a, b)) &= (k-1) \binom{n}{r-1} + o(n^{r-1}) \quad \text{for all odd } r, \\ \text{ex}_r(n, P_{2k}(a, b)) &= (k-1) \binom{n}{r-1} + o(n^{r-1}) \quad \text{for } a > b. \end{aligned}$$

Moreover, if $a \neq b$, $a, b \geq 2$, $\ell = 2k - 1$, then $\Psi_{k-1}(n, r)$ is the only extremal family.

The remaining cases (ℓ is even and $a \leq b$) are still open.

Conjecture 2. $\Psi_{k-1}(n, r)$ gives the correct asymptotic of the Turán number in all the above cases.

2 Trees blown up, our main results

Generalizing the Erdős–Gallai Theorem (4), Ajtai, Komlós, Simonovits and Szemerédi [1] claimed a proof of the Erdős–Sós Conjecture [4], showing that if T is any tree with ℓ edges, where ℓ is large enough, then for all n ,

$$\text{ex}_2(n, T) \leq \frac{1}{2}(\ell - 1)n.$$

A more general conjecture due to Kalai (see in [9]) is about the extremal number for hypergraph trees. A hypergraph T is a *forest* if it consists of edges e_1, e_2, \dots, e_ℓ ordered so that for every $1 < i \leq \ell$, there is $1 \leq i' < i$ such that $e_i \cap (\bigcup_{j < i} e_j) \subseteq e_{i'}$. A connected forest is called a *tree*. If T is r -uniform and for each $i > 1$, $|e_i \cap (\bigcup_{j < i} e_j)| = r - 1$, then we say that T is a *tight tree*.

Conjecture 3. (Kalai) *Let T be an r -uniform tight tree with ℓ edges. Then*

$$\text{ex}_r(n, T) \leq \frac{\ell - 1}{r} \binom{n}{r-1}.$$

When $r = 2$, this is precisely the Erdős–Sós Conjecture. A simple greedy argument shows that

Proposition 4. *If T is an r -uniform tight tree with ℓ edges and G is an r -graph on $[n]$ not containing T , then $|G| \leq (\ell - 1)|\partial(G)|$.*

Here $\partial(G)$ is the family of $(r - 1)$ -sets that lie in some edge of G . We obtain

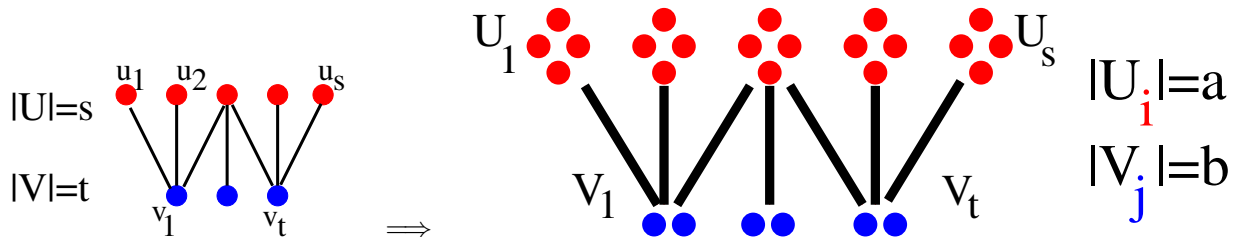
$$\text{ex}_r(n, T) \leq (\ell - 1) \binom{n}{r-1}.$$

Our goal is to prove a nontrivial extension of the Erdős–Gallai Theorem and the Erdős–Sós Conjecture for r -graphs. To define the hypergraph trees we study in this paper, we make the following more general definition:

Definition 5. Let $s, t, a, b > 0$ be integers, $r = a + b$, and let $H = H(U, V)$ denote a bipartite graph with parts $U = \{u_1, u_2, \dots, u_s\}$ and $V = \{v_1, v_2, \dots, v_t\}$. Let U_1, \dots, U_s and V_1, \dots, V_t be pairwise disjoint sets, such that $|U_i| = a$ and $|V_j| = b$ for all i, j . So $|\bigcup U_i \cup V_j| = as + bt$.

The (a, b) -blowup of H , denoted by $H(a, b)$, is the r -uniform hypergraph with edge set

$$H(a, b) := \{U_i \cup V_j : u_i v_j \in E(H)\}$$



Since an (a, b) -blowup of a bipartite graph H $\sigma(H)$ is well defined. Since deleting a vertex cover from a bipartite graph leaves an independent set, each cross cut in a connected bipartite graph is one of its parts, $\sigma(\mathcal{H}) = \min\{s, t\}$. Then the crosscut construction (2), $\Psi_{\sigma-1}^1(n, r) := \{E \subset [n] : |E| = r, |E \cap [\sigma - 1]| = 1\}$, yields that

$$(\sigma - 1) \binom{n}{r - 1} + o(n^{r-1}) = (\sigma - 1) \binom{n - \sigma + 1}{r - 1} = |\Psi_{\sigma-1}^1(n, r)| \leq \text{ex}_r(n, H). \quad (7)$$

Let $\mathcal{T}_{s,t}$ denote the family of trees T with parts U and V where $|U| = s$ and $|V| = t$. We frequently say that T is a tree with $s + t$ vertices. Let $\mathcal{T}_{s,t}(a, b)$ denote the family of (a, b) -blowups of trees $T \in \mathcal{T}_{s,t}$. We frequently suppose that $a \geq b$ (but not always).

