Extremal theory of locally sparse multigraphs

Dhruv Mubayi * Caroline Terry †

September 1, 2016

Abstract

An (n, s, q)-graph is an n-vertex multigraph where every set of s vertices spans at most q edges. In this paper, we determine the maximum product of the edge multiplicities in (n, s, q)-graphs if the congruence class of q modulo $\binom{s}{2}$ is in a certain interval of length about $3s/2$. The smallest case that falls outside this range is $(s, q) = (4, 15)$, and here the answer is $a^{n^2+o(n^2)}$ where a is transcendental assuming Schanuel's conjecture. This could indicate the difficulty of solving the problem in full generality. Many of our results can be seen as extending work by Bondy-Tuza [2] and Füredi-Kündgen [8] about sums of edge multiplicities to the product setting.

We also prove a variety of other extremal results for (n, s, q)-graphs, including product-stability theorems. These results are of additional interest because they can be used to enumerate and to prove logical 0-1 laws for (n, s, q)-graphs. Our work therefore extends many classical enumerative results in extremal graph theory beginning with the Erdős-Kleitman-Rothschild theorem [6] to multigraphs.

1 Introduction

Given a set X and a positive integer t, let $\binom{X}{t} = \{Y \subseteq X : |Y| = t\}$. A *multigraph* is a pair (V, w), where V is a set of vertices and $w : \binom{V}{2} \to \mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition 1. Given integers $s \geq 2$ and $q \geq 0$, a multigraph (V, w) is an (s, q)*-graph* if for every $X \in \binom{V}{s}$ we have $\sum_{xy \in \binom{X}{2}} w(xy) \leq q$. An (n, s, q)-graph is an (s, q)-graph with n vertices, and $F(n, s, q)$ is the set of (n, s, q)-graphs with vertex set $[n] := \{1, \ldots, n\}$.

The goal of this paper is to investigate extremal, structural, and enumeration problems for (n, s, q)-graphs for a large class of pairs (s, q).

Definition 2. Given a multigraph $G = (V, w)$, define

$$S(G) = \sum_{xy \in \binom{V}{2}} w(xy) \qquad \text{and} \qquad P(G) = \prod_{xy \in \binom{V}{2}} w(xy),$$

$$\mathrm{ex}_{\Sigma}(n, s, q) = \max\{S(G) : G \in F(n, s, q)\} \qquad \text{and} \qquad \mathrm{ex}_{\Pi}(n, s, q) = \max\{P(G) : G \in F(n, s, q)\}.$$

An (n, s, q)-graph G is *sum-extremal* (*product-extremal*) if $S(G) = \mathrm{ex}_{\Sigma}(n, s, q)$ ($P(G) = \mathrm{ex}_{\Pi}(n, s, q)$). Let $\mathcal{S}(n, s, q)$ ($\mathcal{P}(n, s, q)$) be the set of all sum-extremal (product-extremal) (n, s, q)-graphs with vertex set $[n]$.

*Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago. Research supported in part by NSF Grant DMS 1300138; mubayi@uic.edu

†Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago; cterry3@uic.edu

In [2], Bondy and Tuza determine the structure of multigraphs in $\mathcal{S}(n,s,q)$ when n is large compared to s and $q \equiv 0, -1 \pmod{\binom{s}{2}}$ and when $s=3$. In [9], Füredi and Kündgen (among other things) determine the asymptotic value of $\mathrm{ex}_{\Sigma}(n,s,q)$ for all s,q with a $O(n)$ error term, and the exact value is determined for many cases. Other special cases of these questions have appeared in [13]. A natural next step from the investigation of extremal problems for (n,s,q)-graphs is to consider questions of structure and enumeration. The question of enumeration for (n,s,q)-graphs was first addressed in [14], where it was shown the problem is closely related extremal results for the product of the edge multiplicities.

Definition 3. Given integers $s \geq 2$ and $q \geq \binom{s}{2}$, define the *asymptotic product density* and the *asymptotic sum density*, respectively, as the following limits (which both exist):

$$\mathrm{ex}_{\Pi}(s,q) = \lim_{n\to\infty}\Big(\mathrm{ex}_{\Pi}(n,s,q)\Big)^{\frac{1}{\binom{n}{2}}} \qquad \text{and} \qquad \mathrm{ex}_{\Sigma}(s,q) = \lim_{n\to\infty}\frac{\mathrm{ex}_{\Sigma}(n,s,q)}{\binom{n}{2}}.$$

In [14], the current authors showed $\mathrm{ex}_{\Pi}(s,q)$ exists for all $s \geq 2$ and $q \geq 0$ and proved the following enumeration theorem for (n,s,q)-graphs in terms of $\mathrm{ex}_{\Pi}(s,q+\binom{s}{2})$.

Theorem 1. **([14])** *Suppose $s \geq 2$ and $q \geq 0$ are integers. If $\mathrm{ex}_{\Pi}(s,q+\binom{s}{2}) > 1$, then*

$$\mathrm{ex}_{\Pi}\Big(s,q+\binom{s}{2}\Big)^{\binom{n}{2}} \leq |F(n,s,q)| \leq \mathrm{ex}_{\Pi}\Big(s,q+\binom{s}{2}\Big)^{(1+o(1))\binom{n}{2}},$$

and if $\mathrm{ex}_{\Pi}(s,q+\binom{s}{2}) \leq 1$, then $|F(n,s,q)| \leq 2^{o(n^2)}$.

This result was used in [14] along with a computation of $\mathrm{ex}_{\Pi}(4,15)$ to give an enumeration of $F(n,4,9)$. This case was of particular interest because it turned out that $|F(n,4,9)| = a^{n^2+o(n^2)}$, where a is transcendental under the assumption of Schanuel's conjecture. In this paper, we continue this line of investigations by proving enumeration results for further cases of s and q, and in some cases proving approximate structure theorems (the particular special case $(s,q)=(3,4)$ was recently studied in [7]). This generalizes many classical theorems about enumeration in extremal graph theory (beginning with the Erdős-Kleitman-Rothschild theorem [6]) to the multigraph setting. All of these results rely on computing $\mathrm{ex}_{\Pi}(n,s,q)$, characterizing the elements in $\mathcal{P}(n,s,q)$, and proving corresponding product-stability theorems, and this is the main content of this paper. Questions about $\mathrm{ex}_{\Pi}(n,s,q)$ and $\mathcal{P}(n,s,q)$ may also be of independent interest, as they are natural "product versions" of the questions about extremal sums for (n,s,q)-graphs investigated in [2,9].

2 Main Results

Given a multigraph $G=(V,w)$ and $xy \in \binom{V}{2}$, we will refer to $w(xy)$ as the *multiplicity* of xy. The *multiplicty of* G is $\mu(G) = \max\{w(xy) : xy \in \binom{V}{2}\}$. Our first main result, Theorem 2 below, gives us information about the asymptotic properties of elements in $F(n,s,q)$, in the case when $\mathrm{ex}_{\Pi}(s,q+\binom{s}{2}) > 1$. Suppose $G=(V,w)$ and $G'=(V,w')$ are multigraphs. We say that G is a *submultigraph* of G' if $V=V'$ and for each $xy \in \binom{V}{2}$, $w(xy) \leq w'(xy)$. Define $G^+ = (V,w^+)$ where for each $xy \in \binom{V}{2}$, $w^+(xy) = w(xy)+1$. Observe that if $G \in F(n,s,q)$, then $G^+ \in F(n,s,q+\binom{s}{2})$.

Definition 4. Suppose $\epsilon > 0$ and n,s,q are integers satisfying $n \geq 1$, $s \geq 2$, and $q \geq 0$. Set

$$\mathbb{E}(n,s,q,\epsilon) = \Big\{G \in F(n,s,q) : P(G^+) > \mathrm{ex}_{\Pi}\Big(s,q+\binom{s}{2}\Big)^{(1-\epsilon)\binom{n}{2}}\Big\}.$$

Then set $E(n,s,q,\epsilon) = \{G \in F(n,s,q) : G \text{ is a submultigraph of some } G' \in \mathbb{E}(n,s,q,\epsilon)\}$.

Theorem 2. *Suppose $s \geq 2$ and $q \geq 0$ are integers satisfying $\mathrm{ex}_\Pi(s, q + \binom{s}{2}) > 1$. Then for all $\epsilon > 0$, there is $\beta > 0$ such that for all sufficiently large n, the following holds.*

$$\frac{|F(n,s,q) \setminus E(n,s,q,\epsilon)|}{|F(n,s,q)|} \leq 2^{-\beta n^2}. \tag{1}$$

Theorem 2 will be proved in Section 4 using a consequence of a version of the hypergraph containers theorem for multigraphs from [14]. Our next results investigate $\mathrm{ex}_\Pi(n,s,q)$ and $\mathcal{P}(n,s,q)$ for various values of (s,q). Observe that if $q < \binom{s}{2}$, then for any $n \geq s$, every (n,s,q)-graph G must contain an edge of multiplicity 0, and therefore $P(G) = 0$. Consequently, $\mathrm{ex}_\Pi(n,s,q) = 0$ and $\mathcal{P}(n,s,q) = F(n,s,q)$, for all $n \geq s$. For this reason we restrict our attention to the cases where $s \geq 2$ and $q \geq \binom{s}{2}$. Suppose $G = (V,w)$ and $G' = (V',w')$. Then $G = (V,w)$ and $G' = (V',w')$ are *isomorphic*, denoted $G \cong G'$, if there is a bijection $f : V \to V'$ such that for all $xy \in \binom{V}{2}$, $w(xy) = w'(f(x)f(y))$. If $V = V'$, set $\Delta(G,G') = \left\{xy \in \binom{V}{2} : w(xy) \neq w'(xy)\right\}$. Given $\delta > 0$, G and G' are *δ-close* if $|\Delta(G,G')| \leq \delta n^2$, otherwise they are *$\delta$-far*. If $X \subseteq V$, $G[X]$ denotes the multigraph $(X, w\restriction_{\binom{X}{2}})$. Suppose that $q \equiv b \pmod{\binom{s}{2}}$. Our results fall into three cases depending on the value of b.

2.1 The case $0 \leq b \leq s-2$

Definition 5. Given $n \geq s \geq 1$ and $a \geq 1$, let $\mathbb{U}_{s,a}(n)$ be the set of multigraphs $G = ([n], w)$ such that there is a partition $A_0, A_1, \ldots, A_{\lfloor \frac{n}{s} \rfloor}$ of $[n]$ for which the following holds.

- For each $1 \leq i \leq \lfloor n/s \rfloor$, $|A_i| = s$, and $|A_0| = n - s\lfloor n/s \rfloor$.
- For each $0 \leq i \leq \lfloor n/s \rfloor$, $G[A_i]$ comprises a star with $|A_i| - 1$ edges of multiplicity $a+1$ and all other edges of multiplicity a.
- For all $xy \notin \bigcup \binom{A_i}{2}$, $w(xy) = a$.

Let $\mathbb{U}_a(n)$ be the unique element of $\mathbb{U}_{1,a}(n)$, i.e. $\mathbb{U}_a(n) = ([n], w)$ where $w(xy) = a$ for all $xy \in \binom{[n]}{2}$.

Theorem 3. *Suppose n, s, q, a are integers satisfying $n \geq s \geq 2$, $a \geq 1$, and $q = a\binom{s}{2} + b$ for some $0 \leq b \leq s-2$.*

- *(Extremal) Then $a^{\binom{n}{2}} \leq \mathrm{ex}_\Pi(n,s,q) \leq a^{\binom{n}{2}}((a+1)/a)^{\lfloor \frac{b}{b+1} n \rfloor}$ and thus $\mathrm{ex}_\Pi(s,q) = a$. Further,*
  - *(a) If $b = 0$, then $\mathcal{P}(n,s,q) = \{\mathbb{U}_a(n)\}$ and $\mathrm{ex}_\Pi(n,s,q) = a^{\binom{n}{2}}$.*
  - *(b) If $b = s-2$, then $\mathbb{U}_{s-1,a}(n) \subseteq \mathcal{P}(n,s,q)$ and $\mathrm{ex}_\Pi(n,s,q) = a^{\binom{n}{2}}\left(\frac{a+1}{a}\right)^{\lfloor \frac{(s-2)n}{s-1} \rfloor}$. Also, $\mathcal{P}(n,3,q) = \mathbb{U}_{2,a}(n)$.*
- *(Stability) For all $\delta > 0$, there is $\epsilon > 0$ and M such that for all $n > M$ and $G \in F(n,s,q)$, if $P(G) > \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$, then G is δ-close to $\mathbb{U}_a(n)$.*

One interesting phenomenon discovered in [2] is that $\mathcal{S}(n,3,3a+1)$ has many non-isomorphic multigraphs when $a \geq 1$ and n is large. In contrast to this, Theorem 3 shows that all the multigraphs in $\mathcal{P}(n,3,3a+1) = \mathbb{U}_{2,a}(n)$ are isomorphic.

2.2 The case $b = \binom{s}{2} - t$ for some $1 \le t \le \frac{s}{2}$

Call a partition $U_1, \dots, U_k$ of a finite set X an *equipartition* if $||U_i| - |U_j|| \le 1$ for all $i \ne j$. Recall the Turán graph, $T_s(n)$, is the complete s-partite graph with n vertices, whose parts form an equipartition of its vertex set.

Definition 6. Given integers $a \ge 2$ and $n \ge s \ge 1$, define $\mathbb{T}_{s,a}(n)$ to be the set of multigraphs $G = ([n], w)$ with the following property. There is an equipartition $U_1, \dots, U_s$ of $[n]$ such that

$$w(xy) = \begin{cases} a-1 & \text{if } xy \in \binom{U_i}{2} \text{ for some } i \in [s]. \\ a & \text{if } (x,y) \in U_i \times U_j \text{ for some } i \ne j \in [s]. \end{cases}$$

We think of elements of $\mathbb{T}_{s,a}(n)$ as multigraph analogues of Turán graphs. Let $t_s(n)$ be the number of edges in $T_s(n)$.

Theorem 4. *Let s, q, a, t be integers satisfying $a \ge 2$, $q = a\binom{s}{2} - t$ and either*

(a) $s \ge 2$ and $t = 1$ or

(b) $s \ge 4$ and $2 \le t \le \frac{s}{2}$.

