Turán problems in pseudorandom graphs

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Abstract

Given a graph F, we consider the problem of determining the densest possible pseudorandom graph that contains no copy of F. We provide an embedding procedure that improves a general result of Conlon, Fox, and Zhao which gives an upper bound on the density. In particular, our result implies that optimally pseudorandom graphs with density greater than $n^{-1/3}$ must contain a copy of the Peterson graph, while the previous best result gives the bound $n^{-1/4}$. Moreover, we conjecture that the exponent $1/3$ in our bound is tight. We also construct the densest known pseudorandom $K_{2,3}$-free graphs that are also triangle-free. Finally, we give a different proof for the densest known construction of clique-free pseudorandom graphs due to Bishnoi, Ihringer, and Pepe that they have no large clique.

1   Introduction

Given a family $\mathcal{F}$ of graphs we say a graph G is $\mathcal{F}$-free if it does not contain any member in $\mathcal{F}$ as a subgraph. A fundamental problem in extremal graph theory is to determine the maximum number $\mathrm{ex}(n, \mathcal{F})$ of edges in an $\mathcal{F}$-free graph on n vertices. Here $\mathrm{ex}(n, \mathcal{F})$ is called the *Turán number* of $\mathcal{F}$, and the limit $\pi(F) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{2}$, whose existence was proved by Katona, Nemetz, and Simonovits [23], is called the *Turán density* of $\mathcal{F}$.

For a graph G we use $V(G)$ to denote the vertex set of G, and use $v(G)$ and $e(G)$ to denote the number of vertices and edges in G, respectively. For a set $S \subset V(G)$ we use $e_G(S)$ to denote the number of edges in the induced subgraph $G[S]$. Given two vertex sets $X, Y \subset V(G)$ we use $e_G(X, Y)$ to denote the number of edges in G that have one vertex in X and one vertex in Y (here edges with both vertices in $X \cap Y$ are counted twice, hence $e_G(X, X) = 2e_G(X)$). We will omit the subscript G if it is clear from the context.

Informally, we say that a graph is pseudorandom if its edge distribution behaves like a random graph. In this note we use the following notation, which was firstly introduced by Thomason in his fundamental papers [38, 39], to quantify the randomness of a graph.

For two real numbers $p \in [0, 1]$ and $\alpha \geq 0$, we say a graph G is (p, α)-*jumbled* if it satisfies

$$\left| e(X, Y) - p|X||Y| \right| \leq \alpha \sqrt{|X||Y|} \tag{1}$$

for all $X, Y \subset V(G)$.

A special family of (p, α)-jumbled graphs are the well-known (n, d, λ)-*graphs*. A graph G is an (n, d, λ)-graph if it is a d-regular graph on n vertices and the second largest eigenvalue in absolute value of its adjacency matrix is λ. The well-known Expander Mixing Lemma (e.g. see [27, Theorem 2.11]) implies that an (n, d, λ)-graph is $(d/n, \lambda)$-jumbled. Conversely, Bilu and Linial [9] proved that an n-vertex d-regular (p, α)-jumbled graph is an (n, d, λ)-graph with $\lambda = O(\alpha \log(d/\alpha))$.

It is known that a random graph $G(n, p)$ is almost surely a (p, α)-jumbled graph with $\alpha = O(\sqrt{np})$ (see e.g. [27, Corollary 2.3]). The proof of Erdős and Spencer in [21] can be extended to show that every (p, α)-jumbled graph on n vertices satisfies that $\alpha = \Omega(\sqrt{np})$ (see e.g. [12, 27]), and, in particular, $\lambda = \Omega(\sqrt{d})$ for an (n, d, λ)-graph. Therefore, an n-vertex (p, α)-jumbled graph with $\alpha = \Theta(\sqrt{np})$ can be viewed as optimally pseudorandom. The tightness of the bound $\lambda = \Omega(\sqrt{d})$ in general is also witnessed by many well-known explicit constructions. For example, the well-known triangle-free (n, d, λ)-graph constructed by Alon [2] satisfies $d = \Theta(n^{2/3})$ and $\lambda = O(\sqrt{d})$.

Constructions of dense pseudorandom graphs that avoid a certain graph as a subgraph are extremely useful for many problems. In particular, the second author and Verstraëte [32] recently showed that for every fixed integer $t \geq 3$, the existence of K_t-free (n, d, λ)-graphs with $d = \Omega(n^{1 - \frac{1}{2t-3}})$ and $\lambda = O(\sqrt{d})$ implies the lower bound $R(t, n) = \Omega^*(n^{t-1})$ for the off-diagonal Ramsey numbers, and this matches the best known upper bound in exponent (see [31] for recent breakthrough result on $R(4, n)$). More generally, [32] shows that the existence of dense F-free pseudorandom graphs implies a good lower bound for the Ramsey number $R(F, n)$. This motivates us to consider the following pseudorandom version of the Turán problem.

Let $\mathcal{F}$ be a family of graphs and $C > 0$ be a real number. Let $\mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})$ be the maximum number of edges in an n-vertex (p, α)-jumbled $\mathcal{F}$-free graph with $\alpha \leq C\sqrt{np}$. Note that in the definition of $\mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})$ we do not have any restriction on p.

In many applications, it suffices to know the exponent of $\mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})$. So we let

$$\exp(\mathcal{F}) = \lim_{C \to \infty} \limsup_{n \to \infty} \frac{\log\left(\mathrm{ex}_{\mathrm{rand}}(n, C, \mathcal{F})/n\right)}{\log n}.$$

In other words, $\exp(\mathcal{F})$ is the supremum of β such that there exist a constant C and a sequence $(G_n)_{n=1}^{\infty}$ of $\mathcal{F}$-free (p_n, α_n)-jumbled graphs with

$$\lim_{n \to \infty} v(G_n) = \infty, \quad \lim_{n \to \infty} \frac{\log(p_n v(G_n))}{\log v(G_n)} \geq \beta, \quad \text{and} \quad \alpha_n \leq C\sqrt{p_n v(G_n)}.$$

Using the Expander Mixing Lemma one can prove that for every integer $t \geq 3$ we have $\exp(K_t) \leq 1 - \frac{1}{2t-3}$. Alon's construction [2] shows that this bound is tight for $t = 3$, that is, $\exp(K_3) = \frac{2}{3}$. It is a major open problem to determine $\exp(K_t)$ in general. Alon and Krivelevich proved in [4] that $\exp(K_t) \geq 1 - \frac{1}{t}$. Recently, Bishnoi, Ihringer, and Pepe [10] improved their bound and proved the following result.