We investigate the problem of determining when crosscut constructions are asymptotically extremal for (a, b) -blowups of trees. For other instances of hypergraph trees for which the crosscut constructions are asymptotically extremal, see [18]. Our main result is the following theorem.

Theorem 6. Suppose $r \geq 3$, $s, t \geq 2$, $a + b = r$, $b < a < r$. Let T be a tree on $s + t$ vertices and let $\mathcal{T} = \mathcal{T}(a, b)$, its (a, b) -blowup. Then (as $n \rightarrow \infty$) any \mathcal{T} -free n -vertex r -graph H satisfies

$$|H| \leq (t - 1) \binom{n}{r - 1} + o(n^{r-1}).$$

This is asymptotically sharp whenever $t \leq s$.

Indeed, in the case $t \leq s$ we have $\sigma(\mathcal{T}) = t$ and (7) provides a matching lower bound.

A vertex x of $T \in \mathcal{T}_{s,t}$ is called a *critical leaf* if $\sigma(T \setminus x) < \sigma(T)$. In case of $t \leq s$ it simply means that $\deg_T(x) = 1$ and $x \in V$. (Similarly, a *critical leaf* of $\mathcal{T} = \mathcal{T}(a, b) \in \mathcal{T}_{s,t}(a, b)$ with $t \leq s$ is a b -set V_j in the part of size t whose degree in \mathcal{T} is one). If such a vertex exists then we have a more precise upper bound.

Theorem 7. *Suppose $r \geq 5$, $2 \leq t \leq s$, $a + b = r$, $b < a < r - 1$. Let T be a tree on $s + t$ vertices and let $\mathcal{T} = T(a, b)$, its (a, b) -blowup. Suppose that T has a critical leaf. Then for large enough n ($n > n_0(T)$)*

$$\text{ex}(n, \mathcal{T}) \leq \binom{n}{r} - \binom{n-t+1}{r}.$$

If, in addition, $\tau(\mathcal{T}) = t$, then equality holds above and the only example achieving the bound is $\Psi_{t-1}(n, r)$.

Since $\tau(\Psi_{t-1}(n, r)) = t - 1$, no r -graph F with $\tau(F) \geq t$ is contained in $\Psi_{t-1}(n, r)$.

3 Asymptotics

In this section we prove the asymptotic version of our main results, i.e., Theorem 6.

3.1 Definition of templates and a lemma.

Throughout this section, $\mathcal{T} \in \mathcal{T}_{s,t}(a, b)$ and we suppose \mathcal{T} is an (a, b) -blowup of a tree T . If H is an r -graph, then an (a, b) -template in H is a pair (A, B) where A is an a -uniform hypergraph on $V(H)$, B is a b -uniform matching on $V(H)$, and $V(A) \cap V(B) = \emptyset$. Define the bipartite graph

$$H_0 = H_0(A, B) = \{(e, f) \in A \times B : e \cup f \in H\}$$

and let $H_1 = H_1(A, B) = \{e \cup f : (e, f) \in H_0\} \subset H$. By construction, $|H_0| = |H_1|$. We claim that *if A and B are both matchings and $H_1(A, B)$ is \mathcal{T} -free, then*

$$|H_1(A, B)| \leq (t-1)|A| + (s-1)|B|. \tag{8}$$

Indeed, otherwise $|H_0(A, B)| = |H_1(A, B)| > (t-1)|A| + (s-1)|B|$ and H_0 has a minimum induced subgraph $H'_0(A', B')$ satisfying $|H'_0(A', B')| > (t-1)|A'| + (s-1)|B'|$. By minimality, H'_0 has minimum degree at least t in A' and minimum degree at least s in B' . This is sufficient to greedily construct a copy of T in H'_0 . Since H_1 is an (a, b) -blowup of $H_0 \supseteq H'_0$, this shows $\mathcal{T} \subset H_1$.

We now prove a version of (8) for templates, i.e., in the case when A may be not a matching:

Lemma 8. *Let $\delta > 0$ and let $\mathcal{T} \in \mathcal{T}_{s,t}(a, b)$. Let H be a \mathcal{T} -free r -graph containing an (a, b) -template (A, B) . If $B = B^0 \sqcup B^1$ and $d_H(e) \leq \delta n^b$ for every a -set $e \subset V(H) \setminus V(B^1)$, then*

$$|H_1(A, B)| \leq (t-1)|A| + asn^{a-1}(\delta|B^0| + |B^1|). \tag{9}$$

Proof. Let $\beta_0 = as\delta n^{a-1}$ and $\beta_1 = asn^{a-1}$. Let $H_1 = H_1(A, B)$ and $H_0 = H_0(A, B)$ and suppose $|H_1| \geq (t-1)|A| + \beta_0|B^0| + \beta_1|B^1|$. By deleting vertices of H_0 , we may assume

$$d_{H_0}(e) \geq t \text{ for all } e \in A \text{ and for } i \in \{0, 1\}, d_{H_0}(e) > \beta_i \text{ for all } e \in B^i. \tag{10}$$

Suppose \mathcal{T} is a blowup of a tree T , where T has a unique bipartition (U, V) with $|U| = s, |V| = t$. We call an embedding of the (a, b) -blowup of a subtree T' of T in $H_1(A, B)$ a *feasible embedding* if

the a -sets corresponding to vertices in U are mapped to members of A and the b -sets corresponding to vertices in V are mapped to members of B . It suffices to prove that any feasible embedding h of the (a, b) -blowup of any proper subtree T' of T can be extended to a feasible embedding h' of the (a, b) -blowup of a subtree of T that strictly contains T' .