- *(Extremal) Then for all $n \ge s$, $\mathbb{T}_{s-t,a}(n) \subseteq \mathcal{P}(n,s,q)$, $\mathrm{ex}_\Pi(n,s,q) = (a-1)^{\binom{n}{2}}(\frac{a}{a-1})^{t_{s-t}(n)}$, and $\mathrm{ex}_\Pi(s,q) = (a-1)(\frac{a}{a-1})^{\frac{s-t-1}{s-t}}$. If (a) holds and $n \ge s$ or (b) holds and n is sufficiently large, then $\mathcal{P}(n,s,q) = \mathbb{T}_{s-t,a}(n)$.*
- *(Stability) For all $\delta > 0$, there is M and ϵ such that for all $n > M$ and $G \in F(n,s,q)$, if $P(G) > \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$ then G is δ-close to an element of $\mathbb{T}_{s-t,a}(n)$.*

2.3 The case $(s,q) = (4,9)$

The case $(s,q) = (4,9)$ is the first pair where $s \ge 2$ and $q \ge \binom{s}{2}$ which is not covered by Theorems 3 and 4, and is closely related to an old question in extremal combinatorics. Let $\mathrm{ex}(n, \{C_3, C_4\})$ denote the maximum number of edges in a graph on n vertices which contains no C_3 or C_4 as a non-induced subgraph.

Theorem 5. $\mathrm{ex}_\Pi(n,4,9) = 2^{\mathrm{ex}(n,\{C_3,C_4\})}$ *for all $n \ge 4$.*

It is known that

$$\left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2} < \mathrm{ex}(n, \{C_3, C_4\}) < \left(\frac{1}{2} + o(1)\right) n^{3/2}$$

and an old conjecture of Erdős and Simonovits [4] states that the lower bound is correct.

The next case not covered here is $(s,q) = (4,15)$ and it was shown in [14] that $\mathrm{ex}_\Pi(n,4,15) = 2^{\gamma n^2 + o(n^2)}$ where γ is transcendental and 2^γ is also transcendental if we assume Schanuel's conjecture from number theory. Many other cases were conjectured in [14] to have transcendental behaviour like the case $(4,15)$. This suggests that determining $\mathrm{ex}_\Pi(s,q)$ for all pairs (s,q) will be a hard problem.

2.4 Enumeration and structure of most (n,s,q)-graphs

Combining the extremal results of Theorems 3, 4, and 5 with Theorem 1 we obtain Theorem 6 below, which enumerates $F(n,s,q)$ for many cases of (s,q).

Theorem 6. *Let s,q,a,b be integers satisfying $s\geq 2$, $a\geq 0$, and $q=a\binom{s}{2}+b$.*

(i) If $0\leq b\leq s-2$, then $|F(n,s,q)|=(a+1)^{\binom{n}{2}}2^{o(n^2)}$.

(ii) If $b=\binom{s}{2}-t$ where $2\leq t\leq \frac{s}{2}$, then $|F(n,s,q)|=(a+1)^{\binom{n}{2}}(\frac{a+2}{a+1})^{t_{s-t}(n)+o(n^2)}$.

(iii) $|F(n,4,3)|=2^{\Theta(n^{3/2})}$.

In our last main result, Theorem 7 below, we combine the stability results of Theorems 3 and 4 with Theorem 2 to prove approximate structure theorems for many (s,q). Given $\delta>0$ and a set $E(n)\subseteq F(n,s,q)$, let $E^{\delta}(n)$ be the set of $G\in F(n,s,q)$ such that G is δ-close to some $G'\in E(n)$.

Definition 7. Suppose n,a,s are integers such that $n,s\geq 1$.

(i) If $a\geq 1$, set $U_a(n)=\{G=([n],w): G$ is a submultigraph of some $G'\in \mathbb{U}_a(n)\}$.

(ii) If $a\geq 2$, set $T_{s,a}(n)=\{G=([n],w): G$ is a submultigraph of some $G'\in \mathbb{T}_{s,a}(n)\}$.

Observe that in each case, $\mathbb{U}_a(n)\subseteq U_a(n)$ and $\mathbb{T}_{s,a}(n)\subseteq T_{s,a}(n)$.

Theorem 7. *Suppose s,q,a,t,b are integers such that $n\geq s\geq 2$, and $E(n)$ is a set of multigraphs such that one of the following holds.*

(i) $a\geq 0$, $q=a\binom{s}{2}+b$ for some $0\leq b\leq s-2$, and $E(n)=U_a(n)$.

(ii) $a\geq 1$, $q=a\binom{s}{2}-t$ for some $1\leq t\leq \frac{s}{2}$, and $E(n)=T_{s-t,a}(n)$.

Then for all $\delta>0$ there exists $\beta>0$ such that for all sufficiently large n,

$$\frac{|F(n,s,q)\setminus E^{\delta}(n)|}{|F(n,s,q)|}\leq 2^{-\beta\binom{n}{2}}. \tag{2}$$

3 Proof of Theorems 6 and 7

In this section we prove Theorems 6 and 7 assuming Theorems 2, 3, and 4.

Proof of Theorem 6. Suppose first that case (i) holds. By Theorem 3 (Extremal),

$$\mathrm{ex}_{\Pi}\Big(s,q+\binom{s}{2}\Big)=\mathrm{ex}_{\Pi}\Big(s,(a+1)\binom{s}{2}+b\Big)=a+1.$$

If $a=0$, then $\mathrm{ex}_{\Pi}(s,q+\binom{s}{2})=1$, so Theorem 1 implies $|F(n,s,q)|=2^{o(n^2)}=(a+1)^{\binom{n}{2}}2^{o(n^2)}$. If $a\geq 1$, then $\mathrm{ex}_{\Pi}(s,q+\binom{s}{2})=a+1>1$, so Theorem 1 implies

$$|F(n,s,q)|=(a+1)^{\binom{n}{2}+o(n^2)}=(a+1)^{\binom{n}{2}}2^{o(n^2)}.$$

Suppose now that case (ii) holds. So $q=a\binom{s}{2}+\binom{s}{2}-t=(a+1)\binom{s}{2}-t$. By Theorem 4 (Extremal),

$$\mathrm{ex}_{\Pi}\Big(s,q+\binom{s}{2}\Big)=\mathrm{ex}_{\Pi}\Big(s,(a+2)\binom{s}{2}-t\Big)=(a+1)\Big(\frac{a+2}{a+1}\Big)^{\frac{s-t-1}{s-t}}.$$

Since $a \geq 0$, this shows $\mathrm{ex}_\Pi(s, q + \binom{s}{2}) > 1$, so Theorem 1 implies

$$|F(n,s,q)| = \Big((a+1)\Big(\frac{a+2}{a+1}\Big)^{\frac{s-t-1}{s-t}}\Big)^{\binom{n}{2}+o(n^2)} = (a+1)^{\binom{n}{2}}\Big(\frac{a+2}{a+1}\Big)^{t_{s-t}(n)+o(n^2)}.$$

For (iii) first observe that any subgraph of a graph of girth at least 5 is a $(4,3)$-graph, and since $\mathrm{ex}(n, \{C_3, C_4\}) \geq c_1 n^{3/2}$ for some constant $c_1 > 0$ (see [4]) we obtain the lower bound. For the upper bound, observe that in a $(4,3)$-graph, there is at most one pair with multiplicity at least two and the set of pairs with multiplicity one forms a graph with no C_4. By the Kleitman-Winston theorem [12], the number of ways to choose the pairs of multiplicity one is at most $2^{c_2 n^{3/2}}$ for some constant $c_2 > 0$ and this gives the upper bound. $\square$

Proof of Theorem 7. Fix $\delta > 0$. Observe that if case (i) holds (respectively, case (ii)), then $(s, q + \binom{s}{2})$ satisfies the hypotheses of Theorem 3 (respectively, Theorem 4). Let

$$\mathbb{E}(n) = \begin{cases} \mathbb{U}_{a+1}(n) & \text{in case (i)} \\ \mathbb{T}_{s-t,a+1}(n) & \text{in case (ii)} \end{cases}$$

By Theorem 3 (Stability) in case (i) and Theorem 4 (Stability) in case(ii), there is $\epsilon > 0$ so that for sufficiently large n, if $G^+ \in F(n, s, q+\binom{s}{2})$ satisfies $P(G^+) > \mathrm{ex}_\Pi(n, s, q+\binom{s}{2})^{1-\epsilon}$, then G^+ is δ-close to some $G' \in \mathbb{E}(n)$. Note that $G' \in \mathbb{E}(n)$ implies there is $H \in E(n)$ such that $H^+ = G$. Combining this our choice of ϵ, we obtain the following. For all sufficienlty large n and $G \in F(n, s, q)$,

$$\text{if } P(G^+) > \mathrm{ex}_\Pi\Big(n, s, q + \binom{s}{2}\Big)^{1-\epsilon}, \text{ then } G^+ \text{ is } \delta\text{-close to } H^+, \text{ for some } H \in E(n). \tag{3}$$

By Theorem 3 (Extremal) in case (i) and Theorem 4 (Extremal) in case(ii), we must have that $\mathrm{ex}_\Pi(s, q + \binom{s}{2}) > 1$. So Theorem 2 implies there is $\beta > 0$ such that for all sufficiently large n the following holds.

$$\frac{|F(n,s,q) \setminus E(n,s,q,\epsilon)|}{|F(n,s,q)|} \leq 2^{-\beta n^2}.$$

So to show (2), it suffices to show that for sufficiently large n, $E(n, s, q, \epsilon) \subseteq E^\delta(n)$. Fix n sufficiently large and suppose $G = ([n], w^G) \in E(n, s, q, \epsilon)$. By definition, this means there is $G' \in F(n, s, q)$ such that $P(G'^+) > \mathrm{ex}_\Pi(n, s, q + \binom{s}{2})^{1-\epsilon}$ and G is a submultigraph of G'. By (3), G'^+ is δ-close to H^+, for some $H \in E(n)$. Define $H' = ([n], w^{H'})$ such that $w^{H'}(xy) = w^G(xy)$ if $xy \in \binom{[n]}{2} \setminus \Delta(G', H)$, and $w^{H'}(xy) = 0$ if $xy \in \Delta(G', H)$. We claim H' is a submultigraph of H. Fix $xy \in \binom{[n]}{2}$. We want to show $w^{H'}(xy) \leq w^H(xy)$. If $xy \in \Delta(G', H)$, then $w^{H'}(xy) = 0 \leq w^H(xy)$ is immediate. If $xy \notin \Delta(G', H)$, then $w^{H'}(xy) = w^G(xy) \leq w^{G'}(xy) = w^H(xy)$, where the inequality is because G is a submultigraph of G' and the last equality is because $xy \notin \Delta(G', H)$. Thus H' is a submultigraph of $H \in E(n)$, which implies H' is also in $E(n)$. By definition of H', $\Delta(G, H') \subseteq \Delta(G', H) = \Delta(G'^+, H^+)$. Since G'^+ and H^+ are δ-close, this implies $|\Delta(G, H')| \leq \delta n^2$, and $G \in E^\delta(n)$. $\square$

4 Proof of Theorem 2

In this section we prove Theorem 2. We will use Theorem 8 below, which is a version of the hypergraph containers theorem of [1, 15] for multigraphs. Theorem 8 was proved in [14].

Definition 8. Suppose $s \geq 2$ and $q \geq 0$ are integers. Set

$$\mathcal{H}(s,q) = \{G = ([s], w) : \mu(G) \leq q \text{ and } S(G) > q\}, \quad \text{and} \quad g(s,q) = |\mathcal{H}(s,q)|.$$

If $G = (V, w)$ is a multigraph, let $\mathcal{H}(G, s, q) = \{X \in \binom{V}{s} : G[X] \cong G' \text{ for some } G' \in \mathcal{H}(s,q)\}$.

Theorem 8. *For every $0 < \delta < 1$ and integers $s \geq 2$, $q \geq 0$, there is a constant $c = c(s, q, \delta) > 0$ such that the following holds. For all sufficiently large n, there is $\mathcal{G}$ a collection of multigraphs of multiplicity at most q and with vertex set $[n]$ such that*

(i) for every $J \in F(n, s, q)$, there is $G \in \mathcal{G}$ such that J is a submultigraph of G,

(ii) for every $G \in \mathcal{G}$, $|\mathcal{H}(G, s, q)| \leq \delta\binom{n}{s}$, and

(iii) $\log |\mathcal{G}| \leq cn^{2-\frac{1}{4s}} \log n$.

We will also use the following two results appearing in [14].