Theorem 1.1 (Bishnoi–Ihringer–Pepe [10])**.** *It holds that $\exp(K_t) \geq 1 - \frac{1}{t-1}$ for all integers $t \geq 4$.*

Mattheus and Pavese [30] give a different construction of K_t-free pseudorandom graphs which also matches the bound in Theorem 1.1. In Section 4, we will present a construction

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that is isomorphic to the construction of Bishnoi, Ihringer, and Pepe [10], and give a new proof to Theorem 1.1.

For bipartite graphs, the pseudorandom version of the Turán problem does not appear to differ much from the ordinary Turán problem since many constructions for the lower bound are pesudorandom. For example, for complete bipartite graphs, the projective norm graphs (see [25, 5]) are optimally pseudorandom (see [36]) and do not contain $K_{s,t}$ with $t \geq (s-1)! + 1$. Therefore, together with the well known Kövari–Sós–Turán Theorem [26], we know that $\exp(K_{s,t}) = 1 - \frac{1}{s}$ for all positive integers s, t with $t \geq (s-1)! + 1$. For even cycles, constructions from generalized polygons [28] and an old result of Bondy and Simonovits [13] imply that $\exp(C_6) = \frac{1}{3}$ and $\exp(C_{10}) = \frac{1}{5}$. The value of $\exp(C_{2k})$ for $k \neq 2, 3, 5$ are still unknown due to the lack of constructions. For non-bipartite graphs, $\exp(F)$ is completely different from the ordinary Turán problem (indeed, $\exp(F) < 2$ while $\pi(F) > 0$) and there are very graphs F for which $\exp(F)$ is known. For example, for odd cycles, a construction due to Alon and Kahale [3] together with Proposition 4.12 in [27] implies that $\exp(C_\ell) = \frac{2}{\ell}$ for all odd integers $\ell \geq 3$.

The first general upper bound on $\exp(F)$, is due to Kohayakawa, Rödl, Schacht, Sissokho, and Skokan [24]. They prove that $\exp(F) \leq 1 - \frac{1}{2\nu(F)-1}$ for every triangle-free graph F. Here $\nu(F) = \frac{1}{2}(d(F) + D(F) + 1)$, where $D(F) = \min\{2d(F), \Delta(F)\}$ and $d(F)$ is the degeneracy of F. This was improved via the following result of Conlon, Fox, and Zhao in [17].

Theorem 1.2 (Conlon–Fox–Zhao [17])**.** *For every graph F, we have $\exp(F) \leq 1 - \frac{1}{2d_2(F)+1}$, where $d_2(F)$ is the minimum real number d such that there is an ordering of the vertices $v_1, \ldots, v_m$ of F so that $N_{<i}(v_i) + N_{<i}(v_j) \leq 2d$ for all edges $v_i v_j \in F$. Here $N_{<i}(v)$ is the number of neighbors of v in $\{v_1, \ldots, v_{i-1}\}$.*

In some cases the bound provided by Theorem 1.2 is sharp (e.g. it is conjectured to be sharp for cliques), but we speculate that for most graphs F it can be improved. Below we give an improvement that holds for many graphs.

Theorem 1.3. *Let F be a fixed graph and d be such that the following holds. There exists an ordering $v_1, v_2, \ldots, v_m$ of the vertices of F and a $1 \leq k \leq m$ such that:*

- *$F[\{v_k, \ldots, v_m\}]$ is a forest,*

- *for all edges $v_i v_j \in F$ with $i < j$, $i < k$, we have $N_{<i}(v_i) + N_{<i}(v_j) \leq 2d$, and*

- *for all edges $v_i v_j \in F$ with $k \leq i < j$, we have $N_{<k}(v_i) + N_{<k}(v_j) \leq 2d$.*

Then, $\exp(F) \leq 1 - \frac{1}{2d+1}$.

Remark. Roughly speaking, Theorem 1.3 says that if there exists an induced forest on an interval of some ordering of $V(F)$, then it is possible to improve the bound of Theorem 1.2 by treating this forest as one edge. In fact, we will see later that the proof of Theorem 1.3 can be extended easily to get a more general result (see Theorem 2.5). In addition, it was observed by a referee that only one-sided jumbledness, namely $e(X, Y) - p|X||Y| \geq -\alpha\sqrt{|X||Y|}$, is necessary to prove all of the results,

As an application of Theorem 1.3, we present an upper bound for $\exp(\mathbf{P})$, where $\mathbf{P}$ is the Petersen graph. The Petersen graph was considered by several researchers in related

contexts. For example, Tait and Timmons [37] proved that the Erdős–Rényi orthogonal polarity graphs [20] (henceforth the Erdős–Rényi graph), which are optimally pseudorandom C_4-free graphs, contain the Petersen graph as a subgraph. Conlon, Fox, Sudakov, and Zhao asked in [16] whether there is a counting lemma for the Petersen graph in an n-vertex C_4-free graph with $\Omega(n^{3/2})$ edges. We know very little about $\exp(\mathbf{P})$, for example, it is not known whether $\exp(\mathbf{P}) \geq \frac{1}{2}$. The only lower bound we have is $\exp(\mathbf{P}) \geq \exp(C_5) = \frac{2}{5}$. In the other direction, the previous best upper bound is $\exp(\mathbf{P}) \leq \frac{3}{4}$ that follows from [17] (it is not too difficult to prove that $d_2(\mathbf{P}) = \frac{3}{2}$).
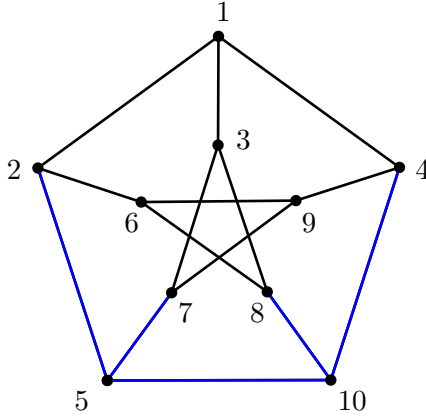

Figure 1: The induced subgraph of $\mathbf{P}$ on $\{2, 4, 5, 7, 8, 10\}$ is a tree.

Theorem 1.4. *We have* $\exp(\boldsymbol{P}) \leq \frac{2}{3}$.

It is easy to observe that Theorem 1.4 follows from Theorem 1.3 by letting the ordering of $V(\mathbf{P})$ be $(v_1, \ldots, v_{10}) = (1, 3, 6, 9, 2, 4, 5, 7, 8, 10)$ and choosing $k = 5$. We conjecture that $\exp(\mathbf{P}) = \frac{2}{3}$. We think that the construction of K_3-free pseudorandom graphs due to Kopparty does not contain the Petersen graph as a subgraph. If this is true, then it will prove the lower bound $\exp(\mathbf{P}) \geq \frac{2}{3}$. For completeness, we include his construction here.