Let T' be given. Then there exists an edge xy in T with $x \in V(T')$ and $y \notin V(T')$. Let h be a feasible embedding of the (a, b) -blowup \mathcal{T}' of T' in $H_1(A, B)$. First suppose that $x \in U$. Let e denote the image under h of a -set in \mathcal{T}' that corresponds to x . By our assumption $e \in A$. Hence by our earlier assumption, $d_{H_0}(e) \geq t$. Thus $|\Gamma_{H_1}(e)| \geq t$. Since $\Gamma_{H_1}(e) \subseteq B$ is a matching of size at least t and the b -sets corresponding to $V - \{y\}$ are mapped to at most $t - 1$ members of B , there exists $f \in B$ such that $f \cap V(h(\mathcal{T}')) = \emptyset$. We can extend h to a feasible embedding of $T' \cup xy$ by mapping the b -set in \mathcal{T} corresponding to y to f .

Next, suppose $x \in V$. Let e denote the image under h of the b -set in \mathcal{T}' that corresponds to x . If there exists $f \in \Gamma_{H_1}(e) - V(h(\mathcal{T}'))$, then $h(\mathcal{T}') \cup \{e \cup f\}$ is a feasible embedding of $T' \cup xy$. Hence we may assume that no such f exists. If $e \in B^0$, then we estimate $d_{H_0}(e)$ by adding $a - b$ new vertices, one from $V(h(\mathcal{T}'))$ and all outside $V(B^1)$. This yields

$$d_{H_0}(e) \leq |V(h(\mathcal{T}')) \cap V(A)| \cdot n^{a-b-1} \cdot \delta n^b \leq a\delta n^{a-1} = \beta_0,$$

a contradiction to (10). Note it is crucial here that $b < a$. Similarly, if $e \in B^1$, then

$$d_{H_0}(e) \leq |V(h(\mathcal{T}')) \cap V(A)| \cdot n^{a-1} \leq a\delta n^{a-1} = \beta_1.$$

This contradicts $d_{H_0}(e) > \beta_1$ for $e \in B^1$. Hence we have shown that each feasible embedding of \mathcal{T}' can be extended. This completes the proof. \square

3.2 Proof of Theorem 6.

In a few places of the proof we will use the following elementary fact or a slight variant of it. Let e be a fixed edge in $\binom{[n]}{p}$ and H a p -graph on at most n vertices. Let L be a copy of H in $\binom{[n]}{p}$ chosen uniformly at random among all copies of H . Then $P(e \in L) = |H|/\binom{n}{p}$.

Let m be an integer satisfying $m > r^r$ and $m = o(\sqrt{n})$. Let $f(m) = m^{-1/r}n^{r-1} + m^2n^{r-2}$. We show that if H is \mathcal{T} -free for some $\mathcal{T} \in \mathcal{T}_{s,t}(a, b)$, then

$$|H| \leq (t-1) \binom{n}{r-1} + O(f(m)).$$

In particular, taking $m = n^{1/3}$, we obtain

$$|H| \leq (t-1) \binom{n}{r-1} + O(n^{r-1-1/(3r)}).$$

In our arguments below, for convenience, we assume b divides n , since assuming so has no effect on the asymptotic bound we want to establish. Let $D = \{e \in \binom{V(H)}{a} : d_H(e) \geq n^b/m\}$ and L be a

smallest vertex cover of D , meaning that every set in D intersects L . We claim

$$|L| = O(m). \tag{11}$$

Indeed, if $|L| \geq asm$, then D has a matching M of size sm . Each set in M forms an edge of H with at least n^b/m different b -sets, and at most $a|M|n^{b-1} = asm n^{b-1}$ of these b -sets intersect $V(M)$. By averaging, there is a matching N of b -sets disjoint from $V(M)$ such that

$$|H_0(M, N)| \geq \frac{|M|(n^b/m - asm n^{b-1})}{\binom{n-1}{b-1}} > |M| \cdot \frac{n}{m} - |M| \cdot asm.$$

Since n is large and $m = o(\sqrt{n})$, this is at least

$$(t-1)|M| + \left(\frac{n}{m} - t + 1 - asm\right)|M| \geq (t-1)|M| + (s-1)n > (t-1)|M| + (s-1)|N|.$$

By (8), we conclude that $\mathcal{T} \subset H_1(M, N) \subset H$, a contradiction. This proves (11).

Let $G = \{e \in H : |e \cap L| \leq 1\}$, so that

$$|G| \geq |H| - |L|^2 n^{r-2} \geq |H| - O(m^2 n^{r-2}). \tag{12}$$

Let $R \subset V(G) \setminus L$ be a set whose elements are chosen independently with probability $\alpha = m^{-1/r}$, and $A = \binom{R}{a}$. Let P be a random partition of $V(G)$ into b -sets. Let B denote the set of b -sets in P that are disjoint from R , and let $H_1 = H_1(A, B)$. If $B^0 = \{e \in B : e \cap L = \emptyset\}$ and $B^1 = \{e \in B : |e \cap L| \geq 1\}$, then by (9) with $\delta = 1/m$, and using $|B^1| \leq |L|$,

$$|H_1| \leq (t-1)|A| + O(n^{a-1}|B^0|/m) + O(n^{a-1}|L|).$$

Taking expectations over all choices of R and P and using (11) and $|B^0| \leq n$, we get

$$E(|H_1|) \leq (t-1)\alpha^a \binom{n}{a} + O(n^a/m). \tag{13}$$

For $i \in \{0, 1\}$, let $G_i = \{e \in G : |e \cap L| = i\}$ and note $G = G_0 \cup G_1$. We observe that for an edge $e \in G_0$,

$$P(e \in H_1) = \frac{\binom{r}{b}\alpha^a(1-\alpha)^b}{\binom{n-1}{b-1}} := p_0$$

and for an edge $e \in G_1$,

$$P(e \in H_1) = \frac{\binom{r-1}{b-1}\alpha^a(1-\alpha)^{b-1}}{\binom{n-1}{b-1}} := p_1.$$