Lemma 1 (Lemma 1 of [14])**.** *Fix integers $s \geq 2$ and $q \geq 0$. For all $0 < \nu < 1$, there is $0 < \delta < 1$ such that for all sufficiently large n, the following holds. If $G = ([n], w)$ satisfies $\mu(G) \leq q$ and $|\mathcal{H}(G, s, q)| \leq \delta\binom{n}{2}$, then G is ν-close to some G' in $F(n, s, q)$.*

Proposition 1 (Proposition 2 in [14])**.** *For all $n \geq s \geq 2$ and $q \geq 0$, $\mathrm{ex}_\Pi(s, q)$ exists and $\mathrm{ex}_\Pi(n, s, q) \geq \mathrm{ex}_\Pi(s, q)^{\binom{n}{2}}$. If $q \geq \binom{s}{2}$, then $\mathrm{ex}_\Pi(s, q) \geq 1$.*

Proof of Theorem 2. Fix $\epsilon > 0$ and set $\nu = (\epsilon \log(\mathrm{ex}_\Pi(s, q + \binom{s}{2})))/(8 \log(q + 1))$. Choose $\delta > 0$ according to Lemma 1 so that the following holds for all sufficiently large n.

$$\text{Any } G = ([n], w) \text{ with } \mu(G) \leq q \text{ and } |\mathcal{H}(G, s, q)| \leq \delta\binom{n}{2} \text{ is } \nu\text{-close to some } G' \text{ in } F(n, s, q). \quad (4)$$

Fix n sufficiently large. Apply Theorem 8 to obtain a constant c and a collection $\mathcal{G}$ of multigraphs of multiplicity at most q and with vertex set $[n]$ satisfying (i)-(iii) of Theorem 8. Suppose that $H = ([n], w^H) \in F(n, s, q) \setminus E(n, s, q, \epsilon)$. By (i), there is $G = ([n], w^G) \in \mathcal{G}$ such that H is a submultigraph of G and $|\mathcal{H}(G, s, q)| \leq \delta\binom{n}{s}$. We claim that $P(G^+) \leq \mathrm{ex}_\Pi(n, s, q + \binom{s}{2})^{1-\epsilon/2}$. Suppose towards a contradiction this is not the case, so $P(G^+) > \mathrm{ex}_\Pi(n, s, q + \binom{s}{2})^{1-\epsilon/2}$. By (4), $|\mathcal{H}(G, s, q)| \leq \delta\binom{n}{s}$ implies there is $G' = ([n], w^{G'}) \in F(n, s, q)$ which is ν-close to G. Define $H' = ([n], w^{H'})$ by setting $w^{H'}(xy) = w^H(xy)$ for all $xy \in \binom{[n]}{2} \setminus \Delta(G, G')$ and $w^{H'}(xy) = 0$ for all $xy \in \Delta(G, G')$. By construction and because H' is a submultigraph of G', we have that H is also a submultigraph of G'. Observe

$$P(G'^+) = P(G^+)\Big(\prod_{xy \in \Delta(H, H')} \frac{w^{G'}(xy) + 1}{w^G(xy) + 1}\Big) \geq P(G^+)(q + 1)^{-|\Delta(G, G')|},$$

where the inequality is because $1 \leq w^{G'}(xy) + 1, w^G(xy) + 1 \leq q + 1$ implies $\frac{w^{G'}(xy)+1}{w^G(xy)+1} \geq \frac{1}{q+1}$. Combining this with the fact that G and G' are ν-close, the definition of ν, and our assumption that $P(G^+) \geq \mathrm{ex}_\Pi(n, s, q + \binom{s}{2})^{1-\epsilon/2}$, we have that $P(G'^+)$ is at least the following.

$$P(G^+)(q+1)^{-\nu n^2} = P(G^+)\mathrm{ex}_\Pi\Big(s, q+\binom{s}{2}\Big)^{-\epsilon n^2/8} \geq \mathrm{ex}_\Pi\Big(n, s, q+\binom{s}{2}\Big)^{1-\epsilon/2}\mathrm{ex}_\Pi\Big(s, q+\binom{s}{2}\Big)^{-\epsilon n^2/8}.$$

Since $\mathrm{ex}_\Pi(n,s,q+\binom{s}{2})^{1/\binom{n}{2}} \geq \mathrm{ex}_\Pi(s,q+\binom{s}{2})$ (see Proposition 1), we obtain that the right hand side is at least

$$\mathrm{ex}_\Pi\Big(n,s,q+\binom{s}{2}\Big)^{1-\epsilon/2}\mathrm{ex}_\Pi\Big(n,s,q+\binom{s}{2}\Big)^{-\epsilon n^2/(8\binom{n}{2})} \geq \mathrm{ex}_\Pi\Big(n,s,q+\binom{s}{2}\Big)^{1-\epsilon},$$

where the inequality is because n large implies $\epsilon n^2/(8\binom{n}{2}) \leq \epsilon/2$. But now H is a submultigraph of G' and $P(G'^+) \geq \mathrm{ex}_\Pi(n,s,q+\binom{s}{2})^{1-\epsilon}$, contradicting that $H \in F(n,s,q) \setminus E(n,s,q,\epsilon)$. Therefore, every element of $F(n,s,q) \setminus E(n,s,q,\epsilon)$ can be constructed as follows.

- Choose some $G \in \mathcal{G}$ with $P(G^+) \leq \mathrm{ex}_\Pi(n,s,q+\binom{s}{2})^{1-\epsilon/2}$. There are at most $cn^{2-\frac{1}{4s}}\log n$ choices. Since n is large and $\mathrm{ex}_\Pi(s,q+\binom{s}{2}) > 1$, we may assume $cn^{2-\frac{1}{4s}}\log n \leq \mathrm{ex}_\Pi(s,q+\binom{s}{2})^{\epsilon\binom{n}{2}/4}$.
- Choose a submultigraph of G. There are at most $P(G^+) \leq \mathrm{ex}_\Pi(n,s,q+\binom{s}{2})^{1-\epsilon/2}$ choices.

This shows

$$\begin{aligned}|F(n,s,q) \setminus E(n,s,q,\epsilon)| &\leq \mathrm{ex}_\Pi\Big(s,q+\binom{s}{2}\Big)^{\epsilon\binom{n}{2}/4}\mathrm{ex}_\Pi\Big(n,s,q+\binom{s}{2}\Big)^{1-\epsilon/2}\\ &\leq \mathrm{ex}_\Pi\Big(s,q+\binom{s}{2}\Big)^{-\epsilon\binom{n}{2}/4}\mathrm{ex}_\Pi\Big(n,s,q+\binom{s}{2}\Big),\end{aligned}$$

where the second inequality is because $\mathrm{ex}_\Pi(n,s,q+\binom{s}{2}) \geq \mathrm{ex}_\Pi(s,q+\binom{s}{2})^{\binom{n}{2}}$. By Theorem 1, $|F(n,s,q)| \geq \mathrm{ex}_\Pi(n,s,q)$, so this implies

$$\frac{|F(n,s,q) \setminus E(n,s,q,\epsilon)|}{|F(n,s,q)|} \leq \mathrm{ex}_\Pi\Big(s,q+\binom{s}{2}\Big)^{-\epsilon\binom{n}{2}/4}.$$

Setting $\beta = \frac{\epsilon}{4}\log_2(\mathrm{ex}_\Pi(s,q+\binom{s}{2})$ finishes the proof (note $\beta > 0$ since $\mathrm{ex}_\Pi(s,q+\binom{s}{2}) > 1$). □

5 Extremal Results

In this section we prove the extremal statements in Theorems 3 and 4. We begin with some preliminaries. Suppose $s \geq 2$ and $q \geq \binom{s}{2}$. It was shown in [9] that $\mathrm{ex}_\Sigma(s,q)$ exists, and the AM-GM inequality implies that

$$\mathrm{ex}_\Pi(s,q) = \lim_{n\to\infty}\mathrm{ex}_\Pi(n,s,q)^{1/\binom{n}{2}} \leq \lim_{n\to\infty}\frac{\mathrm{ex}_\Sigma(n,s,q)}{\binom{n}{2}} = \mathrm{ex}_\Sigma(s,q). \tag{5}$$

The following lemma is an integer version of the AM-GM inequality.

Lemma 2. *If $\ell \geq 2$, $k \in [\ell]$ and $a, x_1, \ldots, x_\ell$ are positive integers such that $\sum_{i=1}^{\ell} x_i \leq a\ell - k$, then $\prod_{i=1}^{\ell} x_i \leq a^{\ell-k}(a-1)^k$. Moreover, equality holds if and only if exactly k of the x_i are equal to $a-1$ and the rest are equal to a.*

Proof. If there are x_i and x_j with $x_i < x_j - 1$, then replacing x_i with $x_i + 1$ and replacing x_j with $x_j - 1$ increases the product and keeps the sum unchanged. So no two of the x_i's differ by more than one when the product is maximized. □

Corollary 1. *Let $n \geq s \geq 2$, $a \geq 2$, and $(a-1)\binom{s}{2} \leq q < a\binom{s}{2}$. Suppose $G \in \mathcal{S}(n,s,q)$ has all edge multiplicities in $\{a, a-1\}$ and contains exactly k edges of multiplicity $a-1$. Then for all other $G' \in F(n,s,q)$, $G' \in \mathcal{P}(n,s,q)$ if and only if G' has k edges of multiplicity $a-1$ and all other edges of multiplicity a. Consequently, $G \in \mathcal{P}(n,s,q) \subseteq \mathcal{S}(n,s,q)$.*

Proof. Fix G so that the hypotheses hold. Then $S(G) = a\binom{n}{2} - k$ and $P(G) = a^{\binom{n}{2}-k}(a-1)^k$. Let $G' = ([n], w)$ be another element of $F(n,s,q)$. Since $G \in \mathcal{S}(n,s,q)$, we have

$$S(G') \le S(G) = a\binom{n}{2} - k.$$

By Lemma 2 with $\ell = \binom{n}{2}$, $P(G') \le a^{\binom{n}{2}-k}(a-1)^k$ with equality if and only if $\{w(xy) : xy \in \binom{[n]}{2}\}$ consists of k elements equal to $a-1$ and the rest equal to a. This shows $G' \in \mathcal{P}(n,s,q)$ if and only if G' has k edges of multiplicity $a-1$ and the rest of multiplicity a. Consequently, $G \in \mathcal{P}(n,s,q)$. To show $\mathcal{P}(n,s,q) \subseteq \mathcal{S}(n,s,q)$, let $G' \in \mathcal{P}(n,s,q)$. Then by what we have shown, $S(G') = a\binom{n}{2} - k = S(G)$, so $G \in \mathcal{S}(n,s,q)$ implies $G' \in \mathcal{S}(n,s,q)$. □

The following is a consequence of Theorem 5.2 in [2] (case $b = 0$) and Theorems 8 and 9 in [9] (cases $0 < b \le s-2$).

Theorem 9 (Bondy-Tuza [2], Füredi-Kündgen [9]). *Let $n \ge s \ge 2$, $a \ge 1$, $0 \le b \le s-2$, and $q = a\binom{s}{2} + b$. Then*

$$\mathrm{ex}_{\Sigma}(n,s,q) \le a\binom{n}{2} + \left\lfloor \frac{b}{b+1} n \right\rfloor.$$

with equality holding when $b = s-2$ and when $b = 0$.

Proof of Theorem 3 (Extremal). Since $\mathbb{U}_a(n) \in F(n,s,q)$, $a^{\binom{n}{2}} \le \mathrm{ex}_{\Pi}(n,s,q)$. On the other hand, let $G \in F(n,s,q)$. Theorem 9 implies that $S(G) \le a\binom{n}{2} + \lfloor \frac{b}{b+1} n \rfloor$. This along with Lemma 2 implies that $P(G) \le a^{\binom{n}{2}}((a+1)/a)^{\lfloor \frac{b}{b+1} n \rfloor}$. Thus $a^{\binom{n}{2}} \le \mathrm{ex}_{\Pi}(n,s,q) \le a^{\binom{n}{2}}((a+1)/a)^{\lfloor \frac{b}{b+1} n \rfloor}$, which implies $\mathrm{ex}_{\Pi}(s,q) = a$.

Case (a): If $b = 0$, then Theorem 9 implies $\mathbb{U}_a(n) \in \mathcal{S}(n,s,q)$. Because $\mathbb{U}_a(n)$ has all edge multiplicities in $\{a\}$, Corollary 1 implies $\mathbb{U}_a(n) \in \mathcal{P}(n,s,q)$ and moreover, every other element of $\mathcal{P}(n,s,q)$ has all edges of multiplicity a. In other words, $\{\mathbb{U}_a(n)\} = \mathcal{P}(n,s,q)$, so $\mathrm{ex}_{\Pi}(n,s,q) = a^{\binom{n}{2}}$.

Case (b): If $b = s-2$, then it is straightforward to check $\mathbb{U}_{s-1,a}(n) \subseteq F(n,s,q)$. Since $S(G) = a\binom{n}{2} + \lfloor \frac{s-2}{s-1} n \rfloor$ for all $G \in U_{s-1,a}(n)$, Theorem 9 implies $\mathbb{U}_{s-1,a}(n) \subseteq \mathcal{S}(n,s,q)$. Because every element in $\mathbb{U}_{s-1,a}(n)$ has all edge multiplicities in $\{a+1, a\}$, Corollary 1 implies $\mathbb{U}_{s-1,a}(n) \subseteq \mathcal{P}(n,s,q)$ and every $G' \in \mathcal{P}(n,s,q)$ contains exactly $\lfloor \frac{s-2}{s-1} n \rfloor$ edges of multiplicity $a+1$, and all others of multiplicity a. Thus $\mathrm{ex}_{\Pi}(n,s,q) = a^{\binom{n}{2}}(\frac{a+1}{a})^{\lfloor \frac{s-2}{s-1} \rfloor}$. Suppose $s = 3$, $b = 1$, and $G' = ([n], w) \in \mathcal{P}(n,s,q)$. If there are $x, y \ne z \in [n]$ such that $w(xy) = w(xz) = a+1$, then because G' contains only edges of multiplicity $a+1$ and a, $S(\{x,y,z\}) \ge 2(a+1) + a = 3a+2 > q$, a contradiction. Thus the edges of multiplicity $a+1$ form a matching of size $\lfloor \frac{n}{2} \rfloor$ in G', so $G' \in \mathbb{U}_{s-1,a}(n)$. This shows $\mathbb{U}_{s-1,a}(n) = \mathcal{P}(n,s,q)$. □

The following is a consequence of Theorem 5.2 of [2].

Theorem 10 (Bondy-Tuza [2]). *Suppose $n \ge s \ge 2$, $a \ge 1$, and $q = a\binom{s}{2} - 1$. Then*

$$\mathrm{ex}_{\Sigma}(n,s,q) = (a-1)\binom{n}{2} + t_{s-1}(n).$$

Proof of Theorem 4(a) (Extremal). Since $\mathbb{T}_{s-1,a}(n) \subseteq F(n,s,q)$ and for all $G \in \mathbb{T}_{s-1,a}(n)$, $S(G) = (a-1)\binom{n}{2} + t_{s-1}(n)$, Theorem 10 implies that $\mathbb{T}_{s-1,a}(n) \subseteq \mathcal{S}(n,s,q)$. Therefore Corollary 1 implies $\mathbb{T}_{s-1,a}(n) \subseteq \mathcal{P}(n,s,q)$ and each $G \in \mathcal{P}(n,s,q)$ has $t_{s-1}(n)$ edges of multiplicity a and the rest of multiplicity $a-1$. Fix $G=([n],w)\in\mathcal{P}(n,s,q)$ and let G' be the graph with vertex set $[n]$ and edge set $E=\{xy\in\binom{[n]}{2}: w(xy)=a\}$. Then G' is K_s-free and has $t_{s-1}(n)$ edges, so by Turán's theorem, $G'=T_{s-1}(n)$ and thus $G\in\mathbb{T}_{s-1,a}(n)$. So we have shown, $\mathcal{P}(n,s,q)=\mathbb{T}_{s-1,a}(n)$. Consequently, $\mathrm{ex}_\Pi(n,s,q)=(a-1)^{\binom{n}{2}}(\frac{a}{a-1})^{t_{s-1}(n)}$ and $\mathrm{ex}_\Pi(s,q)=(a-1)(\frac{a}{a-1})^{\frac{s-2}{s-1}}$. □

To prove Theorem 4(b) (Extremal), we will need the following theorem, as well as a few lemmas.