Let $p \neq 3$ be a prime, and let $\mathbb{F}_q$ be a finite field with where $q = p^h$ for some integer $h \geq 1$. Recall that the *absolute trace function* $\mathrm{Tr} \colon \mathbb{F}_q \to \mathbb{F}_p$ is defined as $\mathrm{Tr}(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{h-1}}$ for every $\alpha \in \mathbb{F}_q$.

Let $V = \mathbb{F}_q^3$, $T = \{x \in \mathbb{F}_q \colon \mathrm{Tr}(x) \in \{1, -1\}\}$, and $S \subset \mathbb{F}_q^3$ be a subset defined as

$$S = \left\{ (xy, xy^2, xy^3) \colon x \in T, \ y \in \mathbb{F}_q \setminus \{0\} \right\}.$$

Kopparty's construction is the graph G on V in which two vertices $\mathbf{u}, \mathbf{v} \in V$ are adjacent iff $\mathbf{u} - \mathbf{v} \in S$. Using some simple linear algebra one can show that G is triangle-free, and using some results about finite fields and abelian groups one can prove that G is an (n, d, λ)-graph with $n = q^3$, $d = \Theta(\frac{q^2}{p})$, and $\lambda = \Theta(\frac{q}{p})$.

Remark. Ferdinand Ihringer informed us that the construction above contains an induced copy of the Petersen graph when $p = 2$ and $h = 3$, and he thinks that, in general, Kopparty's construction contains many copies of the Petersen graph. Nevertheless, it is still might be true that $\exp(\mathbf{P}) = \frac{2}{3}$.

Our next result about $K_{2,3}$ was motivated by an old problem of Erdős [19], which asks if

$$\mathrm{ex}(n, \{K_3, C_4\}) = \left(\frac{1}{2\sqrt{2}} + o(1) \right) n^{3/2}$$

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is true. A construction due to Parsons [33] for the lower bound comes from the Erdős–Rényi graph by removing half of its vertices. Since the Erdős–Rényi graph is optimally pseudorandom, Parsons' construction also implies that $\mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, C_4\}) \geq \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2}$ for some absolute constant C. In [1], Allen, Keevash, Sudakov, and Verstraëte proved that the extremal constructions for $\mathrm{ex}(n, \{K_3, K_{2,t}\})$ cannot be bipartite for every $t \geq 3$ by constructing a $\{K_3, K_{2,t}\}$-free graph whose number of edges is greater than the maximum number of edges in a $\{K_3, K_{2,t}\}$-free bipartite graph. However, their construction is $(t-1)$-partite, and therefore it does not give a lower bound for $\mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, K_{2,3}\})$. The previous best lower bound is

$$\mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, K_{2,3}\}) \geq \mathrm{ex}_{\mathrm{rand}}(n, C, \{K_3, C_4\}) \geq \left(\frac{1}{2\sqrt{2}} - o(1)\right) n^{3/2}$$

that follows from Parsons' construction. We improve this and present a construction of the densest known $\{K_3, K_{2,3}\}$-free pseudorandom graphs.

Theorem 1.5. *We have* $\mathrm{ex}_{\mathrm{rand}}(n, 2, \{K_3, K_{2,3}\}) \geq \left(\frac{1}{2} - o(1)\right) n^{3/2}$.

Remark. Thang Pham pointed out to us that the following construction[1] also provides a lower bound for $\mathrm{ex}_{\mathrm{rand}}(n, 2, \{K_3, K_{2,3}\})$. Let q be an odd prime power. The distance graph D on $\mathbb{F}_q^2$ is a graph whose vertex set is $\mathbb{F}_q^2$, and two points $(x_1, x_2), (y_1, y_2)$ are adjacent iff $(x_1 - y_1)^2 + (x_2 - y_2)^2 = 1$. The $K_{2,3}$-freeness of D follows from the fact that any two cycles have at most two points in the intersection. The K_3-freeness of D follows from results in [8]. The pseudorandomness of D follows from results in [22]. Additionally, a referee informed us that constructions in an even earlier work by Bannai–Shimabukuro–Tanaka [7] also satisfy the desired properties.

In Section 2, we prove Theorems 1.3. In Section 3, we prove Theorem 1.5. In Section 4 we present a new proof of Theorem 1.1. Throughout the paper we will omit the use of floors and ceilings to make the presentation cleaner.

2 Proof of Theorem 1.3

We prove Theorem 1.3 in this section.

Let us present two standard lemmas. We start with the following direct consequence of the definition of a jumbled graph.

Lemma 2.1. *Fix a real number $q > 1$. Let G be a (p, α)-jumbled graph on n vertices and let $X, Y_1, \ldots, Y_m \subseteq V(G)$ be pairwise disjoint subsets. If $|X||Y_i| \geq q^2 m \left(\frac{\alpha}{p}\right)^2$ for all $i \in [m]$, then there exists a vertex $x \in X$ with at least $\frac{q-1}{q} p|Y_i|$ neighbours in Y_i for all $i \in [m]$. In particular, $e(X, Y_i) > 0$ for all $i \in [m]$.*

Proof. Define $X_i := \{v \in X : |N_G(v) \cap Y_i| < \frac{q-1}{q} p|Y_i|\}$ for $i \in [m]$. It suffices to prove that $|X_i| < \frac{|X|}{m}$ for all $i \in [m]$. Suppose to the contrary that $|X_i| \geq \frac{|X|}{m}$ for some $i \in [m]$. By the definition of jumbleness, we get

$$e(X_i, Y_i) \geq p|X_i||Y_i| - \alpha\sqrt{|X_i||Y_i|} = \left(p - \frac{\alpha}{\sqrt{|X_i||Y_i|}}\right)|X_i||Y_i|.$$

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[1] It was pointed out by a referee to us that this construction is essentially a two-dimensional version of Brown's $K_{3,3}$-free construction [15].

It follows from $|X||Y_i| \geq q^2 m \left(\frac{\alpha}{p}\right)^2$ that $\alpha \leq \frac{p}{q}\sqrt{\frac{|X||Y_i|}{m}} \leq \frac{p}{q}\sqrt{|X_i||Y_i|}$. Therefore, it follows from the inequality above that

$$e(X_i, Y_i) \geq \frac{q-1}{q} p |X_i||Y_i|,$$

but our definition of X_i yields $e(X_i, Y_i) < \frac{q-1}{q} p |X_i||Y_i|$, a contradiction. ∎

The next lemma is a simple cleaning procedure which is useful in problems concerning (p, α)-jumbled graphs.