Since $\alpha = m^{-1/r} < 1/r$ and $b \leq (r-1)/2$,

$$p_0 = \frac{r}{b}(1-\alpha)p_1 > 2p_1.$$

Therefore

$$E(|H_1|) \geq p_0|G_0| + p_1|G_1| = (p_0 - p_1)|G_0| + p_1|G| > p_1|G| = \frac{\alpha^a(r-1)!(1-\alpha)^{b-1}}{a!n^{b-1}}|G|. \quad (14)$$

Combining this with (13), using $(1-\alpha)^{-b+1} = 1 - O(m^{-1/r})$ and after some simplification, we find

$$\begin{aligned} |G| &\leq (t-1) \binom{n}{r-1} + O(\alpha n^{r-1}) + O(n^{r-1}/\alpha^a m) \\ &\leq (t-1) \binom{n}{r-1} + O(m^{-1/r} n^{r-1}). \end{aligned}$$

Together with (12), this gives the required bound on $|H|$. □

In fact, the proof of Theorem 6 yields more than the theorem claims. We have the following fact.

Corollary 9. *Let $0 < \gamma < 1/t$, $b < a < r$, $a+b = r$, $t \leq s$. Let n be sufficiently large, $r^r < m \leq n^\gamma$ and $f(m) = m^{-1/r} n^{r-1} + m^2 n^{r-2}$. Let $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ and H be an n -vertex \mathcal{T} -free r -graph. If*

$$|H| = (t-1) \binom{n}{r-1} + O(f(m)) \quad (15)$$

then some $F \subset H$ with $|F| = |H| - O(f(m))$ has a crosscut L of size $O(m)$.

Proof. If $|H| = (t-1) \binom{n}{r-1} + O(f(m))$, then the upper and lower bounds for $E(|H_1|)$ given by (13) and (14) differ by $O(n^a/m)$. By (14) they also differ by at least $(p_0 - p_1)|G_0|$ so

$$(p_0 - p_1)|G_0| = O(n^a/m).$$

Using $p_0 > (1+1/r)p_1$, we get $p_1|G_0| = O(n^a/m)$ and this shows $|G_0| = O(f(m))$. Setting $F = G_1$, L is a crosscut of F and $|F| = |H| - O(f(m))$. □

4 Stability

The aim of this section is to prove the following stability theorem. It is important throughout this section that $t \leq s$, so that for $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$, we have $\sigma(\mathcal{T}) = t$ and therefore $\Psi_{t-1}^1(n,r)$ does not contain \mathcal{T} . The following theorem says that if H is a \mathcal{T} -free r -graph on n vertices and $|H| \sim |\Psi_{t-1}(n,r)|$, then H is obtained by adding or deleting $o(n^{r-1})$ edges from $\Psi_{t-1}(n,r)$.

Theorem 10. *Let $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$, where $b < a < r-1$, $t \leq s$. Let H be a \mathcal{T} -free n -vertex r -graph with $|H| \sim (t-1) \binom{n}{r-1}$. If \mathcal{T} has a critical leaf, then there exists a set S of $t-1$ vertices of H such that $|H - S| = o(n^{r-1})$.*

4.1 Degrees of sets.

By Corollary 9 with $r^r < m = o(n^{1/(t+1)})$ there exists $F \subset H$ such that $|F| \sim |H|$ and F has a crosscut L of size $O(m)$. Our first claim says that most elements of ∂F have degree $t-1$ in F .

Claim 1. *There are $\binom{n}{r-1} - o(n^{r-1})$ sets $e \in \partial F - L$ such that $d_F(e) = t - 1$.*

Proof. Suppose ℓ sets $e \in \partial F - L$ have $d_F(e) \geq t$. By the definition of L , $\Gamma(e) \subseteq L$ for each $e \in \partial F - L$. Let Z be a crosscut of \mathcal{T} with $|Z| = t$ contained in B and let $\mathcal{T}^* = \{e \setminus Z : e \in \mathcal{T}\}$. Then \mathcal{T}^* is an $(a, b - 1)$ -blowup of T . Proposition 4 implies

$$\text{ex}(n, \mathcal{T}^*) < (s + t)n^{r-2}.$$

By the pigeonhole principle, there exists a set $S \subset L$ with $|S| = t$ such that at least $k = \ell/|L|^t$ sets $e \in \partial F - L$ have $\Gamma_F(e) \supseteq S$. If $k > \text{ex}(n, \mathcal{T}^*)$, then $\mathcal{T}^* \subset \partial F - L$ and for all $e \in \mathcal{T}^*$, $\Gamma_G(e) \supseteq S$. Now we can lift \mathcal{T}^* to $\mathcal{T} \subset F$ via S . Indeed, we can greedily enlarge each of the $(b - 1)$ -sets that form \mathcal{T}^* to a b -set by adding an element of S . This contradicts the choice of H . We therefore suppose that