Theorem 11. [Dirac [3], Bondy-Tuza [2]] *Let $n\geq s\geq 4$, $a\geq 1$, and $q=a\binom{s}{2}-t$ for some $2\leq t\leq\frac{s}{2}$. Then $\mathrm{ex}_\Sigma(n,s,q)=\mathrm{ex}_\Sigma(n,s',q')$ where $s'=s-t+1$ and $q'=a\binom{s'}{2}-1$.*

Proof. Let $n\geq s\geq 4$ and $2\leq t\leq s/2$. In [3], Dirac proved that $\mathrm{ex}_\Sigma(n,s,\binom{s}{2}-t)=t_{s-t}(n)$. This along with Lemma 5.1 in [2] implies that for all $a\geq 1$,

$$\mathrm{ex}_\Sigma(n,s,a\binom{s}{2}-t)=\mathrm{ex}_\Sigma(n,s,\binom{s}{2}-t)+(a-1)\binom{n}{2}=t_{s-t}(n)+(a-1)\binom{n}{2}=\mathrm{ex}_\Sigma(n,s',a\binom{s'}{2}-1),$$

where the last equality is by Theorem 10 applied to s' and $a\binom{s'}{2}-1$. □

Lemma 3. *If s,q,a,t are integers satisfying case (b) of Theorem 4, and $s'=s-t+1$, $q'=a\binom{s'}{2}-1$, then for all $n\geq s$, $\mathbb{T}_{s'-1}(n)\subseteq\mathcal{P}(n,s,q)$ and $\mathrm{ex}_\Pi(n,s,q)=\mathrm{ex}_\Pi(n,s',q')$.*

Proof. Set $s'=s-t+1$ and $q'=a\binom{s'}{2}-1$, and fix $n\geq s$. Fix $G\in\mathbb{T}_{s'-1,a}(n)$. It is straightforward to check that $G\in F(n,s,q)$. By Theorem 11, $\mathrm{ex}_\Sigma(n,s',q')=\mathrm{ex}_\Sigma(n,s,q)$. Since $S(G)=(a-1)\binom{n}{2}+t_{s'-1}(n)$, by Theorem 10 applied to s' and q', we have that $S(G)=\mathrm{ex}_\Sigma(n,s',q')=\mathrm{ex}_\Sigma(n,s,q)$. This shows $G\in\mathcal{S}(n,s,q)$. By Corollary 1, since G has all edge multiplicities in $\{a,a-1\}$, $G\in\mathcal{P}(n,s,q)$, so $P(G)=\mathrm{ex}_\Pi(n,s,q)$. Since $G\in\mathbb{T}_{s'-1,a}(n)$ and $\mathbb{T}_{s'-1,a}(n)\subseteq\mathcal{P}(n,s',q')$ by Theorem 4(a) (Extremal), $P(G)=\mathrm{ex}_\Pi(n,s',q')$. Thus $\mathrm{ex}_\Pi(n,s,q)=P(G)=\mathrm{ex}_\Pi(n,s',q')$. □

We now fix some notation. Given $n\in\mathbb{N}$, $z\in[n]$, $Y\subseteq[n]$, and $G=([n],w)$, set

$$S(Y)=\sum_{xy\in\binom{Y}{2}}w(xy),\quad S_z(Y)=\sum_{y\in Y}w(yz),\quad P(Y)=\prod_{xy\in\binom{Y}{2}}w(xy),\quad\text{and}\quad P_z(Y)=\prod_{y\in Y}w(yz)$$

If $X\subseteq[n]$ is disjoint from Y, set $P(X,Y)=\prod_{x\in X,y\in Y}w(xy)$.

Claim 1. *Suppose s,q,a,t are integers satisfying the hypotheses of case (b) of Theorem 4. Then for all $n\geq 2s$ and $s-t+1\leq y\leq s-1$,*

$$\mathrm{ex}_\Pi(n-y,s,q)\leq\mathrm{ex}_\Pi(n,s,q)(a-1)^{-\binom{y}{2}}\Big(a^{y-2}(a-1)^2\Big)^{-(n-y)}\Big(\frac{a-1}{a}\Big)^{\frac{n-y}{s-t}}.$$

Proof. Set $s'=s-t+1$ and $q'=a\binom{s'}{2}-1$. Fix $n\geq s$ and $s'\leq y\leq s-1$. Choose some $H=([n-y],w)\in\mathbb{T}_{s'-1,a}(n-y)$ and let $U_1,\dots,U_{s'-1}$ be the partition of $[n-y]$ corresponding to H. Observe that there is some i such that $|U_i|\geq\frac{n-y}{s'-1}$. Without loss of generality, assume $|U_1|\geq\frac{n-y}{s'-1}$. Assign the elements of $Y':=[n]\setminus[n-y]$ to the U_i in as even a way as possible, to obtain an equipartition $U_1',\dots,U_{s'-1}'$ of $[n]$ extending $U_1,\dots,U_{s'-1}$. Observe that because $s'\leq|Y'|\leq s-1$ and $s'-1=s-t\geq s/2$, for each i, $|U_i'\setminus U_i|\in\{1,2\}$, and there is at least one i such that $|U_i'\setminus U_i|=1$. Since $|U_1|\geq\frac{n-y}{s-t}$, by redistributing Y' if necessary, we may assume that $|U_1'\setminus U_1|=1$. Define a new multigraph $H'=([n],w')$ so that $w'(xy)=a-1$ if $xy\in\binom{U_i'}{2}$ for some $i\in[s'-1]$ and $w'(xy)=a$ if $(x,y)\in U_i'\times U_j'$ for some $i\neq j$. Note that by construction $H'\in\mathbb{T}_{s'-1,a}(n)$ and

$H'[[n-y]] = H$. By Lemma 3, since $n-y \geq s$, $H \in \mathbb{T}_{s'-1,a}(n-y)$ and $H' \in \mathbb{T}_{s'-1,a}(n)$ imply $H \in \mathcal{P}(n-y,s,q)$ and $H' \in \mathcal{P}(n,s,q)$. These facts imply the following.

$$\mathrm{ex}_\Pi(n,s,q) = P(H') = P(H)P(Y')P(Y',[n-y]) = \mathrm{ex}_\Pi(n-y,s,q)P(Y')P(Y',[n-y]). \quad (6)$$

By definition of H', if $|U_i' \setminus U_i| = 2$, then for all $z \in U_i$, $P_z(Y') = a^{y-2}(a-1)^2$ and if $|U_i' \setminus U_i| = 1$, then for all $z \in U_i$, $P_z(Y') = a^{y-1}(a-1)$. Since $|U_1' \setminus U_1| = 1$, this implies

$$P(Y',[n-y]) \geq \left(a^{y-2}(a-1)^2\right)^{n-y-|U_1|}\left(a^{y-1}(a-1)\right)^{|U_1|} = \left(a^{y-2}(a-1)^2\right)^{n-y}\left(\frac{a}{a-1}\right)^{|U_1|}. \quad (7)$$

By construction, $P(Y') \geq (a-1)^{\binom{y}{2}}$. Combining this with (6), (7), and the fact that $|U_1| \geq \frac{n-y}{s-t}$, we obtain

$$\mathrm{ex}_\Pi(n,s,q) \geq \mathrm{ex}_\Pi(n-y,s,q)(a-1)^{\binom{y}{2}}\left(a^{y-2}(a-1)^2\right)^{n-y}\left(\frac{a}{a-1}\right)^{\frac{n-y}{s-t}}.$$

Rearranging this yields $\mathrm{ex}_\Pi(n-y,s,q) \leq \mathrm{ex}_\Pi(n,s,q)(a-1)^{-\binom{y}{2}}(a^{y-2}(a-1)^2)^{-(n-y)}(\frac{a-1}{a})^{\frac{n-y}{s-t}}$. $\square$

Lemma 4. *Let $n \geq s \geq 4$, $a \geq 2$, and $q = a\binom{s}{2} - t$ for some $2 \leq t \leq \frac{s}{2}$. Suppose $G \in F(n,s,q)$ and $Y \in \binom{[n]}{s-t+1}$ satisfies $S(Y) \geq a\binom{s-t+1}{2}$. Then there is $Y \subseteq Y' \subseteq [n]$ such that $s-t+1 \leq |Y'| \leq s-1$ and for all $z \in [n] \setminus Y'$, $S_z(Y') \leq a|Y'|-2$, and consequently, $P_z(Y') \leq a^{|Y'|-2}(a-1)^2$.*

Proof. Suppose towards a contradiction that $Y \in \binom{[n]}{s-t+1}$ satisfies $S(Y) \geq a\binom{s-t+1}{2}$ but for all $Y \subseteq Y' \subseteq [n]$ such that $s-t+1 \leq |Y'| \leq s-1$, there is $z \in [n] \setminus Y'$ with $S_z(Y') > a|Y'|-2$. Apply this fact with $Y' = Y$ to choose $z_1 \in [n] \setminus Y$ such that $S_{z_1}(Y) > a|Y|-2$. Then inductively define a sequence $z_2, \ldots, z_{t-1}$ so that for each $1 \leq i \leq t-2$, $S_{z_{i+1}}(Y \cup \{z_1,\ldots,z_i\}) \geq a(s-t+1+i)-1$ (to define z_{i+1}, apply the fact with $Y' = Y \cup \{z_1,\ldots,z_i\}$). Then $|Y \cup \{z_1,\ldots,z_{t-1}\}| = s$ and

$$\begin{aligned} S(Y \cup \{z_1,\ldots,z_{t-1}\}) &\geq S(Y) + S_{z_1}(Y) + S_{z_2}(Y \cup \{z_1\}) + \ldots + S_{z_{t-1}}(Y \cup \{z_1,\ldots,z_{t-2}\}) \\ &\geq a\binom{s-t+1}{2} + a(s-t+1) - 1 + \ldots + a(s-1) - 1 \\ &= a\binom{s}{2} - (t-1) > a\binom{s}{2} - t, \end{aligned}$$

contradicting that $G \in F(n,s,q)$. Therefore there is $Y \subseteq Y' \subseteq [n]$ such that $s-t+1 \leq |Y'| \leq s-1$ and for all $z \in [n] \setminus Y'$, $S_z(Y') \leq a|Y'|-2$. By Lemma 2, this implies $P_z(Y') \leq a^{|Y'|-2}(a-1)^2$. $\square$

Lemma 5. *Suppose s,q,a,t are integers satisfying the hypotheses of case (b) of Theorem 4. Then there are constants $C > 1$ and $0 < \alpha < 1$ such that for all $n \geq 1$ the following holds. Suppose $G \in F(n,s,q)$ and $k(G)$ is the maximal number of pairwise disjoint elements of $\{Y \in \binom{[n]}{s-t+1} : S(G[Y]) \geq a\binom{s-t+1}{2}\}$. Then*

$$P(G) \leq C^{k(G)}\alpha^{k(G)n}\mathrm{ex}_\Pi(n,s,q). \quad (8)$$

Proof. Set $\alpha = (\frac{a-1}{a})^{\frac{1}{2t(s-t)}}$. Choose $C \geq q^{\binom{s-1}{2}}$ sufficiently large so that $\mathrm{ex}_\Pi(n,s,q) \leq C\alpha^{n^2}$ holds for all $1 \leq n \leq s^3$. We proceed by induction on n. If $1 \leq n \leq s^3$ and $G \in F(n,s,q)$, then (8) is clearly true of $k(G) = 0$. If $k(G) \geq 1$, then by choice of C and since $k(G) \leq n$ and $\alpha < 1$,

$$P(G) \leq \mathrm{ex}_\Pi(n,s,q) \leq C\alpha^{n^2} \leq C\alpha^{k(G)n} \leq C^{k(G)}\alpha^{k(G)n}\mathrm{ex}_\Pi(n,s,q).$$

Now let $n > s^3$ and suppose by induction (8) holds for all $G' \in F(n', s, q)$ where $1 \le n' < n$. If $G \in F(n, s, q)$, then (8) is clearly true if $k(G) = 0$. If $k(G) > 0$, let $Y_1, \dots, Y_k$ be a maximal set of pairwise disjoint elements in $\{Y \in \binom{[n]}{s-t+1} : S(G[Y]) \ge a^{\binom{s-t+1}{2}}\}$. Apply Lemma 4 to find Y' such that $Y_1 \subseteq Y' \subseteq [n]$, $s-t+1 \le |Y'| \le s-1$, and for all $z \in [n] \setminus Y'$, $P_z(Y') \le a^{|Y'|-2}(a-1)^2$. Let $|Y'| = y$. Then note

$$P(Y', [n] \setminus Y') = \prod_{z \in [n] \setminus Y'} P_z(Y') \le \Big(a^{y-2}(a-1)^2\Big)^{n-y}. \tag{9}$$

Observe that $G[[n] \setminus Y']$ is isomorphic to some $H \in F(n-y, s, q)$. Since Y' can intersect at most $t-2$ other Y_i, and since $Y_1, \dots, Y_k$ was maximal, we must have $k(H) + 1 \le k(G) \le k(H) + t - 1$. By our induction hypothesis,

$$P([n] \setminus Y') = P(H) \le C^{k(H)} \alpha^{k(H)(n-y)} \mathrm{ex}_\Pi(n-y, s, q). \tag{10}$$