Lemma 2.2. *Let G be a (p, α)-jumbled graph on n vertices. Then, for all sets $X, Y \subseteq V(G)$ such that $|X||Y| \geq 100(\alpha/p)^2$ the following holds. There exist subsets $X' \subseteq X, Y' \subseteq Y$ respectively of size at least $9|X|/10$ and $9|Y|/10$ such that all $v \in X'$ have $d(v, Y') \geq p|Y'|/10$ and all $u \in Y'$ have $d(u, X') \geq p|X'|/10$.*

Proof. Consider the following process. Start with $X_0 := X, Y_0 := Y$ and at step $i \geq 0$, do the following. Take $G_i := G[X_i, Y_i]$ and if there exists a vertex $v \in X_i$ such that $d(v, Y_i) < p|Y_i|/10$ or a vertex $v \in Y_i$ such that $d(v, X_i) < p|X_i|/10$, remove it from X_i, Y_i respectively, giving new X_{i+1}, Y_{i+1}. We claim that this process stops before $|X_i| \leq 9|X|/10$ or $|Y_i| \leq 9|Y|/10$, which would imply that we are done. Indeed, if it did not stop before that, consider the step at which, w.l.o.g., $|X_i| = 9|X|/10$ and $|Y_i| \geq 9|Y|/10$. By construction, every vertex in $X \setminus X_i$ has less than $p|Y|/10 \leq p|Y_i|/9$ neighbours in Y_i. So,

$$e(X \setminus X_i, Y_i) \leq p|Y_i||X \setminus X_i|/9.$$

On the other hand, the definition of a (p, α)-jumbled graph implies that

$$e(X \setminus X_i, Y_i) \geq p|Y_i||X \setminus X_i| - \alpha\sqrt{|Y_i||X \setminus X_i|}$$

which is a contradiction since $|X \setminus X_i||Y_i| \geq \frac{1}{10} \cdot \frac{9}{10} \cdot |X||Y| \geq 4\left(\frac{\alpha}{p}\right)^2$. ∎

We also need the following embedding lemma for forests.

Lemma 2.3. *Suppose that T is a forest on $[m]$ and G is an n-vertex (p, α)-jumbled graph. Let $X_1, \ldots, X_m \subset V(G)$ be nonempty pairwise disjoint subsets of $V(G)$ that satisfy*

$$|X_i||X_j| \geq 2^m \left(\frac{\alpha}{p}\right)^2$$

for all edges ij in T. Then there exists an embedding of $f : T \to G$ such that $f(i) \in X_i$ for all $i \in [m]$.

Proof. We prove this lemma by induction on m. The base case $m = 1$ is clear since X_1 is nonempty. So we may assume that $m \geq 2$. Without loss of generality, we may assume that the vertex m is a leaf of T and the vertex $m - 1$ is its neighbor in T. Let $T' := T - m$. Let

$$X'_{m-1} := \{v \in X_{m-1} : |N_G(v) \cap X_m| \geq 1\}$$

We claim that $|X'_{m-1}| \geq |X_{m-1}|/2$. Indeed, suppose to the contrary that $|X'_{m-1}| < |X_{m-1}|/2$. Then we would have $|X_m||X_{m-1} \setminus X'_{m-1}| \geq |X_m||X_{m-1}|/2 \geq 2^{m-1}(\alpha/p)^2 \geq$

$2(\alpha/p)^2$, it follows from Lemma 2.1 (with $q = \sqrt{2} > 1$) that $e(X_m, X_{m-1} \setminus X'_{m-1}) > 0$. This contradicts the fact that $e(X_m, X_{m-1} \setminus X'_{m-1}) = 0$. Hence, $|X'_{m-1}| \geq |X_{m-1}|/2$.

Now apply the induction hypothesis to the sets $X_1, \ldots, X_{m-2}, X'_{m-1}$, we obtain an embedding $f : T' \to G$ such that $f(i) \in X_i$ for $i \in [m-2]$ and $f(m-1) \in X'_{m-1}$. By the definition of X'_{m-1}, there exists $v \in X_m$ such that $\{f(m-1), v\} \in G$. Hence we can extend f to get an embedding of T to G by setting $f(m) = v$. This completes the proof of Lemma 2.3. ∎

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let G be a (p, α)-jumbled graph with $\alpha < \frac{p^{d+1}n}{C}$ for an arbitrarily large constant $C > m4^m$. We will show that G contains a copy of F. This implies that $\exp(F) \leq 1 - \frac{1}{2d+1}$. Indeed, if $C_1 > 0$ and $C_2 > (C_1C)^{\frac{2}{2d+1}}$ and G is (p, α)-jumbled with $p > C_2 n^{-\frac{1}{2d+1}}$ and $\alpha < C_1\sqrt{pn}$, then a short calculation shows that $\alpha < \frac{p^{d+1}n}{C}$ and our result will imply the theorem.

Consider an ordering $v_1, v_2, \ldots, v_m$ of the vertices of F and a $1 \leq k \leq m$ such that:

(i) $F[\{v_k, \ldots, v_m\}]$ is a forest,

(ii) for all edges $v_i v_j \in F$ with $i < j$, $i < k$, we have $N_{<i}(v_i) + N_{<i}(v_j) \leq 2d$, and

(iii) for all edges $v_i v_j \in F$ with $k \leq i < j$, we have $N_{<k}(v_i) + N_{<k}(v_j) \leq 2d$.

Let us denote $F[\{v_1, \ldots, v_{k-1}\}]$ by F_1 and $F[\{v_k, \ldots, v_m\}]$ by F_2. We will first embed a copy of F_1 using (b). At the same time, we will also ensure by (c), that the candidate sets for the vertices $v_k, \ldots, v_m$ are still large enough so that the forest F_2 can be embedded in them, thus giving an embedding of F.

Take a partition $V(G) = V_1 \cup \cdots \cup V_m$ such that $|V_i| \geq \frac{n}{2m}$ for all $i \in [m]$.

Claim 2.4. *Let $s \in [k-1]$. Then there exist vertices $u_j \in V_j$ for all $j \in [s]$ such that $G[\{u_1, \ldots, u_s\}]$ contains a copy of $F[\{v_1, \ldots, v_s\}]$ and*

$$V_{i,s} := V_i \cap \left(\bigcap_{j \leq s : \, v_i v_j \in F} N_G(u_j) \right)$$

satisfies the inequality

$$|V_{i,s}| \geq \frac{p^{|N_{\leq s}(v_i)|}|V_i|}{2^s} \geq \frac{p^{|N_{\leq s}(v_i)|}n}{m2^m}$$

for all $i \in [s+1, m]$.