$$\ell/|L|^t = k \leq \text{ex}(n, \mathcal{T}^*) \leq (s + t)n^{r-2}$$

which gives $\ell \leq (s + t)|L|^t n^{r-2} = O(n^{r-2}m^t)$. As $|F| \sim |H| \sim (t - 1)\binom{n}{r-1}$, and the number of $(r - 1)$ -sets in $V(F) - L$ is at most $\binom{n}{r-1}$, the average degree of sets in $\partial F - L$ is at least $t - 1 - o(1)$. We have already argued that at most $O(n^{r-2}m^t)$ of these sets have degree larger than $t - 1$. Furthermore, none of them has degree greater than m . Hence the number of sets in $\partial F - L$ of degree at most $t - 2$ is z , then we have inequality

$$(t - 1)\binom{n}{r - 1} - x + mO(n^{r-2}m^t) \geq (t - 1)\binom{n}{r - 1}(1 - o(1)).$$

Since $m n^{r-2}m^t = o(n^{r-1})$, we conclude that $x = o\left(\binom{n}{r-1}\right)$. This yields the claim. \square

4.2 Proof of Theorem 10

Let S_1, S_2, \dots, S_k be an enumeration of the $(t - 1)$ -element subsets of L , and let F_i denote the family of $(r - 1)$ -element sets e in $V(F) \setminus L$ such that $\Gamma_F(e) = S_i$. By Claim 1, $|F_1 \cup F_2 \cup \dots \cup F_k| \sim \binom{n - |L|}{r - 1}$. Suppose $k \geq 2$. By definition, for $i \neq j$, $F_i \cap F_j = \emptyset$. Therefore,

$$\sum_{i=1}^k |F_i| \sim \binom{n}{r - 1}.$$

For each $i \in [k]$, if $|F_i| = o(n^{r-1}/k)$, let G_i be an empty $(r - 1)$ -graph, if $|F_i| = \Omega(n^{r-1}/k)$, then delete edges of F_i containing a -sets or b -sets of "small" degree until we obtain either an empty $(r - 1)$ -graph or an $(r - 1)$ -graph G_i such that

$$d_{G_i}(e) > r(s + t)n^{r-2-a} \forall a\text{-set } e \in \partial_a G_i, \text{ and } d_{G_i}(f) > r(s + t)n^{r-2-b} \forall b\text{-set } f \in \partial_b G_i. \quad (16)$$

By construction, $|G_i| \geq |F_i| - 2r(s + t)n^{r-2}$ and since $F_i = \Omega(n^{r-1}/k)$ and $k \leq |L|^t \leq O(m^t) = o(n)$, whenever G_i is non-empty we have

$$|G_i| = (1 - o(1))|F_i|.$$

We conclude that if $G = \bigcup G_i$ then $|G| = (1 - o(1))|F| \sim \binom{n}{r-1}$ and

$$\sum_{i=1}^k |G_i| \sim \binom{n}{r-1}. \tag{17}$$

Claim 2. For $i \neq j$, $\partial_a G_i \cap \partial_a G_j = \emptyset$.

Proof. Let W be a tree obtained from the tree T by deleting a leaf vertex x with unique neighbor $y \in T$, such that x is in the part of T of size t . Suppose some a -set e is contained in $\partial_a G_i \cap \partial_a G_j$. By (16), we can greedily grow $W(a, b-1)$ in G_j such that e is the blowup of y . By adding one vertex of S_j to each $b-1$ -set in $W(a, b-1)$, we obtain $W(a, b)$. Now there exists $x' \in S_i \setminus S_j$. Since $d_{G_i}(e) > r(s+t)n^{r-2-a}$, there exists an edge $f \in G_i$ containing e , such that $f \cap V(W(a, b-1)) = \emptyset$, and therefore $f \cup \{x'\} \in F$ plus $W(a, b)$ gives the tree $T(a, b)$, with $f \setminus e$ the blowup of x . This proves the claim. \square

Now we prove Theorem 10. Since $a \leq r-2$, by Claim 2, for all $i \neq j$, $\partial_{r-2} G_i \cap \partial_{r-2} G_j = \emptyset$. Without loss of generality, suppose that for some $0 \leq p \leq k$, $|G_1| \geq |G_2| \geq \dots \geq |G_p| \geq 1$ and $G_i = \emptyset$ for $p+1 \leq i \leq k$. For each $i \in [p]$, let $y_i \geq r-1$ denote the real such that $|G_i| = \binom{y_i}{r-1}$. Then $y_1 \geq y_2 \geq \dots \geq y_p$. By the Lovász form of the Kruskal-Katona theorem, for each $i \in [p]$, $|\partial_{r-2}(G_i)| \geq \binom{y_i}{r-2}$. By the disjointness of the $\partial_{r-2}(G_i)$'s, we have

$$\sum_{i=1}^p \binom{y_i}{r-2} \leq \binom{n}{r-2}.$$

For each $i \in [p]$, since $\binom{y_i}{r-1} = \frac{y_i - r + 2}{r-1} \binom{y_i}{r-2} \leq \frac{y_1 - r + 2}{r-1} \binom{y_i}{r-2}$, by (17) we have

$$(1 - o(1)) \binom{n}{r-1} \leq \sum_{i=1}^p |G_i| = \sum_{i=1}^p \binom{y_i}{r-1} \leq \frac{y_1 - r + 2}{r-1} \sum_{i=1}^p \binom{y_i}{r-2} \leq \frac{y_1 - r + 2}{r-1} \binom{n}{r-2}.$$