Since $\mu(G) \le q$ and $y \le s-1$, and by our choice of C, $P(Y') \le q^{\binom{y}{2}} \le C$. Combining this with (9), (10) and the fact that $\mu(H) \le \mu(G)$ we obtain that

$$\begin{aligned} P(G) = P([n] \setminus Y')P(Y', [n] \setminus Y')P(Y') &\le C^{k(H)} \alpha^{k(H)(n-y)} \mathrm{ex}_\Pi(n-y, s, q)\Big(a^{y-2}(a-1)^2\Big)^{n-y} C \\ &= C^{k(H)+1} \alpha^{k(H)(n-y)} \mathrm{ex}_\Pi(n-y, s, q)\Big(a^{y-2}(a-1)^2\Big)^{n-y}. \end{aligned}$$

Plugging in the upper bound for $\mathrm{ex}_\Pi(n-y, s, q)$ from Claim 1 yields that $P(G)$ is at most

$$C^{k(H)+1} \alpha^{k(H)(n-y)} \mathrm{ex}_\Pi(n, s, q)(a-1)^{-\binom{y}{2}} \Big(\frac{a-1}{a}\Big)^{\frac{n-y}{s-t}} \le C^{k(H)+1} \alpha^{k(H)(n-y)+2t(n-y)} \mathrm{ex}_\Pi(n, s, q), \tag{11}$$

where the last inequality is because $(a-1)^{-\binom{y}{2}} < 1$ and by definition of α, $(\frac{a-1}{a})^{1/(s-t)} = \alpha^{2t}$. We claim that the following holds.

$$k(H)(n-y) + 2t(n-y) \ge (k(H) + t - 1)n. \tag{12}$$

Rearranging this, we see (12) is equivalent to $yk(H) \le tn + n - 2ty$. Since $2 \le t \le s/2$ and $y \le s-1$, $tn + n - 2ty \ge 3n - s(s-1)$, so it suffices to show $yk(H) \le 3n - s(s-1)$. By definition, $k(H) \le \frac{n-y}{s-t+1}$ so $yk(H) \le \frac{y(n-y)}{s-t+1}$. Combining this with the facts that $s-t+1 \le y \le s-1$ and $s/2 < s-t+1$ yields

$$yk(H) \le \frac{(s-1)(n-(s-t+1))}{s-t+1} = n\Big(\frac{s-1}{s-t+1}\Big) - s + 1 < 2n\Big(\frac{s-1}{s}\Big) - s + 1.$$

Thus it suffices to check $2n(\frac{s-1}{s}) - s + 1 \le 3n - s(s-1)$. This is equivalent to $(s-1)^2 \le n(\frac{s+2}{s})$, which holds because $n \ge s^3$. This finishes the verification of (12). Combining (11), (12), and the fact that $k(H) + 1 \le k(G) \le k(H) + t - 1$ yields

$$P(G) \le C^{k(H)+1} \alpha^{(k(H)+t-1)n} \mathrm{ex}_\Pi(n, s, q) \le C^{k(G)} \alpha^{k(G)n} \mathrm{ex}_\Pi(n, s, q).$$

$\square$

Proof of Theorem 4(b) (Extremal). Set $s' = s - t + 1$ and $q' = a\binom{s'}{2} - 1$. Fix $n \geq s$. By Lemma 3 and definition of s', $\mathbb{T}_{s-t,a}(n) = \mathbb{T}_{s'-1,a}(n) \subseteq \mathcal{P}(n, s, q)$ and

$$\mathrm{ex}_\Pi(n, s, q) = \mathrm{ex}_\Pi(n, s', q') = (a-1)^{\binom{n}{2}}(\frac{a}{a-1})^{t_{s'-1}(n)},$$

where the last equality is by Theorem 4(a) (Extremal) applied to s' and q'. By definition, we have $\mathrm{ex}_\Pi(s, q) = (a-1)(\frac{a}{a-1})^{1-\frac{1}{s'-1}}$. We have left to show that $\mathcal{P}(n, s, q) \subseteq \mathbb{T}_{s'-1,a}(n)$ holds for large n. Assume n is sufficiently large and C and α are as in Lemma 5. Note $\mathrm{ex}_\Pi(n, s, q) = \mathrm{ex}_\Pi(n, s', q')$ implies $\mathcal{P}(n, s, q) \cap F(n, s', q') \subseteq \mathcal{P}(n, s', q') = \mathbb{T}_{s'-1,a}(n)$, where the equality is by Theorem 4 (a) (Extremal). So it suffices to show $\mathcal{P}(n, s, q) \subseteq F(n, s', q')$. Suppose towards a contradiction that there exists $G = ([n], w) \in \mathcal{P}(n, s, q) \setminus F(n, s', q')$. Then in the notation of Lemma 5, $k(G) \geq 1$. Combining this with Lemma 5, we have

$$P(G) \leq C^{k(G)}\alpha^{k(G)n}\mathrm{ex}_\Pi(n, s, q) = \Big(C\alpha^n\Big)^{k(G)}\mathrm{ex}_\Pi(n, s, q) < \mathrm{ex}_\Pi(n, s, q),$$

where the last inequality is because n is large, $\alpha < 1$, and $k(G) \geq 1$. But now $P(G) < \mathrm{ex}_\Pi(n, s, q)$ contradicts that $G \in \mathcal{P}(n, s, q)$. □

6 Stability

In this section we prove the product-stability results for Theorems 3 and 4(a). We will use the fact that for any (s, q)-graph G, $\mu(G) \leq q$. If $G = (V, w)$ and $a \in \mathbb{N}$, let $E_a(G) = \{xy \in \binom{V}{2} : w(xy) = a\}$ and $e_a(G) = |E_a(G)|$. In the following notation, p stands for "plus" and m stands for "minus."

$$p_a(G) = |\{xy \in \binom{V}{2} : w(xy) > a\}| \qquad \text{and} \qquad m_a(G) = |\{xy \in \binom{V}{2} : w(xy) < a\}|.$$

Lemma 6. *Let $s \geq 2$, $q \geq \binom{s}{2}$ and $a > 0$. Suppose there exist $0 < \alpha < 1$ and $C > 1$ such that for all $n \geq s$, every $G \in F(n, s, q)$ satisfies*

$$P(G) \leq \mathrm{ex}_\Pi(n, s, q)q^{Cn}\alpha^{p_a(G)}.$$

Then for all $\delta > 0$ there are $\epsilon, M > 0$ such that for all $n > M$ the following holds. If $G \in F(n, s, q)$ and $P(G) \geq \mathrm{ex}_\Pi(n, s, q)^{1-\epsilon}$ then $p_a(G) \leq \delta n^2$.

Proof. Fix $\delta > 0$. Choose $\epsilon > 0$ so that $\frac{2\epsilon \log q}{\log(1/\alpha)} = \delta$. Choose $M \geq s$ sufficiently large so that $n \geq M$ implies $(\epsilon n^2 + Cn)\log q \leq 2\epsilon \log q n^2$. Let $n > M$ and $G \in F(n, s, q)$ be such that $P(G) \geq \mathrm{ex}_\Pi(n, s, q)^{1-\epsilon}$. Our assumptions imply

$$\mathrm{ex}_\Pi(n, s, q)^{1-\epsilon} \leq P(G) \leq \mathrm{ex}_\Pi(n, s, q)q^{Cn}\alpha^{p_a(G)}.$$

Rearranging $\mathrm{ex}_\Pi(n, s, q)^{1-\epsilon} \leq \mathrm{ex}_\Pi(n, s, q)q^{Cn}\alpha^{p_a(G)}$ yields $\Big(\frac{1}{\alpha}\Big)^{p_a(G)} \leq \mathrm{ex}_\Pi(n, s, q)^\epsilon q^{Cn} \leq q^{\epsilon n^2 + Cn}$, where the second inequality is because $\mathrm{ex}_\Pi(n, s, q) \leq q^{n^2}$. Taking logs of both sides, we obtain

$$p_a(G)\log(1/\alpha) \leq (\epsilon n^2 + Cn)\log q \leq 2\epsilon n^2 \log q,$$

where the second inequality is by assumption on n. Dividing both sides by $\log(1/\alpha)$ and applying the definition of ϵ yields $p_a(G) \leq \frac{2\epsilon n^2 \log q}{\log(1/\alpha)} = \delta n^2$. □

We now prove the key lemma for this section.

Lemma 7. *Let s, q, b, a be integers satisfying $s \geq 2$ and either*

(i) $a \geq 1$, $0 \leq b \leq s-2$, *and* $q = a\binom{s}{2} + b$ *or*

(ii) $a \geq 2$, $b = 0$, *and* $q = a\binom{s}{2} - 1$.

Then there exist $0 < \alpha < 1$ *and* $C > 1$ *such that for all* $n \geq s$ *and all* $G \in F(n, s, q)$,

$$P(G) \leq \mathrm{ex}_{\Pi}(n, s, q) q^{Cn} \alpha^{p_a(G)}. \tag{13}$$

Proof. We prove this by induction on $s \geq 2$, and for each fixed s, by induction on n. Let $s \geq 2$ and q, b, a be as in (i) or (ii) above. Set

$$\xi = \begin{cases} 0 & \text{if case (i) holds.} \\ 1 & \text{if case (ii) holds.} \end{cases}$$

Suppose first $s = 2$. Set $\alpha = 1/2$ and $C = 2$. Since G is an $(n, 2, a-\xi)$-graph, $p_a(G) = 0$. Therefore for all $n \geq 2$,

$$P(G) \leq \mathrm{ex}_{\Pi}(n, s, q) \leq \mathrm{ex}_{\Pi}(n, s, q) q^{Cn} = \mathrm{ex}_{\Pi}(n, s, q) q^{Cn} \alpha^{p_a(G)}.$$

Assume now $s > 2$. Let $\mathcal{I}$ be the set of $(s', q', b') \in \mathbb{N}^3$ such that $2 \leq s' < s$ and s', q', b', a satisfy (i) or (ii). Observe that $\mathcal{I}$ is finite. Suppose by induction on s that $(s', q', b') \in \mathcal{I}$ implies there are $0 < \alpha(s', q', b') < 1$ and $C(s', q', b') > 1$ such that for all $n' \geq s'$ and $G' \in F(n', s', q')$, $P(G) \leq \mathrm{ex}_{\Pi}(n, s', q') q^{C(s',q',b')n} \alpha(s', q', b')^{p_a(G)}$. Set

$$\alpha = \max\Big(\Big\{q^{-1}, \Big(\frac{a^{s-2}(a-\xi)-1}{a^{s-2}(a-\xi)}\Big)^{\frac{1}{s-2}}, \Big(\frac{a-1}{a}\Big)^{\frac{1}{s-2}}\Big\} \cup \Big\{\alpha(s', q', b') : (s', q', b') \in \mathcal{I}\Big\}\Big).$$

Observe $0 < \alpha < 1$. Choose $C \geq \binom{s-1}{2}$ sufficiently large so that for all $n \leq s$

$$q^{\binom{n}{2}} \leq q^{Cn} (a-\xi)^{\binom{n}{2}} \Big(\frac{a}{a-\xi}\Big)^{t_{s-1}(n)} \alpha^{\binom{n}{2}}, \tag{14}$$

and so that for all $(s', q', b') \in \mathcal{I}$, $C(s', q', b') \leq C/2$ and $(\frac{a+1}{a})^{(s-3)/(s-2)} \leq q^{C/2}$. Given $G \in F(n, s, q)$, set

$$\Theta(G) = \Big\{Y \subseteq \binom{[n]}{s-1} : S(Y) \geq a\binom{s-1}{2} + (1-\xi)b\Big\},$$

and let $A(n, s, q) = \{G \in F(n, s, q) : \Theta(G) \neq \emptyset\}$. We show the following holds for all $n \geq 1$ and $G \in F(n, s, q)$ by induction on n.

$$P(G) \leq q^{Cn} (a-\xi)^{\binom{n}{2}} \Big(\frac{a}{a-\xi}\Big)^{t_{s-1}(n)} \alpha^{p_a(G)}. \tag{15}$$

This will finish the proof since $(a-\xi)^{\binom{n}{2}} (\frac{a}{a-\xi})^{t_{s-1}(n)} \leq \mathrm{ex}_{\Pi}(n, s, q)$ (by Theorem 3 (Extremal) for case (i) and Theorem 4(a) (Extremal) for case (ii)). If $n \leq s$ and $G \in F(n, s, q)$, then (15) holds because of (14) and the fact that $P(G) \leq q^{\binom{n}{2}}$. So assume $n > s$, and suppose by induction that

(15) holds for all $s \le n' < n$ and $G' \in F(n', s, q)$. Let $G = ([n], w) \in F(n, s, q)$. Suppose first that $G \in A(n, s, q)$. Choose $Y \in \Theta(G)$ and set $R = [n] \setminus Y$. Given $z \in R$, note that

$$a\binom{s-1}{2} + (1-\xi)b + S_z(Y) \le S(Y) + S_z(Y) = S(Y \cup \{z\}) \le a\binom{s}{2} + (1-\xi)b - \xi,$$

and therefore $S_z(Y) \le a(s-1) - \xi$. Then for all $z \in R$, Lemma 2 implies $P_z(Y) \le a^{s-2}(a-\xi)$, with equality only if $\{w(yz) : y \in Y\}$ consists of $s-1-\xi$ elements equal to a and ξ elements equal to $a-1$. Let $R_1 = \{z \in R : \exists y \in Y, w(zy) > a\}$ and $R_2 = R \setminus R_1$. Then $z \in R_1$ implies $P_z(Y) < a^{s-2}(a-\xi)$, so $P_z(Y) \le a^{s-2}(a-\xi) - 1$. Let $k = |R_1|$. Observe that $G[R]$ is isomorphic to an element of $F(n', s, q)$, where $n' = n - |R| \ge 1$. By induction (on n) and these observations we have that the following holds, where $p_a(R) = p_a(G[R])$.