Proof. For every $j \in [m]$ let $I_j := \{\ell \in [j+1, m] : v_j v_\ell \in F\}$. The proof is by induction on s. For the base case $s = 1$, first observe that

$$|V_1||V_j| \geq \left(\frac{n}{2m} \right)^2 \geq 4m \left(\frac{\alpha}{p} \right)^2$$

for all $j \in I_1$. Hence we can apply Lemma 2.1 to V_1 and V_j for all $j \in I_1$ with $q = 2$ to obtain a vertex $u_1 \in V_1$ such that $d(u_1, V_j) \geq p|V_j|/2$ for all $j \in I_1$. Now suppose that $s \geq 2$. Apply the induction hypothesis to get $u_i \in V_i$ for $i \in [s-1]$ such that

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$G[\{u_1, \ldots, u_{s-1}\}]$ contains a copy of $F[\{v_1, \ldots, v_{s-1}\}]$ and for every $i \in [s, m]$ the set $U_i := V_{i,s-1}$ satisfies

$$|U_i| \geq \frac{p^{|N_{\leq s-1}(v_i)|}|V_i|}{2^{s-1}}. \tag{2}$$

Observe that for every $j \in I_s$ we have

$$|U_s||U_j| \geq \frac{p^{|N_{\leq s-1}(v_s)|}n}{m2^m} \cdot \frac{p^{|N_{\leq s-1}(v_j)|}n}{m2^m} = \frac{p^{|N_{\leq s-1}(v_s)|+|N_{\leq s-1}(v_j)|}n^2}{m^2 2^{2m}} \geq \frac{p^{2d}n^2}{m^2 2^{2m}}$$
$$> 2^m \left(\frac{\alpha}{p}\right)^2. \tag{3}$$

In the second last inequality we used (b), and in the last inequality we used the assumption that $\alpha \leq p^{d+1}n/(m4^m)$. So we may apply Lemma 2.1 to U_s and U_j for all $j \in I_s$ with $q = 2$ and obtain $u_s \in U_s$ such that $d(u_s, U_j) \geq p|U_j|/2$ for all $j \in I_s$. Now by (2), for every $i \in I_s$ we have

$$|V_{i,s}| \geq |U_i \cap N(u_s)| \geq \frac{p}{2} \frac{p^{|N_{\leq s-1}(v_i)|}|V_i|}{2^{s-1}} = \frac{p^{|N_{\leq s}(v_i)|}|V_i|}{2^s}.$$

On the other hand, by (2), for every $i \in [s+1, m] \setminus I_s$, we have

$$|V_{i,s}| = |U_i| \geq \frac{p^{|N_{\leq s-1}(v_i)|}|V_i|}{2^{s-1}} \geq \frac{p^{|N_{\leq s}(v_i)|}|V_i|}{2^s}.$$

Finally, it is clear that $G[\{u_1, u_2, \ldots, u_s\}]$ contains a copy of $F[\{v_1, v_2, \ldots, v_s\}]$, so the proof of the claim is complete. ∎

Applying Claim 2.4 with $s = k-1$ we obtain $u_j \in V_j$ for $j \in [k-1]$ such that $G[\{u_1, \ldots, u_{k-1}\}]$ contains a copy of F_1 and

$$|V_{i,k-1}| \geq \frac{p^{|N_{<k}(v_i)|}n}{m2^m}$$

for all $i \in [k, m]$. Now that the first portion of the graph has been embedded, it remains only to embed a forest on the given candidate sets $X_i := V_{i,k-1}$. If we find an embedding $f \colon F_2 \to G$ with $f(v_i) \in X_i$ for all $i \in [k, m]$, then $G[\{u_1, \ldots, u_{k-1}, f(v_k), \ldots, f(v_m)\}]$ contains a copy of F. Similar to (3), by (c) and Claim 2.4, for every $\{v_i, v_j\} \in F_2$ we have

$$|X_i||X_j| \geq \frac{p^{|N_{<k}(v_i)|}n}{m2^m} \cdot \frac{p^{|N_{<k}(v_j)|}n}{m2^m} = \frac{p^{|N_{<k}(v_i)|+|N_{<k}(v_j)|}n^2}{m^2 2^{2m}} \geq \frac{p^{2d}n^2}{m^2 2^{2m}} > 2^m \left(\frac{\alpha}{p}\right)^2.$$

Applying Lemma 2.3 with $T = F_2$ and the sets $X_k, \ldots, X_m$, we know that such an embedding f exists. This completes the proof of Theorem 1.3. ∎

We remark that there are some graphs to which the precise statement of the above theorem cannot be applied in order to get a tight result - for example, odd cycles. However, the proof can be slightly adapted to deal with them. For odd cycles we take $k = 2$, so that $F[v_k, \ldots, v_m]$ is a path; this F_2 can then be embedded in a different way than in the general theorem above (see [17] for more details), in particular, using also the expansion properties of (α, p)-jumbled graphs.

We now give a further generalization of Theorem 1.3 where instead of partitioning the graph into two parts which are dealt with separately, we partition the graph into several parts.

For every graph F on m vertices, let $\hat{d}_2(F)$ denote the smallest number d for which there exists an ordering $v_1, v_2, \ldots, v_m$ of $V(F)$ such that the following statements hold for some $\ell \in \mathbb{N}$ and $1 = k_1 < k_2 < \cdots < k_\ell < k_{\ell+1} = m$:

(i) $F[\{v_{k_s}, \ldots, v_{k_{s+1}-1}\}]$ is a forest for all $s \in [\ell]$,

(ii) for all edges $v_i v_j \in F \setminus \left(\bigcup_{s=1}^{\ell} F[\{v_{k_s}, \ldots, v_{k_{s+1}-1}\}] \right)$ with $i < j$, we have $N_{<i}(v_i) + N_{<i}(v_j) \leq 2d$, and

(iii) for all $s \in [\ell]$ and for all edges $v_i v_j \in F[\{v_{k_s}, \ldots, v_{k_{s+1}-1}\}]$, we have $N_{<k_s}(v_i) + N_{<k_s}(v_j) \leq 2d$.

It is clear that $\hat{d}_2(F) \leq d_2(F)$ since in the definition of $d_2(F)$ we always let $\ell = m-1$ and $k_i = i$ for all $i \in [m]$.

Theorem 2.5. *For every graph F we have* $\exp(F) \leq 1 - \frac{1}{2\hat{d}_2(F)+1}$.

Remark. One can extend Theorem 2.5 to get a counting result for F in pseudorandom graphs that improves Theorem 1.14 in [17] (by replacing $d_2(F)$ there with $\hat{d}_2(F)$ here). This could result in some improvements for the corresponding Turán and Ramsey problems in pseudorandom graphs (see Theorems 1.4, 1.5, and 1.6 in [17]).

3 $\{K_{2,3}, K_3\}$-free pseudorandom graphs

In this section we present a construction of $\{K_{2,3}, K_3\}$-free pseudorandom graphs thereby proving Theorem 1.5.

Suppose that F is a finite group and $S \subset H$ is a symmetric subset, i.e. $S = S^{-1}$. Then the *Cayley graph* $\mathrm{Cay}(H, S)$ is a graph on F with edge set

$$\{\{v, vs\} \colon v \in H \text{ and } s \in S\}.$$

The spectrum, i.e. the eigenvalues of the adjacency matrix, of a Cayley graph can be represented by the characters of of F (see e.g. [29, 6]). For our purpose, we only need the following result for the case that F is an abelian group.