From this, we get $y_1 \geq n - o(n)$. Hence $|F_1| \geq |G_1| = \binom{y_1}{r-1} \geq \binom{n}{r-1} - o(n^{r-1})$. Hence there exists $S = S_1 \subset L$ such that $(t-1)\binom{n}{r-1} - o(n^{r-1})$ edges of F consists of one vertex in S and $r-1$ vertices disjoint from S . \square

5 Exact results

The aim of this section is to prove the following theorem, which completes the proof of Theorem 7:

Theorem 11. Let $t \leq s$, $b < a < r-1$ with $a+b=r$ and $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ such that \mathcal{T} has a critical leaf and $\tau(\mathcal{T}) = t$. If n is large and H is a \mathcal{T} -free n -vertex r -graph with $|H| \geq \binom{n}{r} - \binom{n-t+1}{r}$, then $H \cong \Psi_{t-1}(n, r)$.

To prove this, we aim to show that the set S given by Theorem 10 is a vertex cover of H . We prove the following consequence of Claim 1:

Claim 3. *Let $\Delta_u = (t-1)\binom{n-u}{r-1-u}$. Then for each $\delta > 0$, there exists $G \subset F$ with $|G| \sim |F|$ such that for any u -set $e \subset V(G)$ with $u < r$ and $d_G(e) > 0$, either*

- (i) $|e \cap S| = 0$ and $d_G(e) \geq (1 - \delta)\Delta_u$ or
- (ii) $|e \cap S| = 1$ and $d_G(e) \geq r(s+t)n^{r-1-u}$.

Proof. Let S be the $(t-1)$ -set given by Theorem 10 and K be the set of edges of F containing some $e \in \partial F - S$ with $d_F(e) = t-1$. By Claim 1, $|K| \sim |F|$. Also, every r -set in K has one point in S and $r-1$ points in $V(K) \setminus S$. Since $d_K(e) = t-1$ for all $e \in \partial K - S$, every u -set in $V(K) \setminus S$ has degree at most Δ_u in K .

We repeatedly delete edges from K as follows. Suppose at some stage of the deletion we have a hypergraph K' . If there exists a u -set e for some $u < r$ such that

- (i') $|e \cap S| = 0$ and $d_{K'}(e) < (1 - \delta)\Delta_u$ or
- (ii') $|e \cap S| = 1$ and $d_{K'}(e) < r(s+t)n^{r-1-u}$

then delete all edges of K' containing e . Let G be the hypergraph obtained at the end of this process. We shall prove $|G| \sim |K|$. To this end, suppose that $|G| = |K| - \eta(t-1)\binom{n}{r-1}$, and we show $\eta = o(1)$ to complete the proof. Consider two cases.

Case 1. *At least $\frac{\eta}{2}(t-1)\binom{n}{r-1}$ edges of K were deleted due to (ii').*

In this case, there exists $u < r$ such that the set H' of edges of K deleted due to (ii') on u -sets satisfies $|H'| \geq \frac{\eta}{2r}(t-1)\binom{n}{r-1}$. Then by (ii'), and since the number of u -sets with one vertex in S is $|S|\binom{n-|S|}{u-1}$,

$$|H'| \leq |S|\binom{n-|S|}{u-1} \cdot r(s+t)n^{r-1-u} < |S|r(s+t)n^{r-2}.$$

Since $|H'| \geq \frac{\eta}{2r}\binom{n}{r-1}$ and $|S| = t-1$, this gives $\eta = o(1)$.

Case 2. *At least $\frac{\eta}{2}(t-1)\binom{n}{r-1}$ edges of K were deleted due to (i').*

In this case, there exists $u < r$ such that the set H' of edges of K deleted due to (i') on u -sets satisfies $|H'| \geq \frac{\eta}{2r}(t-1)\binom{n}{r-1}$. Let U_1 be the set of u -sets in $V(K) \setminus S$ on which edges of K were deleted due to (i'), and let U_2 be the remaining u -sets in $V(K) \setminus S$. Then

$$|U_1| > \frac{|H'|}{(1-\delta)\Delta_u} \geq \frac{\eta(t-1)\binom{n}{r-1}}{2r(t-1)\binom{n}{r-1-u}}.$$

If n is large enough, then this is at least $\frac{\eta}{4r\binom{r-1}{u}}\binom{n}{u}$. Let $\gamma = \frac{\eta}{4r\binom{r-1}{u}}$. Then

$$\begin{aligned} |K|\binom{r-1}{u} &= \sum_{e \in (V(K) \setminus L)} d_K(e) \\ &= \sum_{e \in U_1} d_K(e) + \sum_{e \in U_2} d_K(e) \\ &\leq |U_1|(1-\delta)\Delta_u + |U_2|\Delta_u \\ &\leq \gamma(1-\delta)\binom{n}{u}\Delta_u + (1-\gamma)\binom{n}{u}\Delta_u = (1-\gamma\delta)\binom{n}{u}\Delta_u. \end{aligned}$$

Here we used $|U_1| + |U_2| \leq \binom{n}{u}$. Therefore

$$|K| \leq (1-\gamma\delta)\frac{\binom{n}{u}\Delta_u}{\binom{r-1}{u}} = (1-\gamma\delta)(t-1)\binom{n}{r-1}.$$

Since $|K| \sim |F| \sim (t-1)\binom{n}{r-1}$, $\gamma\delta = o(1)$. Since $\delta > 0$ and $\gamma = \frac{\eta}{4r\binom{r-1}{u}}$, this implies $\eta = o(1)$, as required. \square

Let $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ have a critical leaf with $\tau(\mathcal{T}) = t \leq s$, $a+b=r$, $b < a < r-1$, and let H be a \mathcal{T} -free n -vertex r -graph with $|H| \geq \binom{n}{r} - \binom{n-t+1}{r}$. We aim to show that S is a vertex cover of H , which gives $H \cong \Psi_{t-1}(n,r)$, as required. To this end, let $H_i = \{e \in H : |e \cap S| = i\}$. So we have to show $H_0 = \emptyset$.