$$\begin{aligned}
P(G) &= P(R)P(Y) \prod_{z \in R_1} P_z(Y) \prod_{z \in R_2} P_z(Y) \\
&\le q^{C(n-s+1)} (a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)} q^{\binom{s-1}{2}} \left(a^{s-2}(a-\xi) - 1\right)^k \left(a^{s-2}(a-\xi)\right)^{n-s+1-k} \\
&\le q^{C(n-s+2)} (a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)} \left(a^{s-2}(a-\xi) - 1\right)^k \left(a^{s-2}(a-\xi)\right)^{n-s+1-k},
\end{aligned}$$

where the second inequality is because $\binom{s-1}{2} \le C$. Since $\alpha \ge \left(\frac{a^{s-2}(a-\xi)-1}{a^{s-2}(a-\xi)}\right)^{1/(s-2)}$, this is at most

$$q^{C(n-s+2)} (a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)+k(s-1)} \left(a^{s-2}(a-\xi)\right)^{n-s+1}. \tag{16}$$

Because $C(n-s+2) \le Cn - \binom{s-1}{2}$ and $q^{-1} \le \alpha$, we have $q^{C(n-s+2)} \le q^{Cn} \alpha^{\binom{s-1}{2}}$. Combining this with the fact that $p_a(G) \le p_a(R) + k(s-1) + \binom{s-1}{2}$ implies that (16) is at most

$$\begin{aligned}
&q^{Cn} (a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)+k(s-1)+\binom{s-1}{2}} \left(a^{s-2}(a-\xi)\right)^{n-s+1} \\
=&q^{Cn} (a-\xi)^{\binom{n-s+1}{2}+(s-1)(n-s+1)} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)+(s-2)(n-s+1)} \alpha^{p_a(R)+k(s-1)+\binom{s-1}{2}} \\
\le&q^{Cn} (a-\xi)^{\binom{n}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n)} \alpha^{p_a(G)}.
\end{aligned}$$

We now have that $P(G) \le q^{Cn}(a-\xi)^{\binom{n}{2}}(\frac{a}{a-\xi})^{t_{s-1}(n)}\alpha^{p_a(G)}$, as desired. Assume now $G \notin A(n, s, q)$. Then for all $Y \in \binom{[n]}{s-1}$, $S(Y) \le a\binom{s-1}{2} + (1-\xi)b - 1$. Thus G is an (n, s', q')-graph where $s' = s-1$ and $q' = a\binom{s-1}{2} + (1-\xi)b - 1$. Suppose $a = 1$, $\xi = 0$, and $b = 0$. Then $q' = \binom{s'}{2} - 1$ and any (n, s', q')-graph must contain an edge of multiplicity 0. This implies $P(G) = 0$ and (15) holds. We have the following three cases remaining, where $b' = \max\{b-1, 0\}$.

1. $\xi = 0$, $b = 0$, and $a \ge 2$. In this case $q' = a\binom{s'}{2} - 1$ and $b' = 0$.

2. $\xi = 1$, $b = 0$, and $a \ge 2$. In this case $q' = a\binom{s'}{2} - 1$ and $b' = 0$.

3. $\xi = 0$, $1 \le b \le s-2$, and $a \ge 1$. In this case $q' = a\binom{s'}{2} + b'$ and $0 \le b' \le s'-2$.

It is clear that in all three of these cases, $(s', q', b') \in \mathcal{I}$, so by our induction hypothesis (on s), there are $\alpha' = \alpha(s', q', b') \leq \alpha$ and $C' = C(s', q', b)$ such that

$$P(G) \leq \mathrm{ex}_\Pi(n, s', q')(q')^{C'n}(\alpha')^{p_a(G)} \leq \mathrm{ex}_\Pi(n, s', q')q^{C'n}\alpha^{p_a(G)}, \tag{17}$$

where the inequality is because $q' \leq q$ and $\alpha' \leq \alpha$. By Theorem 4(a) (Extremal) if cases 1 or 2 hold, and by Theorem 3 (Extremal) if case 3 holds, we have the following.

$$\mathrm{ex}_\Pi(n, s', q') \leq (a-\xi)^{\binom{n}{2}}\Big(\frac{a}{a-\xi}\Big)^{t_{s'-1}(n)}\Big(\frac{a+1}{a}\Big)^{\lfloor \frac{b'}{b'+1}n\rfloor} \leq (a-\xi)^{\binom{n}{2}}\Big(\frac{a}{a-\xi}\Big)^{t_{s-1}(n)}\Big(\frac{a+1}{a}\Big)^{\frac{s-3}{s-2}n},$$

where the last inequality is because $t_{s'-1}(n) \leq t_{s-1}(n)$ and $\lfloor \frac{b'}{b'+1}n\rfloor \leq \frac{b'}{b'+1}n \leq \frac{s-3}{s-2}n$. By choice of C, $(\frac{a+1}{a})^{\frac{s-3}{s-2}n} \leq q^{Cn/2}$. Thus $\mathrm{ex}_\Pi(n, s', q') \leq (a-\xi)^{\binom{n}{2}}(\frac{a}{a-\xi})^{t_{s-1}(n)}q^{Cn/2}$. Combining this with (17) implies

$$P(G) \leq (a-\xi)^{\binom{n}{2}}\Big(\frac{a}{a-\xi}\Big)^{t_{s-1}(n)}q^{Cn/2}q^{C'n}\alpha^{p_a(G)} \leq (a-\xi)^{\binom{n}{2}}\Big(\frac{a}{a-\xi}\Big)^{t_{s-1}(n)}q^{Cn}\alpha^{p_a(G)},$$

where the last inequality is because $C' \leq C/2$. Thus (15) holds. □

Proof of Theorem 3 (Stability). Let $s \geq 2$, $a \geq 1$, and $q = a\binom{s}{2} + b$ for some $0 \leq b \leq s-2$. Fix $\delta > 0$. Given $G \in F(n, s, q)$, let $p_G = p_a(G)$ and $m_G = m_a(G)$. Note that if $G \in F(n, s, q)$, then $|\Delta(G, \mathbb{U}_a(n))| = m_G + p_G$. Suppose first $a = 1$, so $m_G = 0$. Combining Lemma 7 with Lemma 6 implies there are ϵ_1 and M_1 such that if $n > M_1$ and $G \in F(n, s, q)$ satisfies $P(G) \geq \mathrm{ex}_\Pi(n, s, q)^{1-\epsilon_1}$, then $|\Delta(G, \mathbb{U}_a(n))| = p_G \leq \delta n^2$. Assume now $a > 1$. Combining Lemma 7 with Lemma 6 implies there are ϵ_1 and M_1 such that if $n > M_1$ and $G \in F(n, s, q)$ satisfies $P(G) \geq \mathrm{ex}_\Pi(n, s, q)^{1-\epsilon_1}$, then $p_G \leq \delta' n^2$, where

$$\delta' = \min\Big\{\frac{\delta}{2}, \frac{\delta \log(a/(a-1))}{4\log q}\Big\}.$$

Set $\epsilon = \min\{\epsilon_1, \frac{\delta\log(a/(a-1))}{4\log q}\}$. Suppose $n > M_1$ and $G \in F(n, s, q)$ satisfies $P(G) \geq \mathrm{ex}_\Pi(n, s, q)^{1-\epsilon}$. Our assumptions imply $p_G \leq \delta' n^2 \leq \delta n^2/2$. Observe that by definition of p_G and m_G,

$$P(G) \leq a^{\binom{n}{2}-m_G}(a-1)^{m_G}q^{p_G} = a^{\binom{n}{2}}\Big(\frac{a-1}{a}\Big)^{m_G}q^{p_G}. \tag{18}$$

By Theorem 3(a)(Extremal), $\mathrm{ex}_\Pi(n, s, q) \geq a^{\binom{n}{2}}$. Therefore $P(G) \geq \mathrm{ex}_\Pi(n, s, q)^{1-\epsilon} \geq a^{\binom{n}{2}(1-\epsilon)}$. Combining this with (18) yields

$$a^{\binom{n}{2}(1-\epsilon)} \leq a^{\binom{n}{2}}\Big(\frac{a-1}{a}\Big)^{m_G}q^{p_G}.$$

Rearranging this, we obtain

$$\Big(\frac{a}{a-1}\Big)^{m_G} \leq a^{\epsilon\binom{n}{2}}q^{p_G} \leq q^{\epsilon\binom{n}{2}+p_G} \leq q^{\epsilon n^2+p_G}.$$

Taking logs, dividing by $\log(a/(a-1))$, and applying our assumptions on p_G and ϵ yields

$$m_G \leq \frac{\epsilon n^2\log q}{\log(a/(a-1))} + \frac{p_G\log q}{\log(a/(a-1)} \leq \frac{\delta n^2}{4} + \frac{\delta n^2}{4} = \frac{\delta n^2}{2}.$$

Combining this with the fact that $p_G \leq \frac{\delta n^2}{2}$ we have that $|\Delta(G, \mathbb{U}_a(n))| \leq \delta n^2$. □

The following classical result gives structural information about n-vertex K_s-free graphs with close to $t_{s-1}(n)$ edges.

Theorem 12 (Erdős-Simonovits [5, 16]). *For all $\delta > 0$ and $s \geq 2$, there is an $\epsilon > 0$ such that every K_s-free graph with n vertices and $t_{s-1}(n) - \epsilon n^2$ edges can be transformed into $T_{s-1}(n)$ by adding and removing at most δn^2 edges.*

Proof of Theorem 4(a) (Stability). Let $s \geq 2$, $a \geq 2$, and $q = a\binom{s}{2} - 1$. Fix $\delta > 0$. Given $G \in F(n,s,q)$, let $p_G = p_a(G)$, $m_G = m_{a-1}(G)$. Choose M_0 and μ such that $\mu < \delta/2$ and so that Theorem 12 implies that any K_s-free graph with $n \geq M_0$ vertices and at least $(1-\mu)t_{s-1}(n)$ edges can be made into $T_{s-1}(n)$ by adding or removing at most $\frac{\delta n^2}{3}$ edges. Set

$$A = \begin{cases} 2 & \text{if } a = 2 \\ \frac{a-1}{a-2} & \text{if } a > 2 \end{cases}$$

Combining Lemma 7 with Lemma 6 implies there are ϵ_1, M_1 so that if $n > M_1$ and $G \in F(n,s,q)$ satisfies $P(G) \geq \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon_1}$, then $p_G \leq \delta' n^2$, where

$$\delta' = \min\Big\{\frac{\delta}{3}, \frac{\mu \log(a/(a-1))}{2\log q}, \frac{\delta \log A}{6 \log q}\Big\}. \tag{19}$$

Let

$$\epsilon = \min\Big\{\epsilon_1, \frac{\delta \log A}{6\log q}, \frac{\mu\log(a/(a-1))}{2\log q}\Big\} \qquad \text{and} \qquad M = \max\{M_0, M_1\}.$$

Suppose now that $n > M$ and $G \in F(n,s,q)$ satisfies $P(G) \geq \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$. By assumption, $p_G \leq \delta' n^2 \leq \frac{\delta n^2}{3}$. We now bound m_G. Note that if $a = 2$ and $P(G) \neq 0$, then $m_G = 0$. If $a > 2$, observe that by definition of p_G and m_G,

$$P(G) \leq q^{p_G}(a-2)^{m_G} a^{e_a(G)}(a-1)^{e_{a-1}(G)} \leq q^{p_G}\left(\frac{a-2}{a-1}\right)^{m_G} a^{e_a(G)}(a-1)^{\binom{n}{2}-e_a(G)}, \tag{20}$$

where the last inequality is because $e_{a-1}(G) + m_G \leq \binom{n}{2} - e_a(G)$. Note that Turán's theorem and the fact that G is an (n,s,q)-graph implies that $e_a(G) \leq t_{s-1}(n)$, so

$$a^{e_a(G)}(a-1)^{\binom{n}{2}-e_a(G)} \leq a^{t_{s-1}(n)}(a-1)^{\binom{n}{2}-t_{s-1}(n)} = \mathrm{ex}_\Pi(n,s,q),$$

where the last equality is from Theorem 4(a) (Extremal). Combining this with (20) yields

$$\mathrm{ex}_\Pi(n,s,q)^{1-\epsilon} \leq P(G) \leq q^{p_G}\left(\frac{a-2}{a-1}\right)^{m_G}\mathrm{ex}_\Pi(n,s,q).$$

Rearranging $\mathrm{ex}_\Pi(n,s,q)^{1-\epsilon} \leq q^{p_G}(\frac{a-2}{a-1})^{m_G}\mathrm{ex}_\Pi(n,s,q)$ and using that $\mathrm{ex}_\Pi(n,s,q) \leq q^{n^2}$, we obtain

$$A^{m_G} = \left(\frac{a-1}{a-2}\right)^{m_G} \leq q^{p_G}\mathrm{ex}_\Pi(n,s,q)^{\epsilon} \leq q^{p_G + \epsilon n^2}.$$

Taking logs, dividing by $\log A$, and applying our assumptions on p_G and ϵ we obtain $m_G < \delta n^2/3$. Using (20) and $a^{t_{s-1}(n)}(a-1)^{\binom{n}{2}-t_{s-1}(n)} = \mathrm{ex}_\Pi(n,s,q)$, we have

$$\mathrm{ex}_\Pi(n,s,q)^{1-\epsilon} \leq P(G) \leq q^{p_G} a^{e_a(G)}(a-1)^{\binom{n}{2}-e_a(G)} = q^{p_G}\mathrm{ex}_\Pi(n,s,q)\left(\frac{a}{a-1}\right)^{e_a(G)-t_{s-1}(n)}.$$

Rearranging this we obtain

$$\left(\frac{a}{a-1}\right)^{t_{s-1}(n)-e_a(G)} \leq q^{p_G}\mathrm{ex}_\Pi(n,s,q)^{\epsilon} \leq q^{p_G+\epsilon n^2}.$$

Taking logs, dividing by $\log(a/(a-1))$, and using the assumptions on p_G and ϵ we obtain that

$$t_{s-1}(n) - e_a(G) \le \frac{p_G \log q}{\log(a/(a-1))} + \frac{\epsilon n^2 \log q}{\log(a/(a-1))} \le \frac{\mu n^2}{2} + \frac{\mu n^2}{2} = \mu n^2.$$

Let H be the graph with vertex set $[n]$ and edge set $E = E_a(G)$. Then H is K_s-free, and has $e_a(G)$ many edges. Since $t_{s-1}(n) - e_a(G) \le \mu n^2$, Theorem 12 implies that H is $\frac{\delta}{3}$-close to some $H' = T_{s-1}(n)$. Define $G' \in F(n,s,q)$ so that $E_a(G') = E(H')$ and $E_{a-1}(G') = \binom{n}{2} \setminus E_a(G')$. Then $G' \in \mathbb{T}_{s-1,a}(n)$ and

$$\Delta(G,G') \subseteq (E_a(G)\Delta E_a(G')) \cup \bigcup_{i \notin \{a,a-1\}} E_i(G) = \Delta(H,H') \cup \bigcup_{i \notin \{a,a-1\}} E_i(G).$$

This implies $|\Delta(G,G')| \le |\Delta(H,H')| + p_G + m_G \le \frac{\delta}{3}n^2 + \frac{\delta}{3}n^2 + \frac{\delta}{3}n^2 = \delta n^2$. $\square$

6.1 Proof of Theorem 4(b) (Stability)

In this subsection we prove Theorem 4(b) (Stability). We first prove two lemmas.