Recall that an abelian group H can be represented as $H = \bigoplus_{i=1}^{k} \mathbb{Z}_{n_i}$ for some integers k and $n_1, \ldots, n_k$. For abelian groups we have a simple description of all the characters. For each $\mathbf{a} = (a_1, \ldots, a_k) \in \bigoplus_{i=1}^{k} \mathbb{Z}_{n_i}$ we have a character $\psi_{\mathbf{a}} \colon H \to \mathbb{C}$ defined by

$$\psi_{\mathbf{a}}(h_1, \ldots, h_k) = \prod_{i=1}^{k} \omega_{n_i}^{a_i h_i},$$

where $\omega_t = e^{2\pi i/t}$.

Lemma 3.1 (see e.g. [29, 6]). *Suppose that $H = \bigoplus_{i=1}^{k} \mathbb{Z}_{n_i}$ is an abelian group. Then the spectrum of the Cayley graph $\mathrm{Cay}(H, S)$ is*

$$\left\{ \sum_{s \in S} \psi_{\boldsymbol{a}}(\boldsymbol{s}) \colon \boldsymbol{a} \in H \right\}.$$

Our main result is as follows.

Theorem 3.2. *Suppose that $p \neq 3$ is a prime number, $H = \mathbb{Z}_p^2$, and*

$$S = \left\{ (x, x^3) \colon x \in \mathbb{Z}_p \setminus \{0\} \right\}.$$

Then $\mathrm{Cay}(H, S)$ is a $\{K_3, K_{2,3}\}$-free (n, d, λ)-graph with $n = p^2$, $d = p - 1$, and $\lambda \leq 2\sqrt{p} + 1$.

We will use the following well known estimate of Weil in the proof of Theorem 3.2.

Recall that the *order* of a character χ is the smallest positive integer d such that $\chi^d = \chi_0$, where χ_0 is the trivial character.

Theorem 3.3 (see e.g. [11, Theorem 13.3]). *Let χ be a character of order $d > 1$. Suppose that $f(X) \in \mathbb{F}[X]$ has precisely m distinct zeros and it is not a dth power, that is $f(X)$ is not the form $c\,(g(X))^d$, where $c \in \mathbb{F}$ and $g(X) \in \mathbb{F}[X]$. Then*

$$\left| \sum_{x \in \mathbb{F}} \chi\left(f(x)\right) \right| \leq (m - 1)\sqrt{p}.$$

Proof of Theorem 3.2. Let $G = \mathrm{Cay}(H, S)$. Let $n = p^2$. It is clear that the number of vertices in G is n, and it follows from the definition of Cayley graphs that G is $|S|$-regular. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A_G of G. Since G is regular, we have $\lambda_1 = |S| = p - 1$.

First we prove that G is K_3-free. Suppose to the contrary that there exist three vertices $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{Z}_p^2$ that form a copy of K_3 in G. Assume that $\mathbf{v} - \mathbf{u} = (a, a^3)$, $\mathbf{w} - \mathbf{v} = (b, b^3)$, and $\mathbf{u} - \mathbf{w} = (c, c^3)$. Then

$$a + b + c = 0,$$
$$a^3 + b^3 + c^3 = 0.$$

Therefore,

$$0 = (a + b + c)(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3) = ab(a + b) + ac(a + c) + bc(b + c)$$
$$= -3abc.$$

Since $p \neq 3$, we must have $0 \in \{a, b, c\}$, a contradiction.

Next we prove that G is $K_{2,3}$-free. It is equivalent to show that every pair of vertices $\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{Z}_p^2$ has at most two common neighbors. Let $(a, b) = \mathbf{u} - \mathbf{v}$. A common neighbor of $\mathbf{u}$ and $\mathbf{v}$ implies that there exist $x, y \in \mathbb{Z}_p \setminus \{0\}$ such that

$$y - x = a, \text{ and}$$
$$y^3 - x^3 = b.$$

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These two equations imply that $(x+a)^3 = x^3+b$, which simplifies to $3ax^2+3a^2x+a^3-b = 0$. Since $(a,b) \neq (0,0)$ and $p \neq 3$, this quadratic equation in x has at most two solutions in $\mathbb{Z}_p \setminus \{0\}$. Therefore, $\mathbf{u}$ and $\mathbf{v}$ have at most two common neighbors.

Finally, we prove that $|\lambda_i| \leq 2\sqrt{p}$ for all $i \in [2,n]$. By Lemma 3.1, for every $i \in [n]$ there exists $(a_1,a_2) \in \mathbb{Z}_p^2$ such that

$$\lambda_i = \lambda_{(a_1,a_2)} = \sum_{\mathbf{s} \in S} \varphi_{(a_1,a_2)}(\mathbf{s}) = \sum_{x=1}^{p-1} \omega_p^{a_1 x + a_2 x^3}$$

If $(a_1,a_2) = (0,0)$, then $\lambda_{(a_1,a_2)} = |S| = p-1$, and this corresponds to λ_1. So we may assume that $(a_1,a_2) \neq (0,0)$.

First, it is easy to see that the character $\chi \colon \mathbb{Z}_p \to \mathbb{C}^\times$ defined by $\chi(\alpha) = \omega_p^\alpha$ for all $\alpha \in \mathbb{Z}_p$ has order p. On the other hand, since $(a_1,a_2) \neq (0,0)$ and $p \neq 3$, the polynomial $f(X) = a_1 X + a_2 X^3$ is not of the form $c\,(g(X))^p$ for any $c \in \mathbb{Z}_p$ and for any polynomial $g(X)$. Therefore, it follows from Theorem 3.3 that

$$\left| \sum_{x=1}^{p-1} \omega_p^{a_1 x + a_2 x^3} \right| = \left| \sum_{x=0}^{p-1} \omega_p^{a_1 x + a_2 x^3} - 1 \right| \leq \left| \sum_{x=0}^{p-1} \omega_p^{a_1 x + a_2 x^3} \right| + 1 \leq 2\sqrt{p} + 1.$$

This implies that $|\lambda_i| \leq 2\sqrt{p} + 1$ for all $i \in [2,n]$, and hence completes the proof of Theorem 3.2. ∎

4 K_t-free pseudorandom graphs

In this section, we present a construction that is isomorphic to the construction of Bishnoi, Ihringer, and Pepe [10], and provide a new proof of Theorem 1.1. For simplicity, we omit the short proof of the (p,α)-jumbledness part (which can be found in [4, p.220] and [10, Theorem 9]) and focus only on the K_t-freeness part.