Since \mathcal{T} has a critical leaf, there is a b -set e' of \mathcal{T} in the part of size t with $d_{\mathcal{T}}(e') = 1$. Let \mathcal{T}' be the tree obtained from \mathcal{T} by deleting the edge containing e' . So $V(\mathcal{T}')$ has one part comprising $t-1$ sets, each of size b and the other part comprising s sets, each of size a . It has a crosscut of size $t-1$ by picking one vertex from each of the b -sets above.

Let \mathcal{K}^1 be the set of r -sets of $[n]$ that have exactly one vertex in S . A subfamily $T \subset \mathcal{K}^1$ is a *potential tree* if

- 1) $T \cong \mathcal{T}'$
- 2) the $t-1$ vertices of S play the role of the crosscut vertices of \mathcal{T}' described above
- 3) e_0 is an a -set in $V(T)$ with $e_0 \in \partial_a H_0$
- 4) $e_0 \subset e \in H_0$
- 5) $T \cup e$ is a copy of \mathcal{T} .

Fix an a -set $e_0 \in \partial_a H_0$ and suppose $e_0 \subset e \in H_0$. If $T \subset H_1$ is a potential tree as described above, then $T \cup \{e\}$ is a copy of \mathcal{T} in H , a contradiction. So for each such potential tree T , there exists $f \in T - H_1$. Let us call this a *missing edge*. Let $m = as + bt - b$ be the number of vertices of each potential tree. The number of potential trees containing a fixed missing edge f is at most

$$\binom{n - |S| - (a+b-1)}{m - |S| - (a+b-1)} \cdot c(\mathcal{T}),$$

where $c(\mathcal{T})$ is the number of ways we can put a potential tree using f into the set M with $|M| = m$ and $(S \cup f) \subset M \subset [n]$, (note that $|f \cap S| = 1$).

On the other hand, each $e_0 \in \partial_a H_0$ and a subset M' with $|M'| = m$ and $S \subset M' \subset ([n] - e_0)$ carries at least one potential tree so the total number of potential trees is at least

$$|\partial_a H_0| \binom{n - |S| - a}{m - |S| - a}.$$

It follows that the number of missing edges is at least $c|\partial_a H_0|n^{b-1}$ for some $c > 0$. Therefore

$$|H| = |H_0| + |H_1| + |H_2| + \cdots + |H_r| \leq \binom{n}{r} - \binom{n-t+1}{r} + |H_0| - c|\partial_a H_0|n^{b-1}.$$

By Proposition 4 and the fact that \mathcal{T} is contained in a tight tree on $V(\mathcal{T})$, $|H_0| < c'|\partial H_0|$ for some constant c' .

Next, we observe that $\partial H_0 \cap \partial G = \emptyset$, for otherwise we can use Claim 3 to greedily build a copy of \mathcal{T} using the edge of H_0 , and whose remaining edges form a copy of \mathcal{T}' and come from G . In particular, since $|\partial G| \sim \binom{n}{r-1}$, $|\partial H_0| = o(n^{r-1})$. Writing $|\partial H_0| = \binom{x}{r-1}$ for some real x , we have $|\partial_a H_0| \geq \binom{x}{a}$, by the Kruskal-Katona Theorem. Therefore

$$|H_0| - c|\partial_a H_0|n^{b-1} \leq c'|\partial H_0| - c|\partial_a H_0|n^{b-1} \leq c' \binom{x}{r-1} - cn^{b-1} \binom{x}{a}.$$

Since $x = o(n)$, for large enough n the above expression is negative, unless $|\partial H_0| = |\partial_a H_0| = 0$. We have shown that if $|H| \geq \binom{n}{r} - \binom{n-t+1}{r}$, then $H_0 = \emptyset$ and $|H| = \binom{n}{r} - \binom{n-t+1}{r}$, as required. \square

6 Concluding remarks

In this paper we determined for $b \leq a < r$ the asymptotic behavior of $\text{ex}_r(n, \mathcal{T})$ when $\mathcal{T} \in \mathcal{T}_{s,t}(a, b)$ is an (a, b) -blowup of a tree T with parts of sizes s and t where $s \geq t$ and $\sigma(\mathcal{T}) = t$. The extremal problem appears to be more difficult when $s < t$, in which case the smallest crosscut of \mathcal{T} has size s . We pose Conjecture 12, which covers all cases except $a = r - 1$.