Lemma 8. *Let $s \ge 4$, $a \ge 2$, and $q = a\binom{s}{2} - t$ for some $2 \le t \le \frac{s}{2}$. For all $\lambda > 0$ there are M and $\epsilon > 0$ such that the following holds. Suppose $n > M$ and $G \in F(n,s,q)$ satisfies $P(G) > \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$. Then $k(G) < \lambda n$, where $k(G)$ is as defined in Lemma 5.*

Proof. Fix $\lambda > 0$. Set $\eta = a^{\frac{s-t-1}{s-t}}(a-1)^{\frac{1}{s-t}}$ and choose C and α as in Lemma 5. Choose $\epsilon > 0$ so that $\alpha^{\lambda/2} = \eta^{-\epsilon}$. By Theorem 4(b) (Extremal), $\mathrm{ex}_\Pi(n,s,q) = \eta^{\binom{n}{2}+o(n^2)}$. Assume M sufficiently large so that for all $n \ge M$, (4) holds for all $G \in F(n,s,q)$, $\mathrm{ex}_\Pi(n,s,q) < \eta^{n^2}$, $C^{\lambda n} \le \eta^{\epsilon n^2}$, and $C\alpha^n < 1$. Fix $n \ge M$ and suppose towards a contradiction that $G \in F(n,s,q)$ satisfies $P(G) > \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$ and $k(G) \ge \lambda n$. By Lemma 5 and the facts that $C\alpha^n < 1$ and $k(G) \ge 1$, we obtain that

$$P(G) \le C^{k(G)}\alpha^{nk(G)}\mathrm{ex}_\Pi(n,s,q) = (C\alpha^n)^{k(G)}\mathrm{ex}_\Pi(n,s,q) \le (C\alpha^n)^{\lambda n}\mathrm{ex}_\Pi(n,s,q).$$

By assumption on n and definition of ϵ, $(C\alpha^n)^{\epsilon n} = C^{\lambda n}\alpha^{\lambda n^2} = C^{\lambda n}\eta^{-2\epsilon n^2} \le \eta^{-\epsilon n^2}$. Thus

$$P(G) \le \eta^{-\epsilon n^2}\mathrm{ex}_\Pi(n,s,q) < \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon},$$

where the last inequality is because by assumption, $\mathrm{ex}_\Pi(n,s,q) < \eta^{n^2}$. But this contradicts our assumption that $P(G) > \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$. $\square$

Given a multigraph $G = (V,w)$, let $\mathcal{H}(G,s,q) = \{Y \in \binom{V}{s} : S(Y) > q\}$. Observe that G is an (s,q)-graph if and only if $\mathcal{H}(G,s,q) = \emptyset$.

Lemma 9. *Let $s,q,m \ge 2$ be integers. For all $0 < \delta < 1$, there is $0 < \lambda < 1$ and N such that $n > N$ implies the following. If $G = ([n],w)$ has $\mu(G) \le m$ and $\mathcal{H}(G,s,q)$ contains strictly less than $\lceil \lambda n \rceil$ pairwise disjoint elements, then G is δ-close to an element in $F(n,s,q)$.*

Proof. Fix $0 < \delta < 1$. Observe we can view any multigraph G with $\mu(G) \le m$ as an edge-colored graph with colors in $\{0,\dots,m\}$. By Theorem 1, there is ϵ and M such that if $n > M$ and $G = ([n],w)$ has $\mu(G) \le m$ and $\mathcal{H}(G,s,q) \le \epsilon\binom{n}{s}$, then G is δ-close to an element of $F(n,s,q)$. Let $\lambda := \epsilon/s$ and $N = \max\{M, \frac{s}{1-\lambda s}\}$. We claim this λ and N satisfy the desired conclusions. Suppose towards a contradiction that $n > M$ and $G = ([n],w)$ has $\mu(G) \le m$, $\mathcal{H}(n,s,q)$ contains strictly less than $\lceil \lambda n \rceil$ pairwise disjoint elements, but G is δ-far from every element in $F(n,s,q)$. Then $\mathcal{H}(G,s,q) > \epsilon\binom{n}{s}$ by choice of M and λ. By our choice of N, $\lceil \lambda n \rceil s \le (\lambda n+1)s \le n$. Then Proposition 11.6 in [11] and our assumptions imply $|\mathcal{H}(G,s,q)| \le (\lceil \lambda n \rceil - 1)\binom{n-1}{s-1}$. But now

$$|\mathcal{H}(G,s,q)| \le (\lceil \lambda n \rceil - 1)\binom{n-1}{s-1} < \lambda n \binom{n-1}{s-1} = \left(\frac{\epsilon n}{s}\right)\left(\frac{s}{n}\right)\binom{n}{s} = \epsilon \binom{n}{s},$$

a contradiction. □

Proof of Theorem 4(b) (Stability). Let $s \ge 4$, $a \ge 2$, and $q = a\binom{s}{2} - t$ for some $2 \le t \le \frac{s}{2}$. Fix $\delta > 0$. Let $s' = s-t+1$ and $q' = a\binom{s'}{2} - 1$. Note Theorem 4 (Extremal) implies that for sufficiently large n, $\mathcal{P}(n,s,q) = \mathbb{T}_{s'-1,a}(n)$, $\mathrm{ex}_\Pi(n,s',q') = \mathrm{ex}_\Pi(n,s,q)$, and $\mathrm{ex}_\Pi(s',q') = \mathrm{ex}_\Pi(s,q) = \eta$, where $\eta = (a-1)(\frac{a}{a-1})^{(s'-2)/(s'-1)}$.

Apply Theorem 4 (a) (Stability) for (s',q') to $\delta/2$ to obtain ϵ_0. By replacing ϵ_0 if necessary, assume $\epsilon_0 < 4\delta/\log\eta$. Set $\epsilon_1 = \epsilon_0 \log\eta/(8\log q)$ and note $\epsilon_1 < \delta/2$. Apply Lemma 9 to ϵ_1 and $m = q$ to obtain λ such that for large n the following holds. If $G = ([n], w)$ has $\mu(G) \le q$ and $\mathcal{H}(G,s',q')$ contains strictly less than $\lceil \lambda n \rceil$ pairwise disjoint elements, then G is ϵ_1-close to an element in $F(n,s',q')$. Finally, apply Lemma 8 for s,q,t to λ to obtain $\epsilon_2 > 0$.

Choose M sufficiently large for the desired applications of Theorems 4(a) (Stability) and 4(b) (Extremal) and Lemmas 8 and 9. Set $\epsilon = \min\{\epsilon_2, \epsilon_0/2\}$. Suppose $n > M$ and $G \in F(n,s,q)$ satisfies $P(G) \ge \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$. Then Lemma 8 and our choice of ϵ implies $k(G) < \lambda n$. Observe that by the definitions of s', q',

$$\left\{Y \in \binom{[n]}{s-t+1} : S(Y) \ge a\binom{s-t+1}{2}\right\} = \left\{Y \in \binom{[n]}{s'} : S(Y) \ge q'+1\right\} = \mathcal{H}(G,s',q').$$

Thus $k(G) < \lambda n$ means $\mathcal{H}(G,s',q')$ contains strictly less than $\lceil \lambda n \rceil$ pairwise disjoint elements. Lemma 9 then implies G is ϵ_1-close to some $G' \in F(n,s',q')$. Combining this with the definition of ϵ_1 yields

$$P(G') \ge P(G) q^{-|\Delta(G,G')|} \ge P(G) q^{-\epsilon_1 n^2} = P(G)\eta^{-\epsilon_0 n^2/8} \ge \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}\eta^{-(\epsilon_0/2)\binom{n}{2}}. \tag{21}$$

By Proposition 1, $\mathrm{ex}_\Pi(n,s,q) \ge \mathrm{ex}_\Pi(s,q)^{\binom{n}{2}} = \eta^{\binom{n}{2}}$. Combining this with (21) and the definition of ϵ yields

$$P(G') \ge \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}\eta^{-(\epsilon_0/2)\binom{n}{2}} \ge \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon-\epsilon_0/2} \ge \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon_0}. \tag{22}$$

Since $\mathrm{ex}_\Pi(n,s,q) = \mathrm{ex}_\Pi(n,s',q')$, (22) implies $P(G') \ge \mathrm{ex}_\Pi(n,s',q')^{1-\epsilon_0}$, so Theorem 4(a) (Stability) implies G' is $\delta/2$-close to some $G'' \in \mathbb{T}_{s'-1,a}(n) = \mathbb{T}_{s-t,a}(n)$. Now we are done, since

$$|\Delta(G,G'')| \le |\Delta(G,G')| + |\Delta(G',G'')| \le \epsilon_1 n^2 + \delta n^2/2 \le \delta n^2.$$

□

7 Extremal Result for $(n,4,9)$-graphs

In this section we prove Theorems 5. We first prove one of the inequalities needed for Theorem 5.

Lemma 10. *For all $n \ge 4$, $2^{\mathrm{ex}(n,\{C_3,C_4\})} \le \mathrm{ex}_\Pi(n,4,9)$.*

Proof. Fix $G = ([n], E)$ an extremal $\{C_3, C_4\}$-free graph, and let $G' = ([n], w)$ where $w(xy) = 2$ for all $xy \in E$ and $w(xy) = 1$ for all $xy \in \binom{n}{2} \setminus E$. Suppose $X \in \binom{[n]}{4}$. Since G is $\{C_3, C_4\}$-free, $|E \cap \binom{X}{2}| \leq 3$. Thus $\{w(xy) : xy \in \binom{X}{2}\}$ contains at most 3 elements equal to 2 and the rest equal to 1, so $S(X) \leq 9$. This shows $G' \in F(n, 4, 9)$. Thus $2^{|E|} = 2^{\mathrm{ex}(n,\{C_3,C_4\})} = P(G') \leq \mathrm{ex}_\Pi(n, 4, 9)$. ☐

To prove the reverse inequality, our strategy will be to show that if $G \in F(n, 4, 9)$ has no edges of multiplicity larger than 2, then $P(G) \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$ (Theorem 13). We will then show that all product extremal $(4, 9)$-graphs have no edges of multiplicity larger than 2 (Theorem 14). Theorem 5 will then follow. We begin with a few definitions and lemmas.

Definition 9. Suppose $n \geq 1$. Set $F_{\leq 2}(n, 4, 9) = \{G \in F(n, 4, 9) : \mu(G) \leq 2\}$ and

$$D(n) = F_{\leq 2}(n, 4, 9) \cap F(n, 3, 5).$$

Lemma 11. *For all $n \geq 4$, if $G = ([n], w) \in D(n)$, then $P(G) \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$.*

Proof. If $P(G) = 0$ we are done, so assume $P(G) > 0$. Let $H = ([n], E)$ be the graph where $E = \{xy \in \binom{[n]}{2} : w(xy) = 2\}$. Since $P(G) > 0$ and $\mu(G) \leq 2$, G contains all edges of multiplicity 1 or 2. Consequently, $P(G) = 2^{|E|}$. Since $G \in F(n, 3, 5)$, H is C_3-free and since $G \in F(n, 4, 9)$, H is C_4-free, so $|E| \leq \mathrm{ex}(n, \{C_3, C_4\})$. This shows $P(G) = 2^{|E|} \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$. ☐

The following lemma gives us useful information about elements of $F(n, 4, 9) \setminus F(n, 3, 5)$.

Lemma 12. *Suppose $n \geq 4$ and $G = ([n], w) \in F(n, 4, 9)$ satisfies $P(G) > 0$. If there is $X \in \binom{[n]}{3}$ such that $S(X) \geq 6$, then $P(X) \leq 2^3$ and $w(xy) = 1$ for all $x \in X$ and $y \in [n] \setminus X$. Consequently*

$$P(G) = P(X)P([n] \setminus X) \leq 2^3 P([n] \setminus X).$$

Proof. Let $y \in [n] \setminus X$. Since $P(G) > 0$, every edge in G has multiplicity at least 1, so $S_y(X) \geq 3$. Thus

$$3 + S(X) \leq S_y(X) + S(X) = S(X \cup \{y\}) \leq 9,$$

which implies $S(X) \leq 6$. By Lemma 2, this implies $P(X) \leq 2^3$. By assumption, $S(X) \geq 6$, so we have $6 + S_y(X) \leq S(X) + S_y(X) = S(X \cup \{y\}) \leq 9$, which implies $S_y(X) \leq 3$. Since every edge in G has multiplicity at least 1 and $|X| = 3$, we must have $w(yx) = 1$ for all $x \in X$. Therefore $P(G) = P([n] \setminus X)P(X) \leq P([n] \setminus X)2^3$. ☐

Fact 1. *For all $n \geq 4$ and $1 \leq i < n$, $\mathrm{ex}(n, \{C_3, C_4\}) \geq \mathrm{ex}(n - i, \{C_3, C_4\}) + i$.*

Proof. Suppose $n \geq 4$ and $1 \leq i < n$. Fix $G = ([n - i], E)$ an extremal $\{C_3, C_4\}$-free graph. Let $G' = ([n], E')$ where $E' = E \cup \{n1, (n - 1)1, \ldots, (n - i + 1)1\}$. Then G' is $\{C_3, C_4\}$-free graph because $G = G'[n - i]$ is $\{C_3, C_4\}$-free and because the elements of $[n] \setminus [n - i]$ all have degree 1 in G'. Therefore $\mathrm{ex}(n, \{C_3, C_4\}) \geq \mathrm{ex}(n - i, \{C_3, C_4\}) + |E' \setminus E| = \mathrm{ex}(n - i, \{C_3, C_4\}) + i$. ☐

We now prove Theorem 13. We will use that $\mathrm{ex}(4, \{C_3, C_4\}) = 3$, $\mathrm{ex}(5, \{C_3, C_4\}) = 5$, and $\mathrm{ex}(6, \{C_3, C_4\}) = 6$ (see [10]).