Denote by $\mathrm{PG}(t-1,q)$ the $(t-1)$-dimensional projective space over $\mathbb{F}_q$, i.e. $\mathrm{PG}(t-1,q) = \mathbb{F}_q^t/\sim$, where two vectors $\mathbf{x},\mathbf{y} \in \mathbb{F}_q^t$ are equivalent under $\sim$ if there exists a non-zero element $a \in \mathbb{F}_q$ such that $\mathbf{x} = a\mathbf{y}$. For a vector $\mathbf{x} \in \mathbb{F}_q^t$ we use $[\![\mathbf{x}]\!]$ to denote its equivalence class in $\mathbb{F}_q^t/\sim$. It is easy to see that the number of points in $\mathrm{PG}(t-1,q)$ is $\frac{q^t-1}{q-1} = (1+o(1))q^{t-1}$. Recall that the dot-product $\mathbf{x} \cdot \mathbf{y}$ of two vectors $\mathbf{x},\mathbf{y} \in \mathbb{F}_q^t$ is defined as $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^t x_i y_i$. A point $\mathbf{x} = (x_1,\ldots,x_t) \in \mathbb{F}_q^t$ is called

- *absolute* if $\mathbf{x} \cdot \mathbf{x} = 0$,
- *square* if $\mathbf{x} \cdot \mathbf{x} = a^2$ for some $a \in \mathbb{F}_q$,
- *non-square* if $\mathbf{x} \cdot \mathbf{x} \neq a^2$ for all $a \in \mathbb{F}_q$,

We use $X_0(t,q), X_\square(t,q), X_\boxtimes(t,q)$ to denote the collection of all absolute points, square points, and non-square points in $\mathbb{F}_q^t$, respectively. If t and q are clear from the context, we will omit them and use $X_0, X_\square, X_\boxtimes$ for simplicity. It is easy to see from the definition that if $\mathbf{x} \in X_0$, $\mathbf{x} \in X_\square$, or $\mathbf{x} \in X_\boxtimes$, then $[\![\mathbf{x}]\!] \subset X_0$, $[\![\mathbf{x}]\!] \subset X_\square$, or $[\![\mathbf{x}]\!] \subset X_\boxtimes$, respectively.

Recall that a *character* of a group H is a homomorphism $\psi\colon H \to \mathbb{C}^\times$, and the *quadratic character* $\chi(\cdot)$ of $\mathbb{F}_q$ is defined as

$$\chi(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \text{ is a square}, \\ -1, & \text{if } x \text{ is a non-square}. \end{cases}$$

Let $\mathrm{AK}(t-1, q)$ be the graph whose vertices are non-absolute points of $\mathrm{PG}(t-1, q)$ and two vertices $[\![\mathbf{x}]\!]$ and $[\![\mathbf{y}]\!]$ are adjacent iff $\mathbf{x} \cdot \mathbf{y} = 0$. Note that $\mathrm{AK}(2, q)$ is just the Erdős–Renyi graph. In [4], Alon and Krivelevich proved that $\mathrm{AK}(t - 1, q)$ is a K_{t+1}-free (n, d, λ)-graph with $n = (1 + o(1))q^{t-1}$, $d = \Theta(n^{1-\frac{1}{t-1}})$, and $\lambda = \Theta(\sqrt{d})$.

Parsons [33] proved that for q odd, the induced subgraph of $\mathrm{AK}(2, q)$ on $X_\boxtimes/\sim$ is K_3-free. Indeed, suppose to the contrary that there exist three distinct points $[\![\mathbf{x}_1]\!], [\![\mathbf{x}_2]\!], [\![\mathbf{x}_3]\!] \in X_\boxtimes/\sim$ that induce a copy of K_3 in $\mathrm{AK}(2, q)$. Then $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are pairwise orthogonal, which means that there exists a non-zero element $a \in \mathbb{F}_q$ such that $\mathbf{x}_3 = a\mathbf{x}_1 \times \mathbf{x}_2$, where $\mathbf{x}_1 \times \mathbf{x}_2$ is the *cross-product* of $\mathbf{x}_1$ and $\mathbf{x}_2$. Therefore, we have

$$\mathbf{x}_3 \cdot \mathbf{x}_3 = (a\mathbf{x}_1 \times \mathbf{x}_2) \cdot (a\mathbf{x}_1 \times \mathbf{x}_2) = a^2 \cdot (\mathbf{x}_1 \cdot \mathbf{x}_1) \cdot (\mathbf{x}_2 \cdot \mathbf{x}_2),$$

where in the last equality we used the fact that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. Applying the quadratic character $\chi(\cdot)$ to both sides of the equation above we obtain

$$-1 = \chi(\mathbf{x}_3 \cdot \mathbf{x}_3) = \chi(a^2) \cdot \chi(\mathbf{x}_1 \cdot \mathbf{x}_1) \cdot \chi(\mathbf{x}_2 \cdot \mathbf{x}_2) = 1 \cdot (-1) \cdot (-1) = 1,$$

a contradiction. Therefore, the induced subgraph of $\mathrm{AK}(2, q)$ on $X_\boxtimes/\sim$ is K_3-free.

Our aim in this section is to extend Parsons' proof to the following general symmetric bilinear forms on $\mathbb{F}^t \times \mathbb{F}^t$ for all $t \geq 4$.

Fix a non-degenerate symmetric bilinear form $B\colon \mathbb{F}^t \times \mathbb{F}^t \to \mathbb{F}$ such that

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i \in [t]} a_i x_i y_i \quad \text{for all} \quad (\mathbf{x}, \mathbf{y}) \in \mathbb{F}^t \times \mathbb{F}^t, \tag{4}$$

where $a_1, \ldots, a_t$ are non-zero elements in $\mathbb{F}_q$. A point $\mathbf{x} = (x_1, \ldots, x_t) \in \mathbb{F}_q^t$ is called

- *B-absolute* if $B(\mathbf{x}, \mathbf{x}) = 0$,

- *B-square* if $B(\mathbf{x}, \mathbf{x}) = a^2$ for some $a \in \mathbb{F}_q$,

- *non-B-square* if $B(\mathbf{x}, \mathbf{x}) \neq a^2$ for all $a \in \mathbb{F}_q$,

We use $X_{B,0}(t, q), X_{B,\square}(t, q), X_{B,\boxtimes}(t, q)$ to denote the collection of all B-absolute points, B-square points, and non-B-square points in $\mathbb{F}_q^t$, respectively. Similarly, let $\mathrm{AK}_B(t - 1, q)$ be the graph whose vertices are non-B-absolute points of $\mathrm{PG}(t - 1, q)$ and two vertices $[\![\mathbf{x}]\!]$ and $[\![\mathbf{y}]\!]$ are adjacent iff $B(\mathbf{x}, \mathbf{y}) = 0$.