Conjecture 12. *If $\mathcal{T} \in \mathcal{T}_{s,t}(a, b)$ where $b \leq a < r - 1$, $\sigma = \sigma(\mathcal{T}) = \min\{s, t\}$, and H is a \mathcal{T} -free n -vertex r -graph, then for large enough n , $|H| \leq (\sigma - 1)\binom{n}{r-1} + o(n^{r-1})$, with equality only if H is isomorphic to a hypergraph obtained from $\Psi_{\sigma-1}(n, r)$ by adding or deleting $o(n^{r-1})$ edges.*

The case $a = r - 1$. If $t > s$ (and $n \geq |V(\mathcal{T})|$), then $\Psi_{t-1}^1(n, r)$ contains \mathcal{T} so Conjecture 12 does not hold. Since $\Psi_{s-1}^1(n, r)$ does not contain \mathcal{T} , it is natural to ask whether $\Psi_{s-1}^1(n, r)$ is (asymptotically) extremal for \mathcal{T} . In some cases when $a = r - 1$, this is certainly not so because certain Steiner systems do not contain a blowup of a star $K_{1,t}$ and are denser than $\Psi_{s-1}(n, r)$. More precisely: Let T be a tree on $s + t$ vertices and let $\mathcal{T} = T(a, b)$, its (a, b) -blowup. Suppose $a = r - 1$ and let $\lambda = \max_{x \in U} \deg_T(x)$. Then $\text{ex}(n, \mathcal{T})$ is at least the number of edges in a Steiner $(n, r, r - 1, \lambda - 1)$ -system – an r -graph on n vertices where each $(r - 1)$ -set is contained in exactly $\lambda - 1$ edges. In this case, $\text{ex}(n, T(r - 1, 1)) \geq \frac{\lambda-1}{r} \binom{n}{r-1}$ for infinitely many n (due to the existence of those designs [15]) whereas $\sigma(T) = s$ and it could be much less than $\frac{\lambda-1}{r}$.

No stability for $a = r - 1$. It is important in the above proof that $a \neq r - 1$. If $a = r - 1$, then there is no stability theorem: consider for instance an $(r - 1, 1)$ -blowup \mathcal{T} of a path with four edges. Let H be the n -vertex r -graph constructed as follows. Let $V(H) = [n]$, let $G_1 \sqcup G_2$ be a partition of the edge set of the complete $(r - 1)$ -graph on $\{3, 4, \dots, n\}$, and let H consist of the edges $e \cup \{i\}$ such that $e \in G_i$, for $i \in \{1, 2\}$. Then $|H| = \binom{n-2}{r-1}$ and H does not contain \mathcal{T} .

The case $a = b = r/2$. Let T be a tree on $s + t$ vertices then for $\mathcal{T} = T(r/2, r/2)$ one can use an argument of Frankl [7] (applied by many others, see [20]) to prove that

$$\text{ex}_r(n, \mathcal{T}) \leq \frac{\text{ex}(\lfloor 2n/r \rfloor, T) \binom{n}{r}}{\binom{\lfloor 2n/r \rfloor}{2}} \sim \frac{\text{ex}(\lfloor 2n/r \rfloor, T)}{\lfloor 2n/r \rfloor} \binom{n}{r-1}. \tag{18}$$

Indeed, similarly to the idea of templates, given a \mathcal{T} -free r -graph H on n vertices take a random partition of $[n]$ into $r/2$ -sets, (where for simplicity $r/2$ divides n), and consider only those r -edges of H which are unions of two partite sets. Then this subfamily consists of at most $\text{ex}(2n/r, H)$ edges of H , out of the possible $\binom{2n/r}{2}$.

The bound is asymptotically tight, due to $\Psi_{t-1}^1(n, r)$, if $\sigma(\mathcal{T}) = t$ and T has $2t - 1$ edges. So the inequality (18) completes the proof of Theorem 1 showing that $\text{ex}_r(n, P_{2k-1}(\frac{r}{2}, \frac{r}{2})) \sim (k - 1) \binom{n}{r-1}$ (the other cases follow from Theorems 6 and 7). It also gives a better upper bound for the even length, $\text{ex}_r(n, P_{2k}(\frac{r}{2}, \frac{r}{2})) \leq (1 + o(1)) (k - \frac{1}{2}) \binom{n}{r-1}$.

However, the proof of (18) does not reveal the extremal structure.

The case of forests. Many of our ideas can be generalized for the case of $\mathcal{T} = F(a, b)$, when F is a *forest*, but we do not have a general conjecture.

Problem 13. *Given $a, b \geq 1$ and a forest F on $s+t$ vertices. Determine $\lim_{n \rightarrow \infty} \text{ex}(n, F(a, b)) \binom{n}{r-1}^{-1}$.*

Other bipartite graphs. The class of (a, b) -blowups of bipartite graphs contains well-studied instances including blowups of complete bipartite graphs. In particular, Füredi [10] made the following conjecture for blowups of a 4-cycle. Let $\mathcal{C}_4^r = \{C_4(a, b) : a + b = r, a, b > 0\}$.

Conjecture 14 ([10]). *If $r \geq 3$ then $\text{ex}(n, \mathcal{C}_4^r) \sim \binom{n}{r-1}$.*

The current record is due to Pikhurko and the last author [21], who showed

$$\text{ex}_r(n, \mathcal{C}_4^r) \lesssim \left(1 + \frac{2}{\sqrt{r}}\right) \binom{n}{r-1}$$

and $\text{ex}_3(n, C_4(2, 1)) \lesssim \frac{13}{9} \binom{n}{2}$. When G is an even cycle of length six or more, it is only known [14] that $\text{ex}_r(n, G(a, b)) = \Theta(n^{r-1})$ and the asymptotic behavior of $\text{ex}_r(n, G(a, b))$ is not known. One can show, however, that for $F = K_{s,t}(a, b)$ with $a + b = r$, $b \leq a$, and t sufficiently large as a function of s and r ,

$$\text{ex}_r(n, F) = \Theta(n^{r-\frac{1}{s}})$$

via a randomized algebraic construction.

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