Theorem 13. *For all $n \geq 4$ and $G \in F_{\leq 2}(n, 4, 9)$, $P(G) \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$.*

Proof. We proceed by induction on n. Assume first $4 \leq n \leq 6$ and $G \in F_{\leq 2}(n, 4, 9)$. If $P(G) = 0$ then we are done. If $G \in D(n)$, then we are done by Lemma 11. So assume $P(G) > 0$ and $G \in F_{\leq 2}(n, 4, 9) \setminus D(n)$. By definition of $D(n)$ this means $G \notin F(n, 3, 5)$, so there is $X \in \binom{[n]}{3}$ such that $S(X) \geq 6$. By Lemma 12, this implies $P(G) \leq P([n] \setminus X)2^3 \leq 2^{\binom{n-3}{2}+3}$, where the second inequality is because $\mu(G) \leq 2$. The explicit values for $\mathrm{ex}(n, \{C_3, C_4\})$ show that for $n \in \{4, 5, 6\}$, $2^{\binom{n-3}{2}+3} \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$. Consequently, $P(G) \leq 2^{\binom{n-3}{2}+3} \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$.

Suppose now $n \geq 7$ and assume by induction that for all $4 \leq n' < n$ and $G' \in F_{\leq 2}(n', 4, 9)$, $P(G') \leq 2^{\mathrm{ex}(n',4,9)}$. Fix $G \in F_{\leq 2}(n, 4, 9)$. If $P(G) = 0$ then we are done. If $G \in D(n)$, then we are done by Lemma 11. So assume $P(G) > 0$ and $G \in F_{\leq 2}(n, 4, 9) \setminus D(n)$. By definition of $D(n)$ this means $G \notin F(n, 3, 5)$, so there is $X \in \binom{[n]}{3}$ such that $S(X) \geq 6$. By Lemma 12, this implies $P(G) \leq P([n] \setminus X)2^3$. Clearly there is $H \in F_{\leq 2}(n-3, 4, 9)$ such that $G[[n] \setminus X] \cong H$. By our induction hypothesis applied to H, $P([n] \setminus X) = P(H) \leq 2^{\mathrm{ex}(n-3,\{C_3,C_4\})}$. Therefore

$$P(G) \leq P([n] \setminus X)2^3 \leq 2^{\mathrm{ex}(n-3,\{C_3,C_4\})+3} \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})},$$

where the last inequality is by Fact 1 with $i = 3$. □

We will use the following lemma to prove Theorem 14. Observe for all $n \geq 2$, $\mathrm{ex}_\Pi(n, 4, 9) > 0$ implies that for all $G \in \mathcal{P}(n, 4, 9)$, every edge in G has multiplicity at least 1. We will write xyz to denote the three element set $\{x, y, z\}$.

Lemma 13. *Suppose $n \geq 4$ and $G = ([n], w) \in \mathcal{P}(n, 4, 9)$ satisfies $\mu(G) \geq 3$. Then one of the following hold.*

(i) There is $xyz \in \binom{[n]}{3}$ such that $\mu(G[[n] \setminus xyz]) \leq 2$ and $P(G) \leq 6 \cdot P([n] \setminus xyz)$.

(ii) There is $xy \in \binom{[n]}{2}$ such that $\mu(G[[n] \setminus xy]) \leq 2$ and $P(G) \leq 3 \cdot P([n] \setminus xy)$.

Proof. Suppose $n \geq 4$ and $G = ([n], w) \in \mathcal{P}(n, 4, 9)$ is such that $\mu(G) \geq 3$. Fix $xy \in \binom{[n]}{2}$ such that $w(xy) = \mu(G)$. We begin by proving some preliminaries about G and xy. We first show $w(xy) = 3$. By assumption, $w(xy) \geq 3$. Suppose towards a contradiction $w(xy) \geq 4$. Choose some $u \neq v \in [n] \setminus xy$. Since every edge in G has multiplicity at least 1, $5 + w(xy) \leq S(\{x, y, u, v\}) \leq 9$. This implies $w(xy) \leq 9 - 5 = 4$, and consequently $w(xy) = 4$. Combining this with the fact that every edge has multiplicity at least 1, we have

$$9 \leq 4 + w(uv) + w(ux) + w(vx) + w(yu) + w(yv) = S(\{x, y, u, v\}) \leq 9.$$

Consequently, $w(uv) = w(ux) = w(vx) = w(yu) = w(yv) = 1$. Since this holds for all pairs $uv \in \binom{[n]}{2} \setminus xy$, we have shown $P(G) = w(xy) = 4$. Because $n \geq 4$, Fact 1 implies

$$2^{\mathrm{ex}(n,\{C_3,C_4\})} \geq 2^{\mathrm{ex}(4,\{C_3,C_4\})} = 2^3 > 4 = P(G).$$

Combining this with Lemma 10 shows $P(G) < 2^{\mathrm{ex}(n,\{C_3,C_4\})} \leq \mathrm{ex}_\Pi(n, 4, 5)$, a contradiction. Thus $\mu(G) = w(xy) = 3$. We now show that for all $uv \in \binom{[n]}{2} \setminus xy$, $w(uv) \leq 2$. Fix $uv \in \binom{[n]}{2} \setminus xy$ and suppose towards a contradiction $w(uv) \geq 3$. Choose some $X \in \binom{[n]}{4}$ containing $\{x, y, u, v\}$. Because every edge in G has multiplicity at least 1, we have that $S(X) \geq w(uv) + w(xy) + 4 \geq 10$, a contradiction. Thus $w(uv) \leq 2$ for all $uv \in \binom{[n]}{2} \setminus xy$. We now show that for all $z \in [n] \setminus xy$, at most one of $w(xz)$ or $w(yz)$ is equal to 2. Suppose towards a contradiction there is $z \in [n] \setminus xy$ such that $w(zx) = w(zy) = 2$. Note $S(xyz) \geq 7$. So for each $z' \in [n] \setminus xyz$, $S_{z'}(xyz) \leq 9 - S(xyz) = 9 - 7 = 2$.

But since every edge has multiplicity at least 1 this is impossible. Thus for all $z \in [n] \setminus xy$, at most one of $w(xz)$ or $w(yz)$ is equal to 2.

We now prove either (i) or (ii) holds. Suppose there is $z \in [n] \setminus xy$ such that one of $w(zx)$ or $w(zy)$ is equal to 2. Then by what we have shown, $\{w(xy), w(zx), w(zy)\} = \{3, 1, 2\}$, and consequently $P(xyz) = 6$. By Lemma 12, since $S(xyz) \geq 6$, we have that

$$P(G) = P(xyz)P([n] \setminus xyz) = 6 \cdot P([n] \setminus xyz).$$

By the preceding arguments, $\mu(G[[n] \setminus xyz])) \leq 2$. Thus (i) holds. Suppose now that for all $z \in [n] \setminus xy$, $w(xz) = w(yz) = 1$. Then $P(G) = w(xy)P([n] \setminus xy) = 3 \cdot P([n] \setminus xy)$. By the preceding arguments, $\mu(G[[n] \setminus xy]) \leq 2$. Thus (ii) holds. □

Theorem 14. *For all $n \geq 4$, $\mathcal{P}(n, 4, 9) \subseteq F_{\leq 2}(n, 4, 9)$.*

Proof. Fix $n \geq 4$ and $G = ([n], w) \in \mathcal{P}(n, 4, 9)$. Suppose towards a contradiction $G \notin F_{\leq 2}(n, 4, 9)$. We show $P(G) < 2^{\mathrm{ex}(n,\{C_3,C_4\})}$, contradicting that G is product-extremal (since by Lemma 10, $2^{\mathrm{ex}(n,\{C_3,C_4\})} \leq \mathrm{ex}_\Pi(n, 4, 9)$).

Since $G \notin F_{\leq 2}(n, 4, 9)$, either (i) or (ii) of Lemma 13 holds. If (i) holds, choose $xyz \in \binom{[n]}{3}$ with $\mu(G[[n]\setminus xyz]) \leq 2$ and $P(G) \leq 6 \cdot P([n]\setminus xyz)$. Let $H \in F_{\leq 2}(n-2, 4, 9)$ be such that $G[[n]\setminus xy] \cong H$. If $n \in \{4, 5, 6\}$, then $P(G) \leq 6 \cdot P(H) \leq 6 \cdot 2^{\binom{n-3}{2}} < 2^{\mathrm{ex}(n,\{C_3,C_4\})}$, where the second inequality is because $\mu(H) \leq 2$, and the strict inequality is from the exact values for $\mathrm{ex}(n, \{C_3, C_4\})$ for $n \in \{4, 5, 6\}$. If $n \geq 7$, then by Lemma 13 and because $n - 3 \geq 4$, $P(H) \leq 2^{\mathrm{ex}(n-3,\{C_3,C_4\})}$. Therefore,

$$P(G) \leq 6 \cdot P(H) \leq 6 \cdot 2^{\mathrm{ex}(n-3,\{C_3,C_4\})} < 2^{\mathrm{ex}(n-3,\{C_3,C_4\})+3} \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})},$$

where the last inequality is by Fact 1. If (ii) holds, choose $xy \in \binom{[n]}{2}$ with $\mu(G[[n] \setminus xy])) \leq 2$ and $P(G) \leq 3 \cdot P([n] \setminus xy)$. Let $H \in F_{\leq 2}(n - 2, 4, 9)$ be such that $G[[n] \setminus xy] \cong H$. If $n \in \{4, 5\}$, then $P(G) \leq 3 \cdot P(H) \leq 3 \cdot 2^{\binom{n-2}{2}} < 2^{\mathrm{ex}(n,\{C_3,C_4\})}$, where the second inequality is because $\mu(H) \leq 2$, and the strict inequality is from the exact values for $\mathrm{ex}(n, \{C_3, C_4\})$ for $n \in \{4, 5\}$. If $n \geq 6$, then $n - 2 \geq 4$ and Lemma 13 imply $P(H) \leq 2^{\mathrm{ex}(n-2,\{C_3,C_4\})}$. Therefore,

$$P(G) \leq 3 \cdot P([n] \setminus xy) \leq 3 \cdot 2^{\mathrm{ex}(n-2,\{C_3,C_4\})} < 2^{\mathrm{ex}(n-2,\{C_3,C_4\})+2} \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})},$$

where the last inequality is by Fact 1. □

Proof of Theorem 5. Fix $n \geq 4$ and $G \in \mathcal{P}(n, 4, 9)$. By Theorem 14, $G \in F_{\leq 2}(n, 4, 9)$. By Theorem 13, this implies $P(G) \leq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$. By Lemma 10, $P(G) \geq 2^{\mathrm{ex}(n,\{C_3,C_4\})}$. Consequently, $P(G) = 2^{\mathrm{ex}(n,\{C_3,C_4\})} = \mathrm{ex}_\Pi(n, 4, 9)$. □

References

[1] József Balogh, Robert Morris, and Wojciech Samotij, *Independent sets in hypergraphs*, J. Amer. Math. Soc. **28** (2015), no. 3, 669–709. MR 3327533

[2] J. A. Bondy and Zs. Tuza, *A weighted generalization of Turán's theorem*, J. Graph Theory **25** (1997), no. 4, 267–275. MR 1459892 (99c:05097)

[3] G. Dirac, *Extensions of turán's theorem on graphs*, Acta Mathematica Academiae Scientiarum Hungarica **14** (1963), no. 3, 417–422.

[4] P. Erdős, *Some recent results on extremal problems in graph theory. Results*, Theory of Graphs (Internat. Sympos., Rome, 1966), Gordon and Breach, New York; Dunod, Paris, 1967, pp. 117–123 (English); pp. 124–130 (French). MR 0227049

[5] ______, *On some new inequalities concerning extremal properties of graphs*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 77–81. MR 0232703

[6] Paul Erdős, Daniel J. Kleitman, and Bruce L. Rothschild, *Asymptotic enumeration of K_n-free graphs*, Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II, Accad. Naz. Lincei, Rome, 1976, pp. 19–27. Atti dei Convegni Lincei, No. 17. MR 0463020 (57 #2984)

[7] V. Falgas-Ravry, O'Connell K., Strmberg J., and Uzzell A., *Multicolour containers and the entropy of decorated graph limits*, arXiv:1607.08152 [math.CO], 2016.

[8] Zoltán Füredi and André Kündgen, *Turán problems for integer-weighted graphs*, J. Graph Theory **40** (2002), no. 4, 195–225. MR 1913847 (2003g:05075)

[9] ______, *Turán problems for integer-weighted graphs*, J. Graph Theory **40** (2002), no. 4, 195–225. MR 1913847 (2003g:05075)

[10] David K. Garnick, Y. H. Harris Kwong, and Felix Lazebnik, *Extremal graphs without three-cycles or four-cycles*, Journal of Graph Theory **17** (1993), no. 5, 633–645.

[11] R. L. Graham, M. Grötschel, and L. Lovász (eds.), *Handbook of combinatorics. Vol. 1, 2*, Elsevier Science B.V., Amsterdam; MIT Press, Cambridge, MA, 1995. MR 1373655

[12] Daniel J. Kleitman and Kenneth J. Winston, *On the number of graphs without 4-cycles*, Discrete Math. **41** (1982), no. 2, 167–172. MR 676877

[13] J.A. Kuchenbrod, *Extremal problems on weighted graphs*, Ph.D. thesis, University of Kentucky, 1999.

[14] D. Mubayi and C. Terry, *An extremal graph problem with a transcendental solution*, arXiv:1607.07742, 2016.

[15] David Saxton and Andrew Thomason, *Hypergraph containers*, Inventiones mathematicae **201** (2015), no. 3, 925–992 (English).

[16] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 279–319. MR 0233735