The cross-product used by Parsons [33] in $\mathbb{F}_q^3$ can be extended to t-dimensional space $\mathbb{F}_q^t$ under the general symmetric bilinear form defined by (4) for every $t \geq 4$ by letting

$$\mathbf{x}_1 \times \cdots \times \mathbf{x}_{t-1} := \star(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_{t-1}).$$

Here $\star\colon \bigwedge^{t-1} \mathbb{F}_q^t \to \bigwedge^1 \mathbb{F}_q^t$ is the Hodge star map defined by

$$\mathbf{x} \wedge (\star\mathbf{y}) := B(\mathbf{x}, \mathbf{y})\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_t \quad \text{for all } \mathbf{x}, \mathbf{y} \in \bigwedge^{t-1} \mathbb{F}_q^t,$$

where $\mathbf{e}_1 := (1, 0, \ldots, 0), \cdots, \mathbf{e}_t := (0, \ldots, 0, 1)$ (see [14, Section 3]). We refer the reader to [35, 18] for more discussions on this definition. Here we only list some basic properties of the cross-product defined above.

Fact 4.1 (see e.g. [18]). *Suppose that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{t-1} \in \mathbb{F}_q^t$ are $t-1$ vectors. Then*

(i) $\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1}$ *is skew-symmetric and linear in each $\boldsymbol{x}_i$,*

(ii) $\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1}$ *is a vector that is orthogonal to each of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{t-1}$,*

(iii) $\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1} = 0$ *iff $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{t-1}$ are linearly dependent.*

A proof of the following theorem for the case where $B(\cdot, \cdot)$ is the dot-product can be found in [34, 18]. In fact, their proof[2] works for the symmetric bilinear form defined by (4) (see [34, Proof C] and [18, p.888] for more details).

Theorem 4.2 (see e.g. [18]). *Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{t-1}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{t-1} \in \mathbb{F}_q^t$ and $B\colon \mathbb{F}^t \times \mathbb{F}^t \to \mathbb{F}$ be the non-degenerate symmetric bilinear form defined by (4). Then*

$$B(\boldsymbol{x}_1 \times \cdots \times \boldsymbol{x}_{t-1}, \ \boldsymbol{y}_1 \times \cdots \times \boldsymbol{y}_{t-1}) = \prod_{i \in [t]} a_i \cdot \det \begin{bmatrix} B(\boldsymbol{x}_1, \boldsymbol{y}_1) & \ldots & B(\boldsymbol{x}_1, \boldsymbol{y}_{t-1}) \\ \vdots & \ddots & \vdots \\ B(\boldsymbol{x}_{t-1}, \boldsymbol{y}_1) & \ldots & B(\boldsymbol{x}_{t-1}, \boldsymbol{y}_{t-1}) \end{bmatrix}.$$

Our main result in this section is as follows.

Theorem 4.3. *Suppose that q is an odd prime power, $t \geq 3$ is an integer, and $B\colon \mathbb{F}^t \times \mathbb{F}^t \to \mathbb{F}$ is the non-degenerate symmetric bilinear form defined by (4). Then the following statements hold.*

(i) *If t is odd and $\prod_{i \in [t]} a_i$ is a square, then the induced subgraph of $\mathrm{AK}_B(t-1, q)$ on $X_{B,\boxtimes}/\sim$ is a K_t-free.*

(ii) *If t is even and $\prod_{i \in [t]} a_i$ is a non-square, then the induced subgraph of $\mathrm{AK}_B(t-1, q)$ on $X_{B,\boxtimes}/\sim$ is a K_t-free.*

(iii) *If $\prod_{i \in [t]} a_i$ is a non-square, then the induced subgraph of $\mathrm{AK}_B(t-1, q)$ on $X_{B,\square}/\sim$ is a K_t-free.*

Proof. Suppose that there exists a set $S = \{[\![\mathbf{x}_1]\!], \ldots, [\![\mathbf{x}_t]\!]\} \subset (X_{B,\boxtimes} \cup X_{B,\square})/\sim$ is a set of t distinct points such that the induced subgraph of $\mathrm{AK}_B(t-1, q)$ on S is complete. Then it follows from Fact 4.1 that there exists a nonzero element $a \in \mathbb{F}_q$ such that $\mathbf{x}_t = a\mathbf{x}_1 \times \cdots \times \mathbf{x}_{t-1}$. Therefore, by Theorem 4.2, we have

$$B(\mathbf{x}_t, \mathbf{x}_t) = B(a\mathbf{x}_1 \times \cdots \times \mathbf{x}_{t-1}, \ a\mathbf{x}_1 \times \cdots \times \mathbf{x}_{t-1})$$

$$= a^2 \cdot \prod_{i \in [t]} a_i \cdot \det \begin{bmatrix} B(\mathbf{x}_1, \mathbf{x}_1) & \ldots & B(\mathbf{x}_1, \mathbf{x}_{t-1}) \\ \vdots & \ddots & \vdots \\ B(\mathbf{x}_{t-1}, \mathbf{x}_1) & \ldots & B(\mathbf{x}_{t-1}, \mathbf{x}_{t-1}) \end{bmatrix} = a^2 \cdot \prod_{i \in [t]} a_i \cdot \prod_{i=1}^{t-1} B(\mathbf{x}_i, \mathbf{x}_i).$$

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[2] We did not check the details to confirm whether it works for a general non-degenerate symmetric bilinear form, but it seems that for the general case, one just needs to replace $\prod_{i \in [t]} a_i$ with $\det(M)$ in Theorem 4.2, where $M \in \mathbb{F}_q^{t \times t}$ is the associated matrix of the bilinear form.

In the last equality, we used the fact that $B(\mathbf{x}_i, \mathbf{x}_j) = 0$ for all $i \neq j$. Applying the quadratic character $\chi(\cdot)$ to both sides of the equation above, we obtain

$$\chi\left(B(\mathbf{x}_t, \mathbf{x}_t)\right) = \chi\left(a^2 \cdot \prod_{i \in [t]} a_i \cdot \prod_{i=1}^{t-1} B\left(\mathbf{x}_i, \mathbf{x}_i\right)\right)$$

$$= \chi(a^2) \cdot \chi\left(\prod_{i \in [t]} a_i\right) \cdot \prod_{i=1}^{t-1} \chi\left(B(\mathbf{x}_i, \mathbf{x}_i)\right) = \chi\left(\prod_{i \in [t]} a_i\right) \cdot \prod_{i=1}^{t-1} \chi\left(B(\mathbf{x}_i, \mathbf{x}_i)\right),$$

which is impossible in the following three cases: t is odd, $\prod_{i \in [t]} a_i$ is a square, and $S \subset X_{B,\boxtimes}/\sim$; t is even, $\prod_{i \in [t]} a_i$ is a non-square, and $S \subset X_{B,\boxtimes}/\sim$; $\prod_{i \in [t]} a_i$ is a non-square, and $S \subset X_{B,\square}/\sim$. ∎

Acknowledgment

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